MATH 255 SET THEORY - DR. MCLOUGHLIN'S CLASS
HANDOUT 14
MORE ON CARDINALITY

Common custom characterises a cardinal number as a number used in counting such as 0, 1, 2, 3, ... It is through custom that we understand what is meant; however, rigor is advised.

In set theory, a cardinal number (also called ``the cardinality") is a type of number defined in such a way that any method of counting sets using it gives the same result. (This is not true for the ordinal numbers [see handout 12]).

$\varnothing$ has 0 members
$\{2\}$ has 1 member
$\{1, 3\}$ has 2 members
$\{2, 5, e\}$ has 3 members (as does the set $\{1, 2, 3\}$ [has 3 members], the set $\{10, 117, -5\}$, etc.)

In fact, the cardinal numbers are obtained by collecting all ordinal numbers which are obtainable by counting a given set.

Def. 14.0: The sets A and B are **equinumerous** (or **equipollent**) iff $\exists$ a bijection, j, such that $j : A \longrightarrow B$. Notation: $A \approx B$ or $|A| = |B|$.

Def. 14.1: A set A is finite iff either:

(1) $\exists$ a bijection, f, such that $f : A \longrightarrow \mathbb{N}_k$ for some $k \in \mathbb{N}$

or (2) $A = \varnothing$

Def. 14.2: A finite set A has cardinality k iff $\exists$ a bijection, f, such that $f : A \longrightarrow \mathbb{N}_k$

for some $k \in \mathbb{N}$. We denote the cardinality of A as $|A| = k$.

Def. 14.3: A finite set A has cardinality 0 iff $A = \varnothing$.
We denote the cardinality of $\varnothing$ as $|\varnothing| = 0$.

Theorem 14.1: Let $U = \mathbb{R}$. Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$ $\left| A \right| = 7$.

Proof: Assume the premises. Define $g : A \longrightarrow \mathbb{N}_7 \ni g(x) = x + 4$.

Exercises:
1. g is a well defined function. (prove this on your own)
2. g is an injection. (prove this on your own)
3. g is a surjection. (prove this on your own)

Hence, g is a bijection.

Thus, $A \approx \mathbb{N}_7$

So, $|A| = 7$.

Q. E. D.

Consider the concept ∞ which I get so upset about. It stands for the idea of infinity (not finite); but, what does it really mean. We attempt to get some notion of said in this handout.

Def. 14.4: A set A is infinite if it is not finite.

Def. 14.5: A set A is denumerable iff $\exists$ a bijection, g, such that $g : A \longrightarrow \mathbb{N}$.

Def. 14.6: A denumerable set A has cardinality $\aleph_0$ iff $\exists$ a bijection, g, such that $g : A \longrightarrow \mathbb{N}$. We denote the cardinality of A as $|A| = \aleph_0$.

(obviously, it is therefore the case that $|\mathbb{N}| = \aleph_0$)

Def. 14.7: A set A is countable iff it is finite or denumerable.

Def. 14.8: A set A is uncountable iff it is not countable.

Theorem 14.2: $|\mathbb{Z}| = \aleph_0$.

Proof: Assume the premises. Define $h : \mathbb{Z} \longrightarrow \mathbb{N} \ni$

$$h(x) = \begin{cases} -2x+1 & x \leq 0 \\ 2x & x > 0 \end{cases}$$

Exercises:

4. h is a well defined function. (prove this on your own)
5. h is an injection. (prove this on your own)
6. h is a surjection. (prove this on your own)

Hence, h is a bijection.

Thus, $\mathbb{Z} \approx \mathbb{N}$

So, $|\mathbb{Z}| = \aleph_0$.

Q. E. D.

Let $\mathbb{E}$ denote the set of even natural numbers

Theorem 14.3: $\left| \mathbb{E} \right| = \aleph_0$.

Proof: Assume the premises. Define $h_1 : \longrightarrow \mathbb{N} \ni h_1(x) = \frac{x}{2}$

Exercises:

7. h_1 is a well defined function. (prove this on your own)
8. h_1 is an injection. (prove this on your own)
9. h_1 is a surjection. (prove this on your own)

Hence, h_1 is a bijection.

Thus, $\mathbb{E} \approx \mathbb{N}$

So, $\left| \mathbb{E} \right| = \aleph_0$.

Q. E. D.

Let $\mathbb{O}$ denote the set of odd natural numbers

Theorem 14.4: $\left| \mathbb{O} \right| = \aleph_0$. (prove this on your own)

Let $\mathcal{E}$ denote the set of even integers.

Theorem 14.5: $\left| \mathcal{E} \right| = \aleph_0$.(prove this on your own)

Let $\mathcal{O}$ denote the set of odd natural numbers

Theorem 14.6: $\left| \mathcal{O} \right| = \aleph_0$. (prove this on your own)

Theorem 14.7: $\left| \mathbb{Q} \right| = \aleph_0$.

