Handout $1\frac{1}{2}$

A Tad More on the Concept of Limit

Dr. M. P. M. M. McLoughlin

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Let our universe be $\mathbb{R} \times \mathbb{R}$ which is the Cartesian plane. Let $D \subseteq \mathbb{R}$

§6 Some Points About Evaluating Limits of a Function

We have four distinct methods for finding limits:

(1) The plug-in method.
(2) Graphing.
(3) The bulldozer.
(4) Algebraic manipulation.

So, do the following:

Exercise 6.1. *Which is the 'best' or 'preferred' method for problems? Answer this in your own words.*

Exercise 6.2. *Evaluate* $\lim_{x \to 4} (x^2 + 3)$.

Exercise 6.3. *Evaluate* $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right)$.

Exercise 6.4. *Evaluate* $\lim_{x \to 2} \left(\frac{x^2 - 4}{x + 2} \right)$.

Exercise 6.5. *Evaluate* $\lim_{x \to 2} \left(\frac{3x^3 + x^2 - 23x - 21}{2x^3 - 9x^2 + 10x - 3} \right)$.

Exercise 6.6. *Evaluate* $\lim_{x \to 1} \left(\frac{3x^3 + x^2 - 23x - 21}{2x^3 - 9x^2 + 10x - 3} \right)$.

Exercise 6.7. *Evaluate* $\lim_{x \to 3} \left(\frac{3x^3 + x^2 - 23x - 21}{2x^3 - 9x^2 + 10x - 3} \right)$.

Exercise 6.8. *Evaluate* $\lim_{x \to -1} \left(\frac{3x^3 + x^2 - 23x - 21}{2x^3 - 9x^2 + 10x - 3} \right)$.

Exercise 6.9. *Evaluate* $\lim_{x \to 1} (\lfloor x - 1 \rfloor)$.

Exercise 6.10. *Evaluate* $\lim_{x \to \frac{1}{2}} (\lfloor x - 1 \rfloor)$.

Exercise 6.11. *Evaluate* $\lim_{x \to \frac{1}{2}} (\lfloor 2x - 1 \rfloor)$.

Exercise 6.12. *Evaluate* $\lim_{x \to \infty} \left(\frac{5}{x^6 - 3} \right)$.

One thing bears repeating:

Theorem 6.1. *Let $D \subseteq \mathbb{R}$, a be a real number such that $[a, \infty) \subseteq D$. Let $f : D \longrightarrow \mathbb{R}$ be a well defined function. Let b be a real number and c be a positive real number. Let $f(x) = \dfrac{b}{x^c}$.*
It is the case that $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to \infty} f(x) = 0$.

§7 Some Theorems and Useful Heuristics About Limits of a Function

Theorem 6.2. *Let $D \subseteq \mathbb{R}$, a be a real number such that there exists a real number b where $(a - b, a) \cup (a, a + b) \subseteq D$. Let $f : D \longrightarrow \mathbb{R}$ be a well defined function[1]. Suppose f is a quotient of polynomials (meaning $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x) \wedge q(x)$ are polynomials). Suppose when a person tries the plug in method it gets $\dfrac{k}{0}$ where $k \in \mathbb{R}$ but $k \neq 0$. Then it MUST be the case that $\lim\limits_{x \to a} f(x)$ does not exist.*

Theorem 6.3. *Let $D \subseteq \mathbb{R}$, a be a real number such that there exists a real number b where $(a - b, a) \cup (a, a + b) \subseteq D$. Let $f : D \longrightarrow \mathbb{R}$ be a well defined function[2]. Suppose f is a quotient of polynomials (meaning $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x) \wedge q(x)$ are polynomials). Suppose when a person tries the plug in method it gets $\dfrac{0}{k}$ where $k \in \mathbb{R}$ but $k \neq 0$. Then it MUST be the case that $\lim\limits_{x \to a} f(x) = 0$.[3]*

Heuristic 6.1. *Let $D \subseteq \mathbb{R}$, a be a real number such that there exists a real number b where $(a - b, a) \cup (a, a + b) \subseteq D$. Let $f : D \longrightarrow \mathbb{R}$ be a well defined function[4]. Suppose f is a quotient of polynomials (meaning $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x) \wedge q(x)$ are polynomials). Suppose when a person tries the plug in method it gets $\dfrac{0}{0}$. Then the plug in method DOES NOT WORK and the person needs to try one of the other three methods to determine if $\lim\limits_{x \to a} f(x)$ exists or does not exist.*

Heuristic 6.2. *Let $D \subseteq \mathbb{R}$, a be a real number such that there exists a real number b where $(a - b, a) \cup (a, a + b) \subseteq D$. Let $f : D \longrightarrow \mathbb{R}$ be a well defined function[5]. Suppose f is a quotient of polynomials (meaning $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x) \wedge q(x)$ are polynomials). Suppose when a person tries the plug in method it gets $\dfrac{0}{0}$. Then it could be the case there is a hole at $x = a$, a vertical asymptote at $x = a$ or some wacky combination of the two. No matter remember all we can say from this is that the plug in method DOES NOT WORK and the we need to try one of the other three methods to determine if $\lim\limits_{x \to a} f(x)$ exists or does not exist.*

Now, we can say something quite profound; however, if there is a vertical asymptote at $x = a$ or some wacky combination of vertical asymptote at $x = a$ and a hole, then the limit does not exist vat $x = a$. Nonetheless, if there is a hole at $x = a$, we may or may not have a limit at $x = a$.

[1]This means a does not need to be in the domain, D of the function f.
[2]This means a does not need to be in the domain, D of the function f.
[3]Duh.
[4]This means a does not need to be in the domain, D of the function f.
[5]This means a does not need to be in the domain, D of the function f.

Exercise 6.13. *What kind of holes can there be a limit and what sorrt of holes would be such that there is not a limit?*

Using our naïve concept of 'connected' curves in the plane we can note:

Heuristic 6.3. *Let $\mathcal{C}$ be a planarly connected (or just connected for short) curve. Supppose a is a real number and b is a real number and $(a, b) \in \mathcal{C}$. Further, suppose there is a positive real number δ such that across the segment $(a - \delta, a + \delta)$ it is the case that $\mathcal{C}$ is planarly connected (we need this so (a, b) is not an endpoint of the curve). It is the case that $\lim_{x \to a} \mathcal{C} = b$.*