Chapter 2

Introduction to the Theory of Sets

2.1 Introduction

Set Theory was rigourously developed over the past 150 years. It is a vibrant area of research to this day. The theory of sets was developed by many different mathematicians, but reached a rigorous level by the nineteenth and early twentieth centuries through the work of Boole, Cantor, Zermelo, Fraenkel, Dedekind, Frege, Zorn, von Neumann, etc. It is the one of the basic building blocks and a foundation of higher level mathematics and gives the mathematician the power to communicate abstract ideas and thoughts succinctly, clearly, and in an organised manner. There are many different approaches to an introduction to sets. One approach is to intuitively introduce the subject; another is to rigorous introduce the subject axiomatically. We shall discuss the subject using a bit of both manners of introduction.

2.2 The Basics of Naïve Set Theory

One must begin with defining a **universe**. That is to say that a universe is the beginning building block of discussion (akin to a domain of definition for logic). Without it there is no set theory. Contained within the universe are **elements**.
A **set** and an **element** are rather nebulous concepts much that same way as **point**, **line**, and **plane** are in Euclidean geometry. However, we can intuit an understanding of a set by thinking about a well defined group or collection of well-defined objects. Any object in the set is called an **element** of the set and is said to be a **member** of the set or the element **belongs** to the set. We say that a set **consists** of it elements or a set **contains** its elements. So, in terms of a hierarchy, we should consider the objects as secondary to the set - - the set is the grouping and the elements the individuals much as matter is made up of elements. However, for a set to be well-defined must exist within a realm that

has been specified. That specification of all possible elements to be discussed is called the **universe** (or domain of discourse). The universe is also an undefined concept in so far as we axiomatically allow for its existence (e.g.: a premise of our discussion of sets is always that a well defined universe has been specified). No discussion of sets can properly take place before the universe has been defined. Remember the most important beginning point of this is if one discussed a set without specifying what the universe is, then ambiguity can enter into the discussion and the theory collapses.

Definition 2.2.1. A **Universe** must be specified. Within it are **elements**; and any group of elements (including none) constitutes a **well-defined set**. Universe, set, and element are primitive concepts that we accept.

Notation 2.2.1. *We use U for universe; English capitals letters for sets; and lower case English alphabet letters for elements typically; but not exclusively.*

Suppose a well defined universe has been defined.[1] An example of a universe is the natural numbers, $\{1, 2, 3, \ldots\}$; greek letters $\{\alpha, \beta, \gamma, \ldots, \omega\}$; the collection of all students enrolled at Kutztown University in 2002, etc. Then examples of sets in order from the respective universes mentioned previously are $\{1, 2\}$; $\{\eta, \nu, \zeta, \phi\}$; and the set of all Kutztown University freshmen enrolled in 2002. Note when $U = \{1, 2, 3, \ldots\}$ one cannot have a set of students; one cannot have a set of all real numbers greater than or equal to 5 but less than 8 since $5\frac{1}{2}$ is not in the universe but is (supposedly) in the 'set.' When $U = \{1, 2, 3, \ldots\}$ the set consisting of the multiplicative identity (that is to say the set consisting of 1) exists; but, the 'set' of protons in a carbon atom does not; etc. Moreover, examples of aggregates that are not sets since there is ambiguity, inconsistency, opinion, subjectivity, or such nebulous understanding of the concept that rigour cannot be achieved include the "set" of all my hopes and dreams, the "set" of all good presidents who served in the twentieth century; the "set" of all numbers; the "set" of all sets; and, the "set" of all 'real' men.

An example of the ambiguity that arises with the concept of set is when one forgets to specify a universe. Thus, when described, the "set" is not well defined because a full accounting of elements has not preceded the discussion. Let us say we are in an arithmetic class and the instructor asks us to describe the set of numbers between 1 and 4. Many children would say, "2 and 3." Others might say, "2, 1.5, 2.5, and so forth." Others might express the numbers as fractions. Very few would consider that , π, e, etc. could be in the 'set.' The ambiguity arises because the universe has not been specified and the term "number" is not a singular concept in this context (indeed, how many children would realise that $3 + 2i$ is a number (albeit complex))? So we can see that specification of a universe is important. Note that the universe, since it is a collection of

[1] Note the rather odd nature of the use of words in the sentence. It is a rather circular notion that a "well-defined" universe has been defined; but, that is the language used - - it was defined well.

elements that will be discussed, is a set. It is in our context the "biggest" set (big, large, etc. are very dangerous words when applied to sets - -be careful the term is *not* well defined; but, it should be intuitively appealing and can do no harm so long as one realises that "big" like "tall" is subjective).

