HANDOUT 5 ½
MATH 017
EULER DIAGRAMMES

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Not every argument is of the form or type (i.e.: given the premises A ⇒ ¬B, ¬A ⇒ ¬C, C, D ⇒ B, it is the case that ¬D follows as a conclusion) we studied in chapter one. Some argument types are claims of quantification. Now this seems especially apropos for a discussion in a mathematics class. Many mathematical arguments hinge on an understanding of sets and involve questions of what is in a set or not, how many elements are in the set, etc. Thus, as this section illustrates we use terms such as 'not every,' 'some,' 'many,' 'none,', 'all,' for every,' etc. in our discussions, arguments, and applications. Each of the aforementioned words are examples of quantifiers and are part of the area of logic known as syllogistic logic.

Arguments of the type we have studied to date were examples of propositional or symbolic arguments; that is to say, arguments that use the conditional, biconditional, conjunction, and disjunction. However, there are other types of arguments known as syllogistic arguments which differ from symbolic argument in that syllogistic arguments use quantifying words such as all, none, or some.

Note the difference in the following two arguments:

Example 5h.1 : All frogs have warts.
All creatures that have warts are blue.
Therefore, all frogs are blue.

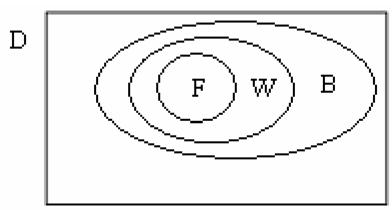
Example 5h.2: If Malcolm is talking, then people listen.
Malcolm is not talking.
Therefore, no one is listening.

The former is a syllogistic argument (or syllogisms); the latter is a symbolic argument.

To determine evidence to suggest that a syllogistic argument is valid or invalid one may use a diagramming technique known as Euler's diagrammes (named after the mathematician Leonhard Euler). Euler's Diagrams are really an application of set theory to logic for they are really just Venn diagrammes set to logic problems.[1]

Let us consider example 5h.1 from above. Since all frogs have warts, the set of frogs is a subset of the set of creatures that have warts. Since all creatures that have warts are blue, the set of creatures that have warts is a subset of blue things. Ergo, all frogs have warts.

We need to assume there is a well defined universe, D, defined that in this instance we shall reference as the **domain of discourse**. We need to also the sets we shall use (i.e.: the symbols we shall use). Let F denote the set of frogs, W denote the set of creatures that have warts, and B be the set of blue things. The diagramme (Euler diagramme solution) illustrates this:



Note that we are simply showing a Venn diagramme such that the universe is D, F ⊆ W, W ⊆ B,

[1] For historical purposes, the Euler diagrammes or circles came first; Venn adapted them to the (then) newly emerging study in a rigorous sense of set theory. We shall adopt the more rigorous requirements of the Venn diagramme convention and insist on specification (presumed) of a well defined universe.

and so, $F \subseteq B$. It must be the case since there is not a way to illustrate this in another manner. Hence, the Euler diagramme illustrates that the syllogistic argument is valid (but remember that this *does not prove* the argument is true).

In terms of quantification, the symbol to represent each element of the set F is an element of the set W is $\forall\ x \in F, x \in W$. The translation of this symbol ($\forall$) is 'for all,' 'for each,' 'every,' etc. So, the following statements are symbolized as:

All frogs are amphibians. $\forall\ f \in F, f \in A$ where F represents the set of frogs and A represents the set of amphibians.

Each student is attentive. $\forall\ s \in S, s \in A$ where S represents the set of students and A represents the set of attentive people.

Every professor is funny. $\forall\ p \in P, p \in F$ where P represents the set of professors and F represents the set of funny people.

Note the use of the lower case English letter for elements and upper case for sets. Note that one does not have to use x as the variable representing an element of the set. Finally note the difference between example 5h.1 and example 5h.2. Example 5h.2 is solved using a truth table (which is left to the student as an exercise).

We can rewrite example 5h.1 in symbols as:

$\forall\ x \in F, x \in W.$

$\underline{\forall\ y \in W, y \in B.}$

$\therefore\ \forall\ x \in F, x \in B.$

The traditional way to represent therefore in syllogistic argument for is the "$\therefore$" rather than the "$\Rightarrow$," but either is (of course) fine.

Consider the following example:

<u>Example 5h.3</u>: No Morehouse students are Spelman students.
<u>Some Spelman students work.</u>
Therefore, no Morehouse students work.

