MATH 017 INTRODUCTION TO MATHEMATICS
HANDOUT 8
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HANDOUT ON INTRODUCTION TO COMBINATORICS

We learned to count before elementary school (one hopes); but, the formal theory of counting is oft called number theory (not yet offered here) or Combinatorics (not yet offered here).

At an introductory level for mathematics, combinatorics is considered a branch of discrete mathematics (Math 125) in which the main focus is the number of ways to choose or arrange objects from a finite set (Math 224). It is a branch of number theory insofar as the axioms of number theory create the building blocks from which combinatorics arises. Much work in numerical analysis (Math 322) requires a rudimentary understanding of number theory as well as Analysis (Math 351-352).

Most of the counting techniques that we will concern ourselves with are of the type that assist us in understanding the groundwork for an intuitive understanding of probability theory (Math 301-302).

We have already discussed at least one method of counting from a finite set and that was the applications of Venn Diagrammes to surveys in Math 017. Now, we will extend our understanding to some more complex problems.

Definition: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots, (k - 1), k, (k + 1), \ldots\}$

Theorem (Archimedian property of $\mathbb{N}$ in $\mathbb{R}$): The natural numbers are *unbounded above* in the reals.

The properties of addition of natural numbers can be derived from a short set of axioms. The axioms are called the Peano Axioms:

There exists a set, P, which is defined by the following four axioms.

Axiom 1: There exists a natural number, call it 1, that is not the successor of any other natural number.

Axiom 2: Every natural number has a unique successor. If $k \in P$, then let k' denote the successor of k.

Axiom 3: Every natural number except one is the successor of exactly one natural number.

Axiom 4: If M is a set of natural numbers such that

(i) $1 \in M$ and

(ii) for each $k \in P$, if $k \in M$, then $k' \in P$,

then $P = M$.

P, of course is $\mathbb{N}$.

From these axioms arise the natural numbers by defining what addition by one means.

Definition: For every $k \in \mathbb{N}$, define $k + 1 = k'$.

Then, note inductively, the entire understanding of addition flows from this definition.
Likewise, then multiplication, etc.

So, it seems the basis of our understanding of counting will be $\mathbb{N}$

No, pity. We need : $\mathbb{N}^*$

Recall : $\mathbb{N}^* = \{0, 1, 2, 3, 4, 5, \ldots, (k - 1), k, (k + 1), \ldots\} = \mathbb{N} \cup \{0\}$.

Finally, before proceeding we need to define factoral.

Let $k \in \mathbb{N}^*$

recursively define $k!$ Such that

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1$

$3! = 3 \cdot 2 \cdot 1$

.

.

.

$k! = k \cdot (k-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ where $k \geq 3$.

A more succinct definition is $k! = \prod_{j=1}^{k} j$ when k is a natural number.

<u>Theorem</u>: Let $k \in \mathbb{N}$. It is the case that $k! = k \cdot (k-1)!$

Now, to the business at hand:

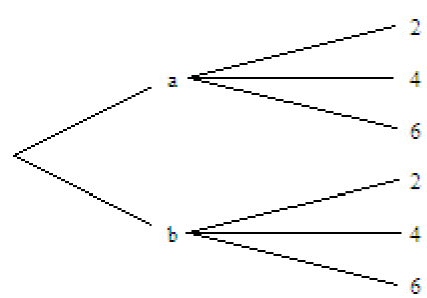
<u>Theorem (The Fundamental Principle of Counting)</u>: If activities $1, 2, 3, \ldots, k$ can be performed in $n_1, n_2, n_3, \ldots n_k$ ways, respectively, such that $k \in \mathbb{N}$, then the k activities can be performed in

$$\prod_{j=1}^{k} n_j = n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k \text{ ways.}$$

Consider we choose one of 2 objects from the set {a, b}, then we choose one of three objects from the set {2, 4, 6}. Hence, the number of ways to do this is (2)(3) = 6.

The sequence of activities can be illustrated with a set of ordered pairs since the activities are in order: {(a, 2), (a, 4), (a, 6), (b, 2), (b, 4), (b, 6)}.