Outline of "Proof": [not a proof since we are going to have difficulty defining rigorously the function] Assume the premises.

$$\mathbb{Q} = \{ \frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{Z},\ \wedge n \neq 0 \}$$

So,

$\mathbb{Q} = \{ \ 0,$

$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \ . \ . \ .$

$\frac{-1}{1}, \frac{-2}{1}, \frac{-3}{1}, \frac{-4}{1}, \frac{-5}{1}, \ . \ . \ .$

$\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ . \ . \ .$

$\frac{-1}{2}, \frac{-2}{2}, \frac{-3}{2}, \frac{-4}{2}, \frac{-5}{2}, \ . \ . \ .$

$\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ . \ . \ .$

$\frac{-1}{3}, \frac{-2}{3}, \frac{-3}{3}, \frac{-4}{3}, \frac{-5}{3}, \ . \ . \ .$

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Now, delete all the repetitions in the list
(example: delete $\frac{2}{2}$ since it was previously represented as $\frac{1}{1}$)

Define $h_2 : \mathbb{Q} \longrightarrow \mathbb{N} \ni$

$h_2(0) = 1$
$h_2(1) = 2$
$h_2(-1) = 3$
$h_2(2) = 4$
$h_2(\frac{1}{2}) = 5$
$h_2(-2) = 6$
$h_2(3) = 7$
$h_2(-\frac{1}{2}) = 8$
$h_2(-3) = 9$
$h_2(4) = 10$
$h_2(\frac{1}{3}) = 11$
$h_2(\frac{3}{2}) = 12$
$h_2(-4) = 13$
$h_2(5) = 14$ etc.

h_2 seems a well defined function.
h_2 seems an injection.

h_2 seems a surjection.

Hence, *h* seems a bijection.

Which leads us to conclude [inductively, which is not a proof] that it seems that $\mathbb{Q} \approx \mathbb{N}$

So, we opine that $|\mathbb{Q}| = \aleph_0$.

"Q. E. D."

Actually to prove Theorem 14.7 rigorously, the following set of Theorem 14.s suffices and can be shown rigorously to prove Theorem 14.7 as a result. If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

Theorem 14.9: Let *U* be a well defined universe. Let A and B be denumerable sets. $A \cup B$ is denumerable.

Theorem 14.10: Let *U* be a well defined universe. Let A and B be finite sets. $A \cup B$ is finite.

Theorem 14.11: Let *U* be a well defined universe. Let A and B be countable sets. $A \cup B$ is countable.

Theorem 14.12: Let *U* be a well defined universe. Let A_i be denumerable sets $\forall\ i \in I \ni I \subseteq \mathbb{N}$

Let $\Phi = \{ A_i \mid i \in I\}$
$\cup\, \Phi$ is denumerable.

Theorem 14.13: Let *U* be a well defined universe. Let A_i be finite sets $\forall\ i \in I \ni I \subseteq \mathbb{N}_k$

for some $k \in \mathbb{N}$ Let $\Phi = \{ A_i \mid i \in I\}$. $\cup\, \Phi$ is finite.

Theorem 14.14: Let *U* be a well defined universe. Let A_i be finite sets $\forall\ i \in I \ni I \subseteq \mathbb{N}$

Let $\Phi = \{ A_i \mid i \in I\}$ $\cup\, \Phi$ is denumerable.

Theorem 14.15: Let *U* be a well defined universe. Let A_i be countable sets $\forall\ i \in I \ni I \subseteq \mathbb{N}$

Let $\Phi = \{ A_i \mid i \in I\}$ $\cup\, \Phi$ is countable.

Note that

$$\mathbb{Q} = \{ 0 \} \cup \{ \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \ . \ . \ . \} \cup \{ \frac{-1}{1}, \frac{-2}{1}, \frac{-3}{1}, \frac{-4}{1}, \frac{-5}{1}, \ . \ . \ . \}$$
$$\cup \{ \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ . \ . \ . \} \cup \{ \frac{-1}{2}, \frac{-2}{2}, \frac{-3}{2}, \frac{-4}{2}, \frac{-5}{2}, \ . \ . \ . \} \cup$$
$$\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ . \ . \ . \} \cup \{ \frac{-1}{3}, \frac{-2}{3}, \frac{-3}{3}, \frac{-4}{3}, \frac{-5}{3}, \ . \ . \ . \} \cup \ldots .$$

and hopefully one can see the outline of the proof such that using the Theorem 14.s, we can establish

$|\mathbb{Q}| = \aleph_0$.

<u>Theorem 14.16</u>: $|(0,1)| = |\mathbb{R}|$

<u>Proof:</u> Assume the premises.

Define $h_3 : (0,1) \longrightarrow \mathbb{R} \ni h_3(x) = \tan(\pi(x - \frac{1}{2}))$

Exercises:
10. h_3 is a well defined function. (prove this on your own)
11. h_3 is an injection. (prove this on your own)
12. h_3 is a surjection. (prove this on your own)

Hence, h_3 is a bijection.