Subjectivity is not a hallmark of rigourous mathematics. Whilst the are of Applied Mathematics is a worthy area to study; that is not what is being studied in this course. Hence, objectivity is the characteristic of Pure Mathematics that we are engaging in during this course.

Now let us move on to some basic definitions. Recall in general, we will use lower case English letters to signify elements, upper case English letters to signify sets, and U to signify the special set, the universe.

Let U be a pre-specified well defined universe. If a is an element of the set S, then we shall write a $a \in A$. The negation of this statement is, "a is not an element of the set S, and we shall write a $a \notin S$.[2] So, the symbol "$\in$" is read as "belongs to," "is in," or "is a member of." It is standard notation to use braces to enclose elements to signify a set. Let $U = \{1, 2, 3, 4, 5, \ldots\}$. So, if one wishes to refer to the set consisting of the elements one, two, and three, then one would write, $\{1, 2, 3\}$. Also, it is standard notation to write a set with braces, use a variable to denote a generalised element of a set, and then describe the set thereafter based on axioms or previous definitions. We shall see an example of that later.

Notation 2.2.2. *Let* U *be a well defined universe. If* a *is an element of the set* S*, then it is symbolised as* $a \in A$.

Example 2.2.1. *Let* U $= \{1,\ 2,\ 3,\ 4,\ 5\}$*. Let* $A = \{1,\ 3,\ 5\}$*. Thus,* $1 \in A, 2 \notin A$*, etc. Note every element in* A *must be in the universe; so, for the sake of this particular universe one could not discuss standard multiplication since,* $3 \cdot 2 = 6$ *but* $6 \notin U$ *so it does not exist (in this context (meaning with this universe)). Every possible set constructed cannot have more than five elements with regard to this universe (repetitions are not wrong; but, are unnecessary). A set can have no elements.*

Definition 2.2.2. Let U be a well defined universe. If there are no elements of the set S, then S is the empty set (or null set).

Notation 2.2.3. *Let* U *be a well defined universe. If the set* S *is empty it is denoted by the symbol* $\varnothing$.

There are some special standard sets and symbols for them to denote sets that we use often. The sets under discussion are formulated from the real line (are either points of the line or are generalisations of the line. So, for this discussion let U be the real line.

[2] As with logic, a slash through a symbol means not the symbol. This is standard throughout mathematics.

Your knowledge of high school geometry will, no doubt, be of use in making concrete these abstract ideas that follow. As previously stated in the text, one of the most basic of sets is called the **natural numbers**. It has been with us since antiquity, and we will denote it as $\mathbb{N} = \{1, 2, 3, 4, ...\}$ where the set never ends and includes all the whole or counting numbers that the student learnt in kindergarten or before. We shall denote the set of natural numbers along with zero as the set $\mathbb{N}^* = \{0, 1, 2, 3, 4, ...\}$. We shall denote the set $\{1\}$ as $\mathbb{N}_1$, the set $\{1, 2\}$ as $\mathbb{N}_2$, the set $\{1,\ 2,\ 3\}$ as $\mathbb{N}_3$, and so forth so that $\mathbb{N}_k = \{1, 2, 3, 4, ..., (k-1), k\}$. All of these are referred to as *initial segments* of the natural numbers. This definition is known as a **recursive definition** since we are inductively defining a myriad of sets at once; the three dots signify that the enumeration of the elements continues.[3] Likewise, we shall denote the set $\{0\}$ as $\mathbb{N}_0^*$, the set $\{0, 1\}$ as $\mathbb{N}_1^*$the set $\{0,\ 1,\ 2\}$ as $\mathbb{N}_2^*$, etc. so that $\mathbb{N}_p^* = \{0, 1, 2, 3, ..., (p-1), p\}$.

