Handout $11\frac{1}{2}$

Lemmas N' More[1]

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Let U be a well defined universe, let V be another well defined universe, and let V_2 be another well defined universe. For product sets let the universe be $U \times V$, for compositions of relations let the universe be $U \times V \times V_2$ and let the universe be $\mathcal{P}(U)$ for collections. Let $A \subseteq U$, $D \subseteq U$, $B \subseteq V$, and $C \subseteq V$.

Lemma 7: Let $D \subseteq U$. It is the case that $\emptyset \subseteq D$.

Lemma 10: $D \times B = \emptyset$ if and only if $(D = \emptyset \quad \vee \quad B = \emptyset)$

Claim 1: Let $A \ \wedge \ D \subseteq \ U$ whilst $C \ \wedge \ B \ \subseteq V$.
It is the case that $(A \cap D) \times (C \cap B) = (A \times C) \cap (D \times B)$.

Claim 2: Let $A \ \subseteq \ U \ \wedge B \ \subseteq \ V$.
It is the case that $(A \times B)^C = A^C \times B^C$.

Claim 3: Let $A \ \wedge \ D \subseteq \ U$ whilst $C \ \wedge \ B \ \subseteq V$.
It is the case that $(A \times C) \cup (D \times B) \subseteq (A \cup D) \times (C \cup B)$.

Definition 1: Let $D \ \subseteq \ U$ and $C \ \subseteq V$.
We define the set R as the **relation from** A **to** C if and only if $R \subseteq (D \times C)$.

Definition 1.1: Let R be a relation from D to C.
We define the set D as the **domain** of the relation R.
We denote the domain of the relation R as $dom(R)$.

Definition 1.2: Let R be a relation from D to C.
We define the set C as the **codomain** of the relation R.
We denote the codomain of the relation R as $cod(R)$.

Definition 1.3: Let R be a relation from D to C.
We define the set $M = \{x | \ (x, y) \in R\}$ as the **corange**[2] of the relation R.
We denote the corange of the relation R as $cor(R)$.

Definition 1.4: Let R be a relation from D to C.
We define the set $W = \{y | \ (x, y) \in R\}$ as the **range** of the relation R.
We denote the range of the relation R as $ran(R)$.

[1] Not to be confused with Linens N' Things which went bankrupt (I think).

[2] This is **non-standard**. We define a corange for two reasons; first, because it make it 'nice' to have the domain and codomain then the range and the corange. Second it makes defining functions, in my mind, more facile.

Example of an Exercise 0: Let $U = \mathbb{N}$ and $V = \mathbb{Z}$.
Let $D = \mathbb{N}_{10}$ and $C = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
Let $Q = \{(1,2), (1,3), (2,1), (4,-1), (2,0), (5,-1)\}$
A. Find the domain of Q. B. Find the codomain of Q.
C. Find the corange of Q. D. Find the range of Q.
Look at Q. It should be clear the solutions are:
Solution A. $dom(Q) = D$.
Solution B. $cod(Q) = C$.
Solution C. $cor(Q) = \{1, 2, 4, 5\}$.
Solution D. $ran(Q) = \{-1, 0, 1, 2, 3\}$.

Example of an Exercise 1: Let $U = \mathbb{Z}$ and $V = \mathbb{N}^*$.
Let $D = \mathbb{N}_4$ and $C = \mathbb{N}_2^*$ Let $J = \{(x,y) | x < y\}$
A. Find the domain of J. B. Find the codomain of J.
C. Find the corange of J. D. Find the range of J.
Look at J. It is $\{(1,2)\}$.
$\Longrightarrow$
Solution A. $dom(J) = D$ (duh).
Solution B. $cod(J) = C$ (duh).
Solution C. $cor(J) = \{1\}$.
Solution D. $ran(J) = \{2\}$.