Thus, $(0,1) \approx \mathbb{R}$

Q. E. D.

Astonishingly, there are as many numbers between zero and one as there are on the whole line! Think about that: all we have between zero and one is equinumerous with the entire line!

<u>Theorem 14.17</u>: $|\mathbb{R}| > |\mathbb{N}|$

<u>'Proof'</u>: Assume the premises. Suppose $|\mathbb{R}| \leq |\mathbb{N}|$

Suppose further $|\mathbb{R}| < |\mathbb{N}|$. But, $\mathbb{N} \subseteq \mathbb{R}$, which is a contradiction.

So, $|\mathbb{R}| \not< |\mathbb{N}|$.

Hence, $|\mathbb{R}| = |\mathbb{N}|$

Nonetheless, $(0,1) \approx \mathbb{R}$ which implies $|(0,1)| = |\mathbb{N}|$

Let $(0,1) = \{a_i$ such that $i \in \mathbb{N}\}$.

$(0,1) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots\}$
a_i is a decimal such that a_i is expressible as an infinite string of digits.
Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Define a_i as the decimal $.\, a_{i1}\, a_{i2}\, a_{i3}\, a_{i4}\, a_{i5}\, a_{i6}\, a_{i7} \ldots \ni a_{ij} \in D\ \forall\ i \in \mathbb{N}\quad \forall\ j \in \mathbb{N}$

So,
$a_1 = .\, a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, a_{17} \ldots$
$a_2 = .\, a_{21}\, a_{22}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, a_{27} \ldots$
$a_3 = .\, a_{31}\, a_{32}\, a_{33}\, a_{34}\, a_{35}\, a_{36}\, a_{37} \ldots$
$a_4 = .\, a_{41}\, a_{42}\, a_{43}\, a_{44}\, a_{45}\, a_{46}\, a_{47} \ldots$
$a_5 = .\, a_{51}\, a_{52}\, a_{53}\, a_{54}\, a_{55}\, a_{56}\, a_{57} \ldots$
$a_6 = .\, a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, a_{66}\, a_{67} \ldots$
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Define $b = .\, b_{01}\, b_{02}\, b_{03}\, b_{04}\, b_{05}\, b_{06}\, b_{07} \ldots \ni b_{0j} \in D\ \forall\ j \in \mathbb{N}$

AND $b_{0j} = 3$ if $a_{jj} = 4$
$b_{0j} = 4$ if $a_{jj} \neq 4$

Now, $b \in (0,1)$ by line five of this proof .

However, $b \neq a_j\quad \forall\ j \in \mathbb{N}$

Therefore, $b \notin (0,1)$.

So, $b \in (0,1) \wedge\ b \notin (0,1)$, which is a contradiction.

Hence, $|(0,1)| \neq |\mathbb{N}|$.

So, $|\mathbb{R}| \neq |\mathbb{N}|$.

Ergo, $|\mathbb{R}| > |\mathbb{N}|$.

'Q. E. D.'

Notice we used the trichotomy law in this proof. However, the trichotomy law was an axiom of the reals. Hence, this 'proof' is not compleat for we need to make rigorous transfinite trichotomy laws, etc. If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

But, we can see there is a cardinal larger than $\aleph_0$! This is most counter-intuitive and is a major 'headache' for many [including myself many years ago when I started to realise that this was indeed the case.

<u>Def. 14.9</u>: $\left| \mathbb{R} \right| = \aleph_1$. (in many texts $\aleph_1$ is denoted as *c* [the cardinality of the continuum]).

Cantor opined that there was no cardinal between $\aleph_0$ and $\aleph_1$. This is the continuum hypothesis. Further, there is another nifty Theorem 14.(beyond the scope of the course) which generalises the following:

<u>Theorem 14.18</u>: Let *U* be a well defined universe. Let A be a finite set. $\mathcal{P}(A)$ is finite.

<u>Theorem 14.19</u>: Let *U* be a well defined universe. Let A be a finite set.
Let $|A| = k$. $|\mathcal{P}(A)| = 2^k$.

<u>Corollary 14.19</u>:
Let *U* be a well defined universe. Let A be a finite set. $|A| < |\mathcal{P}(A)|$.

It is:
<u>Theorem 14.20</u>: Let *U* be a well defined universe. Let A be a set. $|A| < |\mathcal{P}(A)|$ and $|\mathcal{P}(A)| = 2^{|A|}$.

Exercises:
13. Consider how one would argue such. Does it make sense? If so, why? If not, why not? If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

So, what is ∞? $\aleph_0$? $\aleph_1$? Or is it just a concept?

Great references that delve into the subject in greater detail:

Goldrei, D. (1996). *Classic Set Theory*. New York, NY: Chapman & Hall USA.

Hamos, P. (1974). *Naïve Set Theory*. New York, NY: Springer-Verlag.

Lipschutz, S. (1964). *Schaum's Outline Series: Theory and Problems of Set Theory and Related Topics*. New York, NY: McGraw-Hill.

Suppes, P. (1972). *Axiomatic Set Theory*. New York, NY: Dover Publications.