Definition 2.2.3. Let U be the reals.

1. $\mathbb{N} = \{1, 2, 3, ..., (p-1), p, (p+1), \ldots\}$.
2. $\mathbb{N}^* = \{0, 1, 2, 3, ..., (p-1), p, (p+1), \ldots\}$.
3. $\mathbb{N}_p = \{1, 2, 3, ..., (p-1), p\}$.
4. $\mathbb{N}_p^* = \{0, 1, 2, 3, ..., (p-1), p\}$.

Another of the most basic of sets is called the **integers**. It has been with us quite a long time (the people of India invented the symbol of zero and were really the first to use it and negative numbers (in fact the number system that we use is of course the Hindu-Arabic number system since the Hindus created it, the Arabs adopted it and brought it west); another interesting fact is the Mayans also invented a zero independent of the Hindus). We will denote the set of integers as $\mathbb{Z}$ such that $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, ...\}$.

Definition 2.2.4. Let U be the reals.
$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$.

Generalising from the integers, we have the **rational numbers**. The rationals are denoted as $\mathbb{Q}$ such that $\mathbb{Q} = \{x | x = \frac{m}{n}$ where $a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$. This statement is read as the rationals are the set consisting of elements x such that x is equal to a divided by b where a is an integer, b is an integer, and b is not zero. Thus, the symbol "|" in this context means **such that**. Indeed, as I type this opus I am getting very tired of writing

[3]The three dot symbol (the ellipsis) is most problematic. Many students think that the ellipsis establishes a pattern. It does not. Consider . is not 3.14. It is not 3.141592. It is 3.14159265359 . . . (the decimal does not repeat or have a pattern); thus, the three dot symbol means on and on but not necessarily in a pattern.

the words, "such that," (I suppose it is an occupational hazard, I think we mathematicians are a lazy lot so we have invented many symbols to create a short hand to ease the amount of words necessary to communicate). A more general symbol for such that is, "$\ni$" or "∍" and will be used liberally from this point onward. Typically it is not used in the notation inside the braces for a set only as a free-standing symbol. However, it is not incorrect to use it. Therefore, it is technically correct to write:

Definition 2.2.5. Let U be the reals.

$\mathbb{Q} = \{x \ni x = \frac{m}{n}$ where $a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$

We can simplify the definition of the rationals so that it does not have to depend solely on all the integers. Note the definition $\mathbb{Q} = \{x : x = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N}\}$ is logically equivalent to the previous definition of the rationals and in this case the colon means **such that** (which is the third of the standard notations for such that). Generalising from the rational numbers, we have the **real numbers**. However, this is axiomatically executable, but practically most difficult to do in a basic introduction to sets. Therefore, we shall consider the set of reals from a geometric standpoint. The set of real numbers are denoted as , $\mathbb{R}$ such that $\mathbb{R} = \{x|x$ is a point on the line$\}$. One could also define the reals from a sequential (or decimal) perspective by defining the reals to be $\mathbb{R} = \{x|x$ is a number where x is an integer followed by a decimal and then a sequence of digits where each digit belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Yes, this is a rather cumbersome definition, but one can prove that the sequential definition and the geometric definitions are equivalent. One other way to write $\mathbb{R}$ is by writing $(-\infty, \infty)$. The symbols ∞ and $-\infty$ are **not numbers**, they simply represent that the line goes on ad infinitum to the left in the case of $-\infty$and to the right in the case of ∞.

Definition 2.2.6. Let U be well defined.

$\mathbb{R} = \{x|x$ is a number where x is an integer followed by a decimal and then a sequence of digits where each digit belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$.