Before defining some new symbols, let us not this is a syllogistic claim since it is in terms of sets, elements, and quantification. Let us represent the domain of discourse as D, the set of Morehouse students as M, the set of Spelman students as S, and the set of people who work as W.

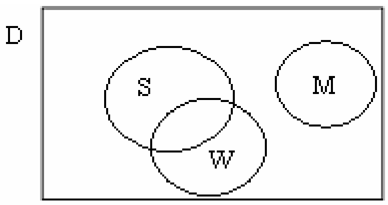
The first sentence means that $M \cap S = \varnothing$ because if there was an element of M that was in the set S, then there would be a Morehouse student who was a Spelman student. Further, if there was an element of S that was in M, then there would be a Spelman student who was a Morehouse students which implies that this person is a Morehouse student who is a Spelman student.

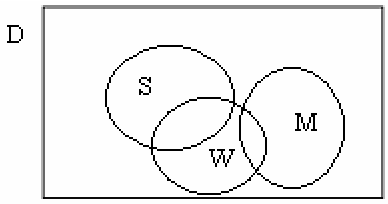
The second sentence is a tad more challenging. We need a new symbol to represent the statement. In terms of quantification, the symbol to represent, 'some,' $\exists$. The translation of this symbol ($\exists$) is 'some,' 'there exists,' 'there is at least one,' etc. So, the second sentence quantified symbolically would be $\exists\ x \in S, x \in W$.

The third sentence symbolized would be: $\therefore\ M \cap W = \varnothing$.

Now, do you think the claim is true or false? Let us reflect on the claim.
No Morehouse students are Spelman students. We represent the set of Morehouse students as M and the set of Spelman students as S. $M \cap S = \varnothing$. Thus, the sets are disjoint. Some Spelman students work. Notice that $S \cap W \neq \varnothing$ (since $\exists\ x \in S \ni x \in W$) . So, S and W intersect. The two statements

are the suppositions (the premises). We are now faced with drawing any possible Euler diagramme that illustrates these two suppositions. Note that for the argument to be valid, it must be the case that $M \cap W = \varnothing$. Let us consider the following two diagrammes.





Both *could be* true for the premises. There is nothing in the premises that forces either to be the only possibility. However, note the second diagramme represents an instance such that the premises hold and is more generalised. The first represents a diagramme where one is assuming the conclusion (a logical fallacy from chapter one, recall). Therefore, there is evidence to conclude that the syllogism is ***invalid***. The justification for that is the second Euler diagramme (thus, the first one need not be drawn).

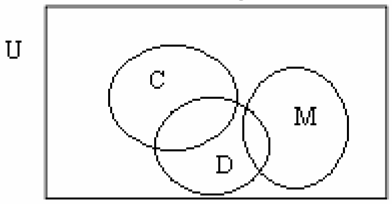
One might believe the next example is obvious. Well, let's consider it.

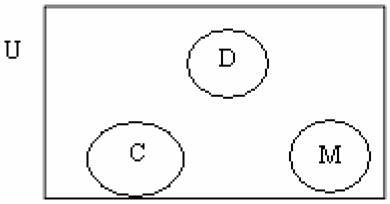
Example 5h.4: Some cats are not dogs.
Some dogs are not mice.
Therefore, some cats are not mice.

Let us define our sets. Let U be the domain of discourse, C be the set of cats, D be the set of dogs, and M be the set of mice. Note we are not going to use D as the domain of discourse since it might cause confusion with the set of dogs. So, we will use our old friend, U, for that.

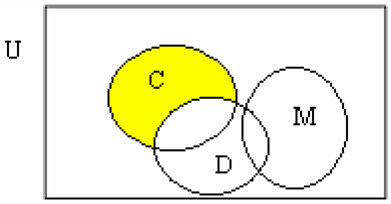
The first sentence means that $C \cap D^C \neq \varnothing$ because there is at least one cat that is not a dog. The second sentence means $D \cap M^C \neq \varnothing$. The last sentence means $C \cap M^C \neq \varnothing$. In terms of quantification, the first sentence translated reads: $\exists\, x \in C, x \in D^C$ or $\exists\, x \in C, x \notin D$ (either is fine). The second sentence translated reads: : $\exists\, y \in D, y \in M^C$ or $\exists\, y \in D, y \notin M$. The third sentence translated into symbols would be: $\therefore \exists\, c \in C, c \in M^C$ or $\exists\, c \in C, c \notin M$.