Further, the sequence of activities can be illustrated with a tree diagramme:



Remember a tree diagramme is a simple graphical illustration of an ordered sequence of activities.

In many counting problems, the task assigned is one involving arranging a set of objects. Now, the arrangement may or may not involve order.

Definition: Suppose the set A has n objects such that $n \in \mathbb{N}^*$

and we wish to order k of the objects $\ni k \leq n$ where $k \in \mathbb{N}^*$

The number of ways to do this is referred to as the permutation of n things taken k at a time and is symbolised as $P_{n,k}$ where $P_{n,k} = \dfrac{n!}{(n-k)!}$

Alternate notation: $P_{n,k} = {}_nP_k = P_k^n = {}_kP^n = P(n, k)$

Definition: Suppose the set A has n objects such that $n \in \mathbb{N}^*$ and we wish to choose k of the objects

(without regard to order) $\ni k \leq n$ where $k \in \mathbb{N}^*$

The number of ways to do this is referred to as the combination of n things taken k at a time and is symbolised as $\binom{n}{k}$ where $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$

Alternate notation: $\binom{n}{k} = C_{n,k} = {}_nC_k = C_k^n = {}_kC^n = C(n, k)$.

Theorem: Let $n \in \mathbb{N}^*$ and $k \in \mathbb{N}_0 \ni k \leq n$. $P_{n,k} = k! \cdot \binom{n}{k}$.

Note the following theorems which are of use:

Let $n \in \mathbb{N}$ Let $k \in \mathbb{N}$ where $k \leq n$ $\binom{n}{k} = \binom{n}{n-k}$

Let $n \in \mathbb{N}^*$ $\binom{n}{0} = 1$

Let $n \in \mathbb{N}^*$ $\binom{n}{1} = n$

Let $n \in \mathbb{N}$ $\binom{n}{n-1} = n$

Let $n \in \mathbb{N}^*$ $\binom{n}{n} = 1$

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(please print legibly)

Show all work. If an answer DNE, explain why it does not exist.

Part A (factorals):

1. Compute $7\,!$ 2. Compute $(7-3)\,!$ 3. Compute $(3-7)\,!$ 4. Compute $(7-4)\,!$

4. Compute $(7 \div 3)\,!$ 5. Compute $(8 \div 2)\,!$ 6. Compute $(2 \div 8)\,!$ 8. Compute $(2 \cdot 3)\,!$

7. Compute $(3 \cdot 2)\,!$ 8. Compute $(2)! \cdot (3)!$ 9. Compute $7! - 3!$ 10. Compute $2! \div 8!$

Part B (combinations or permutations):

11. Compute C_4^7 12. Compute $\binom{12}{7}$ 13. Compute $\binom{6}{11}$ 14. Compute $\binom{11}{6} \cdot \binom{5}{5}$

15. Compute $\binom{11}{6} \cdot \binom{7}{5} \cdot \binom{12}{7}$ 16. Compute P_5^8 17. Compute P_{18}^{17}

18. Compute $P(7, 4)$ 19. Compute $C(7, 4)$ 20. Compute $C(17, 3)$ 21. Compute $C(17, 14)$

Part C (word problems):

22. Six boys and eleven girls are in a club. Suppose an election is to be held such that a boy will be chosen president, then a girl vice president, then a girl as secretary. How many ways can this be done?

23. Six boys and eleven girls are in a club. Suppose an election is to be held such that two boys and three girls will be chosen for a committee. How many ways can this be done?

24. Six boys and eleven girls are in a club. Suppose an election is to be held such that five of these kids will be chosen for a committee. How many ways can this be done?

25. Six boys and eleven girls are in a club. Suppose an election is to be held such that a girl will be chosen president, then a girl vice president, then a girl as secretary. How many ways can this be done?

26. A fair coin is to be flipped five times. How many unique sequences of heads / tails are there?

27. A fair coin is to be flipped five times. How many ways are there to get exactly three of the flips to be heads?

28. A fair coin is to be flipped five times. How many ways are there to get at least three of the flips to be heads?