Note, gentle reader that I skipped another standard set; that is because in a basic introduction to sets it is oft easier to 'jump' up to the reals, then return back to describe another set. That set is, of course, the irrationals. By the very nature of its name one can understand that it is composed of elements that are not rational. But, recall our discussion of the need for defining a domain of discourse or a universe. To say what something is not presupposes that everything has been specified! Putting it another way saying that the irrationals are numbers that are not rational is wrong since is a number that is not rational; but, it is also not irrational. Thus, the set of **irrational numbers** is denoted as $\mathbb{I}$ such that $\mathbb{I} = \{x : x \in \mathbb{R}$ and $x \ni \mathbb{Q}\}$. So, the irrationals are the set of all real numbers that are *not* rational. Note that this is a definition of something such that it is defined by what it is not. This definition by negation is oft quite useful; but one must understand what the first thing is (a real number) and the second thing is (a rational number) in order to understand what the third thing is (an irrational number) by way of what it isn't.

Definition 2.2.7. Let U be the reals.
$\mathbb{I} = \{x | x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$.

Constructing sets from this perspective leaves us with the feeling that all is known and specified previously, but consider people before they thought of these sets. Consider the man or woman who first thought of these sets. Is it not rather astonishing to think that such was not known nor conceived, but someone thought of these ideas first? A facile way of considering the wonderful experience it must have been is by specify a set that is not contained within the set of reals. Laying aside the important principle of consideration of the specification of a universe for the moment, let us look at the idea of set from a construction standpoint. Note that we did this by specifying $\mathbb{N}$, then $\mathbb{N}^*$, moving to $\mathbb{Z}$ and then $\mathbb{Q}$ then $\mathbb{R}$. We deviated from it when we specified $\mathbb{I}$.

Let us do it again. Define $\mathbb{C}$ to be the **complex numbers** such that $\mathbb{C} = \{z | z = a + bi \text{ where } a \in \mathbb{R}, b \in \mathbb{R}, \text{ and } i \text{ is defined as the } \sqrt{-1}\}$. From a geometric standpoint recall that $\mathbb{R}$ is a line and then we see that $\mathbb{C}$, the complex numbers, are really the plane (the horizontal axis consists of all points corresponding to the real part of the complex number and the vertical axis consists of all points corresponding to the i ('imaginary') part). So, why do so many people call complex number imaginary numbers when they correspond to things not so imagined but real? Indeed, if one argues that the reals correspond to real things and the complex numbers are 'not real' then why are both simply concepts corresponding to geometric forms (which recall are axiomatically given [point, line, and plane]). So, how real are the reals and imaginary are the complex numbers? But, I digress.[4]

Definition 2.2.8. Let U be well defined.
$\mathbb{C} = \{z | z = a + bi \text{ where } a \in \mathbb{R}, b \in \mathbb{R}, \text{ and } i \text{ is defined as the } \sqrt{-1}\}$

We can also define $\mathbb{C} = \{(x, y) | \text{ where } x \in \mathbb{R}, y \in \mathbb{R}\}$. This corresponds to the standard Cartesian coordinate system.

With that being said note we can define other types of numbers such as
$A_3 = \{(x, y, z) | \text{ where } x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$,
$A_4 = \{(w, x, y, z) | \text{ where } w \in \mathbb{R}, x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$, and so forth inductively. There are infinitely many universes that could be defined an infinitely many sets constructed. Truly magnificent!

[4] Hopefully this rudimentary waxing upon the nature of mathematics and ideas is not something which causes you, dear reader, to be distressed. If it does cause you discomfort, I am sorry. It might indicate (dare I say?) a lack of enthusiasm for studying mathematics. If this is the case, then perhaps one should spend time asking oneself why exactly he is studying, minoring in, or majoring in mathematics.

Example 2.2.2. *Let* $\mathrm{U} = \mathbb{N}$*. Specify* $A = \{x|5 < x \leq 11\}$ *in list form. Note the solution to this is:* $A = \{6, 7, 8, 9, 10, 11\}$*.*
However, let $\mathrm{U} = \mathbb{R}$*. The question, specify* $A = \{x|5 < x \leq 11\}$ *in list form,* cannot *produce a solution to this since there is no way to list out all the elements. So, we can use the same symbol for two sets that are different (even though they seem quite the same, the difference in the universes made radical difference in the sets).*

Let us consider another example.