Example of an Exercise 2: Let $U = \mathbb{Z}$ and $V = \mathbb{N}^*$.
Let $D = \mathbb{N}_4$ and $C = \mathbb{N}_2^*$ Let $K = \{(x,y) | x \geq y\}$
A. Find the domain of K. B. Find the codomain of K.
C. Find the corange of K. D. Find the range of K.
Look at K. It is
$\{(1,0), (2,0), (3,0), (4,0), (1,1), (2,1), (3,1), (4,1), (2,2), (3,2), (4,2)\}$.
$\Longrightarrow$
Solution A. $dom(K) = D$ (duh).
Solution B. $cod(K) = C$ (duh).
Solution C. $cor(K) = \{1, 2, 3, 4\}$.
Solution D. $ran(K) = \{0, 1, 2, 3\}$.

Example of an Exercise 3: Let $U = \mathbb{Z}$ and $V = \mathbb{N}^*$.
Let $D = \mathbb{N}_4$ and $C = \mathbb{N}_2^*$ Let $M = \{(x,y) | x > y\}$
A. Find the domain of M. B. Find the codomain of M.
C. Find the corange of M. D. Find the range of M.
Look at M. It is $\{(1,0), (2,0), (3,0), (4,0), (2,1), (3,1), (4,1), (3,2), (4,2)\}$.
$\Longrightarrow$
Solution A. $dom(M) = D$ (duh).
Solution B. $cod(M) = C$ (duh).
Solution C. $cor(M) = \{1, 2, 3, 4\}$.
Solution D. $ran(M) = \{0, 1, 2\}$.

Example of an Exercise 4: Let $U = \mathbb{Z}$ and $V = \mathbb{N}^*$.
Let $D = \mathbb{N}_4$ and $C = \mathbb{N}_2^*$ Let $M = \{(x,y) | x \leq y\}$
A. Find the domain of P. B. Find the codomain of P.
C. Find the corange of P. D. Find the range of P.
Look at P. It is $\{(1,1), (1,2), (2,2)\}$.
$\Longrightarrow$
Solution A. $dom(P) = D$ (duh).
Solution B. $cod(P) = C$ (duh).
Solution C. $cor(P) = \{1, 2\}$.
Solution D. $ran(P) = \{1, 2\}$.

Example of an Exercise 5: Let $U = \mathbb{Z}$ and $V = \mathbb{N}^*$.
Let $A = \mathbb{N}_8$ and $B = \mathbb{N}_9$ Let $W = \{(x, y) \,|\exists p \in \mathbb{N} \ni x = p \cdot y \quad \wedge \quad x = 2 \cdot m \ni m \in \mathbb{N}\}$
A. Find the domain of W. B. Find the codomain of W.
C. Find the corange of W. D. Find the range of W.
Look at W. It is
$\{(2,1), (2,2), (4,1), (4,2), (4,4), (6,1), (6,2), (6,3), (6,6), (8,1), (8,2), (8,4), (8,8)\}$.
$\Longrightarrow$
Solution A. $dom\,(W) = A$ (duh).
Solution B. $cod\,(W) = B$ (duh).
Solution C. $cor\,(W) = \{2, 4, 6, 8\}$.
Solution D. $ran\,(W) = \{1, 2, 3, 4, 6, 8\}$.

Definition 2: Let $D \subseteq U$.
We define the set R as the **relation from** D **to** D if and only if $R \subseteq (D \times D)$.
We call the set R as the **relation on** D also.

Example of an Exercise 6: Let $U = \mathbb{Z}$.
Let $A = \mathbb{N}_{12}$. Let $X = \{(x, y) \,|\exists w \in \mathbb{N} \ni y = w \cdot x \quad \wedge \quad y = 3 \cdot m \ni m \in \mathbb{N}\}$
Find X and note the domain, codomain, range, and corange.
The domain of $X = A$ (duh) The codomain of $X = A$ (duh^2).
We find X and it is
$\{(1,3), (1,6), (1,9), (1,12), (2,6), (2,12), (3,3), (3,6), (3,9), (3,12), (4,12), (6,6), (6,12), (9,9), (12,12)\}$.
$\Longrightarrow$
So, $cor\,(X) = \{1, 2, 3, 4, 6, 9, 12\}$.
$ran\,(W) = \{3, 6, 9, 12\}$.