Consider the following two Euler diagrammes:

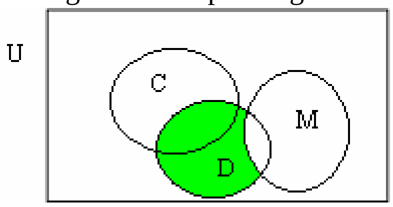




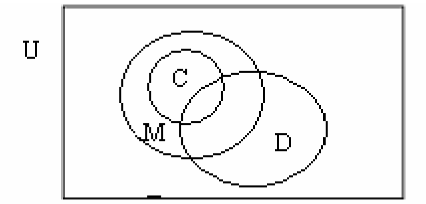
Many people would decide on the second Euler based on empirical evidence. They would be *wrong* since we are trying to judge the veracity of the syllogism, not the state of nature. Many people would decide on the first Euler since for sentence one there is a region corresponding to the sentence



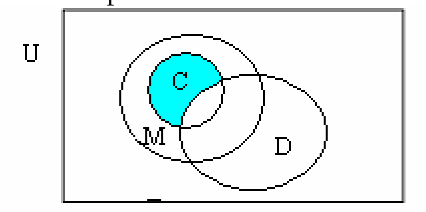
shaded so that represents the idea that there is at least one cat that is not a dog. For sentence two there are regions corresponding to the sentence



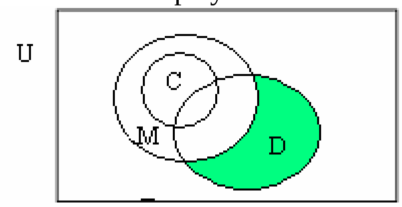
shaded so that represents the idea that there is at least one dog that is not a mouse. However, one can reasonably draw an Euler diagramme such that the premises hold, but the conclusion does not:



Note the premises hold for this since it displays that $C - D \neq \varnothing$ (sentence one):



and since it displays that $D - M \neq \varnothing$ (sentence two):



but there is not a display such that $C - M \neq \varnothing$! So, the syllogistic argument is ***invalid***.

The difficulty for many with syllogisms such as example 5h.4 is that experience dictates in the real world the claim seems true (indeed can be strengthened). However, we are not interested in the existential question of life and mice, cats, and dogs! We are interested in the argument form because it should be clear from the Euler diagramme that one can change the wording of example 5h.4 to:

Some students are not men.
Some men are not tall.
Therefore, some students are not tall.

and **the argument is still invalid**!

Suffice it to say, there are some important principles to consider in this section.
Assume D is a well defined domain of discourse, A is a set and B is a set.

Assume $\forall$ $x \in A, x \in B$. Does this mean $\exists$ $x \in A, x \in B$? The answer is , ‘no,’ since a counterexample exists to the claim.
Let $A = \varnothing$ and all elements of $\varnothing$ are in B since there are no elements in $\varnothing$!

Another principle that is of import is negation of quantifiers. What does it mean to say $\neg(\forall\ x \in A, x \in B)$? Consider $\forall\ x \in A, x \in B \Leftrightarrow A \subseteq B$. So, $\neg(\forall\ x \in A, x \in B) \Leftrightarrow A \quad B$. Hence for $A \quad B$, it must be the case there is an element in A that is not in B. So, $\exists\ x \in A, x \notin B$. To sum up $\neg(\forall\ x \in A, x \in B) \Leftrightarrow \exists\ x \in A, x \notin B$.

One can reason that the other negation principle is $\neg(\exists\ x \in A, x \in B) \Leftrightarrow \forall\ x \in A, x \notin B$ since $\neg(\exists\ x \in A, x \in B)$ implies that $A \subseteq B^C$.

From these two principles and the law of double negation, it should be clear to the reader that $\neg(\forall\ x \in A, x \notin B) \Leftrightarrow \exists\ x \in A, x \in B$ and $\neg(\exists\ x \in A, x \notin B) \Leftrightarrow \forall\ x \in A, x \in B$.

So, from a vernacular standpoint one must be quite attentive to what is being said so that we can understand a statement. For example, what is the opposite of the statement, "All Atlantans are citizens of Fulton County?" If you were to say, "no Atlantan is a citizen of Fulton County," you would be wrong. The correct statement would be, "not all Atlantans are citizens of Fulton County;" which can also be stated as, "some Atlantans are not citizens of Fulton County." This is because some can still be citizens of Fulton, but *at least one* is not.

