

Math 021 Fundamentals of Math Fall 2005 Worksheet III Name: _____
 (please print legibly)

Why do we count as we do? What is the reason? We use the base ten system and we define digits to be the universe $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So our digits that are from the set natural numbers star ($\mathbb{N}^* = \{0, 1, 2, 3, 4, 5, \dots\}$) simply show positional meaning to the powers of ten. Hopefully, we all recall that the number 1,237 simply means that we have one 10^3 , two 10^2 , three 10^1 , and seven 10^0 . So our positional method (the Hindu-Arabic numeration system so named based on the symbols being developed in India then transmitted through the Muslim caliphate to Africa and Europe) is an elegant and useful method for expressing the natural numbers (with zero) and is extended to the integers, rationals, etc.

We were probably told that the Hindu-Arabic numeration system is that which it is and it is the *best* way to do it. But in the course of our existence we use other systems; for example, modular or base 12 and base 60 (sort of) for time, base 12 and base 3 (sort of) for English measurement of distance, etc. It is the best system; for if one studies other numeration systems one would find they lack the ease of operation and the rigor of the Hindu-Arabic numeration system. You are free to study other systems and compare them to this one (Ancient Egyptian or Sumerian, the Roman system or Chinese system). Our discussion will centre on arithmetic using the Hindu-Arabic numeration system or the digits of the Hindu-Arabic numeration system in different bases than ten.

Formally, a natural number can be expressed in any base system such that the base is well defined so that each position represents groups of powers of the base.

Consider 1,237 in base ten means one 10^3 , two 10^2 , three 10^1 , and seven 10^0 but if this were base nine then 1,237 would mean one 9^3 , two 9^2 , three 9^1 , and seven 9^0 . This is meaningful, but for base five, for example it would not because since $7 > 5$ how could one have 7 in base 5? Thus the digits for each type of base depend on the base.

For base 2 the set of digits is $T = \{0, 1\}$;

for base 3 the set of digits is $H = \{0, 1, 2\}$;

for base 4 the set of digits is $F = \{0, 1, 2, 3\}$;

for base 5 the set of digits is $V = \{0, 1, 2, 3, 4\}$;

for base 6 the set of digits is $X = \{0, 1, 2, 3, 4, 5\}$;

for base 7 the set of digits is $S = \{0, 1, 2, 3, 4, 5, 6\}$;

for base 8 the set of digits is $E = \{0, 1, 2, 4, 5, 6, 7\}$; and,

for base 9 the set of digits is $N = \{0, 1, 2, 4, 5, 6, 7, 8\}$.

One can extend past base 10 in more than one way. Let us use the 'alpha-numeral' system such that

for base eleven the set of digits is $L = \{0, 1, 2, 4, 5, 6, 7, 8, 9, T\}$;

for base twelve the set of digits is $W = \{0, 1, 2, 4, 5, 6, 7, 8, 9, T, E\}$; etc.

Now in each system the value in each position represents the number of powers of the base from right to left.

Hence, the number 11010 in base 2 is well defined and for clarity when we are referencing a number in an alternate base system let us use a subscript to clarify the meaning; so, let us let 'one one zero one zero base two,' be written as 11010_2 .

Now, it is 1 of 2^4 , 1 of 2^3 , 0 of 2^2 , 1 of 2^1 , 0 of 2^0 . Hence it is

$1 \times 2^4, 1 \times 2^3, 0 \times 2^2, 1 \times 2^1, \text{ and } 0 \times 2^0$ using the elementary sign for multiplication.

So, it is $(1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)$.

Hopefully, it is facile to see that we therefore have $(16) + (8) + (0) + (2) + (0)$.

So, $11010_2 \equiv 26$ in standard decimal (base ten) form; we shall use the symbol for logically equivalent (\equiv) since that is what we are expressing that the two concepts are indeed the same only they are expressed in different systems.

So, suppose we are presented with 113241_5 . What is it?

Clearly it is $(1 \times 5^5) + (1 \times 5^4) + (3 \times 5^3) + (2 \times 5^2) + (4 \times 5^1) + (1 \times 5^0)$; which is by the axioms of the natural numbers, $(1 \times 5^0) + (4 \times 5^1) + (2 \times 5^2) + (3 \times 5^3) + (1 \times 5^4) + (1 \times 5^5) = (1) + (20) + (50) + (375) + (625) + (3,125) = 4,196$.

Now consider 3,401. Suppose we wish to convert it to base 6 (yes, I am aware this is rather odd but bear with me). Since conversion from base 'A' to base ten was done through expansion and multiplication it is logical to conclude that this process will require division (explain why this seems reasonable to yourself). However, we need the powers of 6. Note that

$6^0 = 1$, $6^1 = 6$, $6^2 = 36$, $6^3 = 216$, $6^4 = 1,296$, $6^5 = 7,776$, and so forth. We only need those powers less than or equal to 3,401 since there can be no groups of size 7,776 or more to allot.

Now, note the algorithm we shall use.

$$\begin{array}{r}
 \overline{)3401} \\
 \underline{-2596} \\
 805
 \end{array}$$

Note that $1,296 \times 2 = 2,596$. So, we subtract leaving 805 as a remainder.