Definition 2.1: Let R be a relation on D.
We define the relation R as a **reflexive** relation on D iff $\forall x \in D$, $(x, x) \in R \wedge \exists x \in D$.

Definition 2.2: Let R be a relation on D.
We define the relation R as a **symmetric** relation on D iff $(x, y) \in R \Longrightarrow (y, x) \in R$.

Definition 2.3: Let R be a relation on D.
We define the relation R as a **antisymmetric** relation on D iff $x \neq y$, $(x, y) \in R \Longrightarrow (y, x) \notin R$.

Definition 2.4: Let R be a relation on D.
We define the relation R as a **transitive** relation on D iff $(x, y) \in R \wedge (y, z) \in R \Longrightarrow (x, z) \in R$.

Alternate Definition 2.3: Let R be a relation on D.
We define the relation R as a **antisymmetric** relation on D iff $(x, y) \in R \wedge (y, x) \in R \Longrightarrow x = y$.

Example Exercise 6: Let $U = \mathbb{Z}$.
$A = \mathbb{N}_{12}$. $X = \{(x, y) \,|\exists w \in \mathbb{N} \ni y = w \cdot x \quad \wedge \quad y = 3 \cdot m \ni m \in \mathbb{N}\}$
Note X is not reflexive on A.
Note X is not symmetric on A.
Note X is antisymmetric on A.
Is X is transitive on A or is it not? You figure it out.

Exercise 7: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$
Let $Q = \{(1,2),(1,3),(2,1),(4,3),(2,2),(5,5)\}$
A. Determine if Q is reflexive on D. B. Determine if Q is symmetric on D.
C. Determine if Q is antisymmetric on D. D. Determine if Q is transitive on D.

Exercise 8: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $M = \{(x,y)\,|x \leq y\}$
A. Determine if M is reflexive on D. B. Determine if M is symmetric on D.
C. Determine if M is antisymmetric on D. D. Determine if M is transitive on D.

Exercise 9: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $X = \{(x,y)\,|\exists w \in \mathbb{N} \ni y = w \cdot x \quad \wedge \quad y = 3 \cdot m \ni m \in \mathbb{N}\}$
A. Determine if X is reflexive on D. B. Determine if X is symmetric on D.
C. Determine if X is antisymmetric on D. D. Determine if X is transitive on D.

Exercise 10: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $I = \{(x,y)\,|x = y\}$
A. Determine if I is reflexive on D. B. Determine if I is symmetric on D.
C. Determine if I is antisymmetric on D. D. Determine if I is transitive on D.

Exercise 11: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $Y = \{(x,y)\,|\exists w \in \mathbb{N} \ni y = w \cdot x\}$
A. Determine if Y is reflexive on D.
B. Determine if Y is symmetric on D.
C. Determine if Y is antisymmetric on D.
D. Determine if Y is transitive on D.

Exercise 12: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $S = \{(x,y)\,|x = y \vee y = 2 \cdot x \vee x = 2 \cdot y\}$
A. Determine if S is reflexive on D.
B. Determine if S is symmetric on D.
C. Determine if S is antisymmetric on D.
D. Determine if S is transitive on D.

Definition 3: Let R be a relation from D to C.
We define the relation $R^{-1} = \{(y,x)\,|\,(x,y) \in R\}$.
We note that $R^{-1} \subseteq C \times D$.

Definition 4: Let R be a relation from D to C and S be a relation from C to E.
We define the relation $W = S \circ R$ as $W = \{(x,y)\,|\exists p \in C \ni (x,p) \in R \wedge (p,y) \in S\}$.
We note that $W \subseteq D \times E$.

Exercise 13: Let $U = \mathbb{N}$.
Let $D = \mathbb{N}_{10}$. Let $X = \{(x,y)\,|\exists w \in \mathbb{N} \ni y = w \cdot x \quad \wedge \quad y = 3 \cdot m \ni m \in \mathbb{N}\}$
and $Y = \{(x,y)\,|\exists w \in \mathbb{N} \ni y = w \cdot x\}$
A. Find X^{-1}.
B. Find $X \circ Y$.
C. Find $Y \circ X$.
D. Find $X \circ X$.