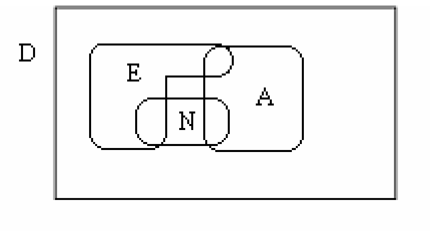
Likewise, the opposite of "no people wear hearing aids" is "some people wear hearing aids." It is incorrect to say, "all people wear hearing aids." That kind of extremism, it should be noted, is incorrect, and many times, creates many real problems. For example, to negate the statement, "all of us are poor," does not mean all of us will be rich! It simply means some of us will not be poor anymore.

Let us consider one more example using Euler's Circles.

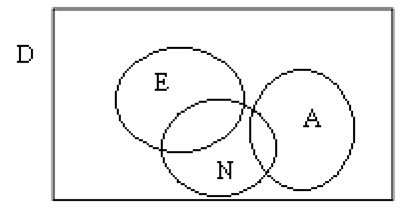
Example 5h.5: Some Estonians are not Nigerians.
Some Nigerians are not Argentines.
Therefore, some Estonians are not Argentines.

Let D denote the domain of discourse, E be the set of Estonians, N be the set of Nigerians, and A be the set of Argentines for the Euler's diagramme. What does the Euler diagrammes indicate, true or false?

Note, you can draw an Euler diagramme to show that a counterexample should exist to this argument form since:



So, the argument is invalid, because this diagramme indicates there exists a counter-example to the argument. Indeed further note that you can draw a diagramme such that there could be an instance where it can be true:



so, the two Euler diagrammes which both hold for the premises but the first is contrary to the conclusion whilst the second is supportive of the conclusion should assist you in noting that the argument is **fallacious** (and that you *don't* ***have*** to use circles for the sets).

A couple of other notes: in many logic texts the form used is not set-theoretic but functional form. So, considering

Example 5h.1 : All frogs have warts.
All creatures that have warts are blue.
Therefore, all frogs are blue.

You might see it written as $\forall$ x , F(x) $\Rightarrow$ W(x).

$\forall$ x , W(x) $\Rightarrow$ B(x).

$\therefore \forall$ x , F(x) $\Rightarrow$ B(x).

This is indeed helpful (in a way) for it assists us in understanding that subsethood from set theory has the same intrinsic property to the conditional from logic. However, since we are studying logic in order to help us better learn, remember, and understand mathematics we shall use the convention from set-theory.

Also, the four basic quantification statements that we have studied in this section lead to some interesting relationships (when viewed in functional form). Let Φ(x) be a statement where x is some element of the domain of discourse. Note that $\forall$ x , Φ(x); $\forall$ x , ¬Φ(x); $\exists$ x , Φ(x); and, $\exists$ x , ¬Φ(x) have interesting properties.

$\forall$ x , Φ(x) and $\forall$ x , ¬Φ(x) are **contraries**; that is to say they *might* both be false or both be true.

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$\forall$ x , Φ(x) and $\exists$ x , ¬Φ(x) are **contradictories**; that is to say one *must* false and the other true.

$\exists$ x , Φ(x) and $\forall$ x , ¬Φ(x) are **contradictories**; that is to say one *must* false and the other true.

$\exists$ x , Φ(x) being true does not necessarily imply $\forall$ x , Φ(x) is true.

$\exists$ x , ¬Φ(x) being true does not necessarily imply $\forall$ x , ¬Φ(x) is true.

$\forall$ x , Φ(x) being true does not necessarily imply $\exists$ x , Φ(x) is true.

$\forall$ x , ¬Φ(x) being true does not necessarily imply $\exists$ x , ¬Φ(x) is true.

So, you can see we have extremes: $\forall$ x , Φ(x) and $\forall$ x , ¬Φ(x) (the contraries) and a middle (the contradictories to the extremes) which may be viewed as a "continuum."

$\forall$ x , ¬Φ(x)	$\exists$ x , ¬Φ(x)	$\exists$ x , Φ(x)	$\forall$ x , Φ(x)
negative universal	negative existential	positive existential	positive universal