Now,

$$\begin{array}{r}
 \overline{)805} \\
 \underline{-648} \\
 157
 \end{array}$$

Note that $216 \times 3 = 648$. So, we subtract leaving 157 as a remainder.

Now,

$$\begin{array}{r}
 \overline{)157} \\
 \underline{-144} \\
 13
 \end{array}$$

Note that $36 \times 4 = 144$. So, we subtract leaving 13 as a remainder.

So,

$$\begin{array}{r}
 \overline{)13} \\
 \underline{-12} \\
 1
 \end{array}$$

Note that $6 \times 2 = 12$. So, we subtract leaving 1 as a remainder.

Finally,

$$\begin{array}{r}
 \overline{)1} \\
 \underline{-1} \\
 0
 \end{array}$$

Note that $1 \times 1 = 1$. So, we subtract leaving *no* remainder.

Hence, it is the case that $3401 = 2596 + 648 + 144 + 12 + 1 =$

$$(2 \times 1296) + (3 \times 216) + (4 \times 36) + (2 \times 6) + (1 \times 1) =$$

$$(2 \times 6^4) + (3 \times 6^3) + (4 \times 6^2) + (2 \times 6^1) + (1 \times 6^0) \Rightarrow$$

$$3401 \equiv 23421_6.$$

Now, before proceeding any further explain what the algorithm was; how it was used; why it was used; and, opine as to its generalisation.

Consider 711. Let us convert this to base 2. You can create the algorithm yourself; suffice it to say the powers of two are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, etc. We only need begin our work with 512. $711 - 512 = 199$. $199 - 128 = 81$. $81 - 64 = 17$. 32 into 17 yields 0 with remainder 17 so proceed to 2^4 . $17 - 16 = 1$. We divide by 8, 4, and 2 and also get zeros. Finally 1 divided by 1 is 1 with *remainder* zero. So we get 111010001_2

Let us consider 711 again only this time let us convert it to base 12. The powers of twelve are 1, 12, 144, 1,728, etc.

$$\begin{array}{r}
 \overline{)711} \\
 \underline{-576} \\
 135
 \end{array}$$

Note that $144 \times 4 = 576$. So, we subtract leaving 135 as a remainder.

Now,

$$\begin{array}{r}
 \overline{)135} \\
 \underline{-132} \\
 3
 \end{array}$$

Note that $12 \times 11 = 132$. But, we cannot use '11' so we use E. Subtract leaving 3 as a remainder.

Now,

$$\begin{array}{r}
 \overline{)3} \\
 \underline{-3} \\
 0
 \end{array}$$

Note that $1 \times 3 = 3$ So, we are done.

Hence, $711 \equiv 4E3_{12}$.

Now, the easiest way to convert from base A to base B (where A and B are not 10) is to convert to base ten then out of it.

For example converting 312_4 to base 7 would entail considering that 312_4 is $(3 \times 4^2) + (1 \times 4^1) + (2 \times 4^0) = (3 \times 16) + (1 \times 4) + (2 \times 1) = 48 + 4 + 2 = 54$.

Now the powers of seven are 1, 7, 49, 343, 2,401, 16,807, etc. We could begin with 343; but that would be foolish. We need only begin with 49.

$$\begin{array}{r} 1 \\ 49 \overline{)54} \\ \underline{-49} \\ 5 \end{array} \quad \text{Note that } 49 \times 1 = 49. \text{ So, we subtract} \\ \text{leaving 5 as a remainder.}$$

Then $\begin{array}{r} 0 \\ 7 \overline{)5} \end{array}$ Note that $7 \times 0 = 0$. So, we don't subtract. *still* leaving 5 as a remainder.

Now, $\begin{array}{r} 5 \\ 1 \overline{)5} \\ \underline{-5} \\ 0 \end{array}$ Note that $1 \times 5 = 5$. Subtract So, we are done.

Hence, $312_4 \equiv 105_7$. Please note that we are using transitivity to deduce this.

There are shortcuts and other tricks that (long ago) we studied in Sister Rose Dominic's fifth grade class at St. Margaret's School; but, suffice it to say that is the gist of base systems. So, try some!

For addition in other bases one can use the method of translation to do it: To compute $31423_5 + 12344_5$ we can translate each to base ten $(2118 + 974) = 3092_{10}$ and then translate that back to base 5 to get 44332_5 or one can do it directly by vertically adding and remembering that each position in base 5 represents groups of powers of 5:

$$\begin{array}{r} 3 \ 1 \ 4 \ 2 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline \end{array}$$

$3_5 + 4_5 = 12_5 (1 \times 5) + (2 \times 1)$ write down the 2 carry the one group of 5

$$\begin{array}{r} 3 \ 1 \ 4 \ 2^1 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline 2 \end{array}$$

$1_5 + 2_5 + 4_5 = 12_5 (1 \times 5) + (2 \times 1)$ write down the 2 carry the one group of 25

$$\begin{array}{r} 3 \ 1 \ 4^1 \ 2 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline 2 \ 2 \end{array}$$

$1_5 + 4_5 + 3_5 = 13_5 (1 \times 5) + (3 \times 1)$ write down the 3 carry the one group of 125

$$\begin{array}{r} 3 \ 1^1 \ 4 \ 2 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline 3 \ 2 \ 2 \end{array}$$

$1_5 + 1_5 + 2_5 = 4_5 (0 \times 5) + (4 \times 1)$ write down the 4 nothing is carried (groups of 3125)

$$\begin{array}{r} 3 \ 1^1 \ 4 \ 2 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline 4 \ 3 \ 2 \ 2 \end{array}$$

$3_5 + 1_5 = 4_5 (0 \times 5) + (4 \times 1)$ write down the 4 nothing is carried (groups of 15,625)

$$\begin{array}{r} 3 \ 1 \ 4 \ 2 \ 3_5 \\ + 1 \ 2 \ 3 \ 4 \ 4_5 \\ \hline 4 \ 4 \ 3 \ 3 \ 2_5 \end{array}$$

Multiplication is a bit trickier (keeping track of larger numbers) but the process is the same): a modified standard base 10 multiplication as was our addition in other bases (which is a modification of standard base 10 addition).

§ 3.4 EXERCISES.

1. Convert the following numbers to base ten.

- | | | | | | |
|----|------------|----|-------------|----|-------------|
| A. | 813_9 | G. | 313_4 | M. | 6611_8 |
| B. | 110311_4 | H. | 3103013_4 | N. | 1166_8 |
| C. | 110101_2 | I. | 1010101_2 | O. | 6116_8 |
| D. | 03413_5 | J. | 34013_5 | P. | $3ET1_{12}$ |
| E. | 110311_7 | K. | 110131_7 | Q. | 1562_{12} |
| F. | 1661_8 | L. | 6161_8 | R. | 111_{11} |

2. Convert 913 the following bases

- | | | | |
|----|--------|----|-----------------------------------|
| A. | base 2 | G. | base 8 |
| B. | base 3 | H. | base 9 |
| C. | base 4 | I. | base 12 |
| D. | base 5 | J. | base 16 (define the symbols used) |
| E. | base 6 | | |
| F. | base 7 | | |

3. Convert the following numbers to the specified base.

- | | | | | | |
|----|-----------------------|----|-----------------------|----|-----------------------|
| A. | 813_9 to base 3 | G. | 313_4 to base 5 | M. | 6611_8 to base 6 |
| B. | 110311_4 to base 2 | H. | 3103013_4 to base 9 | N. | 1166_8 to base 6 |
| C. | 110101_2 to base 3 | I. | 1010101_2 to base 8 | O. | 6116_8 to base 6 |
| D. | 03413_5 to base 8 | J. | 34013_5 to base 7 | P. | $3ET1_{12}$ to base 2 |
| E. | 110311_7 to base 12 | K. | 110131_7 to base 5 | Q. | 1562_{12} to base 5 |
| F. | 1661_8 to base 7 | L. | 6161_8 to base 4 | R. | 111_{11} to base 5 |

4. Convert 11001010101010101_2 to base 10.

5. Let the digits for base 16 be {0, 1, 2, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F} where A is ten B is eleven, C is twelve, D is thirteen, E is fourteen, and F is fifteen. Convert the following to base 10:

- | | | | | | |
|----|---------------|----|------------|----|-------------|
| A. | 813_{16} | C. | BAD_{16} | E. | $FACE_{16}$ |
| B. | 110311_{16} | D. | $5F3_{16}$ | F. | 111_{16} |

6. Determine which is greater 1304_5 or 503_7 and justify your conclusion.

7. Define addition and multiplication in other bases to be as we would naturally assume them to be

(e.g.: $a_b + c_b = d_b$ if and only if $a_b \equiv x_{10}$ and $c_b \equiv y_{10}$ yields $x + y = z$ and $z \equiv d_b$
 $a_b \times c_b = f_b$ if and only if $a_b \equiv x_{10}$ and $c_b \equiv y_{10}$ yields $x \times y = w$ and $w \equiv f_b$).

Compute the following:

- | | | | | | |
|----|-----------------------|----|--------------------|----|---------------------------|
| A. | $813_9 + 712_9$ | H. | $31_4 + 12_4$ | O. | $1101_2 \times 1001_2$ |
| B. | $110311_4 + 110311_4$ | I. | $31_4 + 21_4$ | P. | $6_8 \times 7_8$ |
| C. | $110101_2 + 110101_2$ | J. | $33_4 + 32_4$ | Q. | $3_8 \times 2_8$ |
| D. | $110101_2 + 1101_2$ | K. | $31_4 \times 1_4$ | R. | $3ET1_{12} \times 7_{12}$ |
| E. | $110101_2 + 1001_2$ | L. | $31_4 \times 0_4$ | S. | $110101_2 \times 1001_2$ |
| F. | $110101_2 + 1111_2$ | M. | $31_4 \times 11_4$ | T. | $212_3 \times 120_3$ |
| G. | $31_4 + 11_4$ | N. | $31_4 \times 12_4$ | U. | $212_3 \times 121_3$ |