Example 2.2.3. *Let* $\mathrm{U} = \mathbb{R}$*. Consider* $B = \{x|5 < x \leq 11,$ *where* $x \in \mathbb{Z}\}$ *and consider* $C = \{x|5 < x \leq 11,$ *where* $x \in \mathbb{Q}\}$*. Note* $6 \in B$*;* $6 \in C$*; but,* $\frac{15}{2} \in C$*; whereas,* $\frac{15}{2} \notin B$*.*

Example 2.2.4. *Let* $\mathrm{U} = \mathbb{R}$*. Consider* $D = \{x|5 < x \leq 11,$ *where* $x \in \mathbb{Q}\}$ *and consider* $E = \{x|5 < x \leq 11,$ *where* $x \in \mathbb{R}\}$*. Note* $8\frac{1}{2} \in D$*;* $8\frac{1}{2} \in E$*;* $\pi \in E$*;* $\pi \notin D$*;* $\sqrt{47} \in E$*; but,* $\sqrt{47} \notin D$*.*

So, what does it mean for two sets to be the same; or a part of another; or different? Let U be a well defined universe, A, B, C, D, E, and F be sets of elements from the well defined universe. The statement $A = B$ means that every element of A is in B and every element of B is in A. There is a weaker notion than equality of sets. That notion is the idea of subset. A set C is a **subset** of D if for each element x in C, it is the case that x is in D. When C is a **subset** of D we say D is a **superset** of C. C is a subset of D is denoted as $C \subseteq D$ or $D \supseteq C$ (the same notation means D is a superset of C).[5] There is another notion between the idea of subset and equality, which is the notion of proper subset-hood. A set E is said to be a **proper subset** of F if $E \subseteq F$ and $E \neq F$. Denote E is a proper subset of F as $E \subset F$.[6] In rather a circular manner (granted) we can return to the definition of equality of two sets and state that two set A and B are **equal** if and only if $A \subseteq B$and $B \subseteq A$.

Definition 2.2.9. Let U be a well-defined universe whilst A and B are sets. $A \subseteq B$ if and only if for each element $x \in A$ it is the case that $x \in B$.

Definition 2.2.10. Let U be a well-defined universe whilst A and B are sets. $A \subset B$ if and only if $A \subseteq B$ and there exists at least one element, y, such that $y \in B$ and $y \notin A$.

Definition 2.2.11. Let U be a well-defined universe whilst A and B are sets. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

---

[5] Note the similarity of the symbol ‘ $\subseteq'$ to the symbol '$\leq$.’ In a sense (naïvely) C is somehow less than or equal to D.

[6] Note that some mathematicians use the symbol ‘$\subset$’ to mean subset rather than ‘$\subseteq$.’ So, it is important to always check to see how the symbol is being used. A student once remarked that this is rather tedious and silly. He thought that mathematicians should standardise all the notation. He has a point, but it would probably easier to fly an aeroplane to Alpha Centauri than to get even three academicians to agree on much.

Example 2.2.5. *Let* U $=\mathbb{N}$. *Let*
$A=\{x|5<x\leq 11\}, B=\{x|5<x\leq 9\}, C=\{x|7\leq x\leq 11\}$, *and* $D=\{x|5<x<12\}$.
Note the following: $A=\{6,7,8,9,10,11\}$*;* $B=\{6,7,8,9\}$*;* $C=\{7,8,9,10,11\}$*; and,*
$D=\{6,7,8,9,10,11\}$.

Thus, this example illustrates every set is a subset of itself. Note every set is not *a proper subset of itself.*
Note $A=D$.
Note $B\subseteq A$.
Note $B\subseteq D$. *Moreover* $B\subset D$ *because* $B\subseteq D$ *and* $11\in D$ *but* $11\notin B$.
Note $C\subseteq A$.
Note $C\subseteq D$. *Moreover* $C\subset D$ *because* $C\subseteq D$ *and* $6\in D$ *but* $6\notin C$.
Please note $B\nsubseteq C$ *since* $6\in B$ *and* $6\notin C$.
Finally, please note $C\nsubseteq B$ *since* $11\in C$ *and* $11\notin B$.

There are ten basic axioms for set theory (see chapter 3), we shall introduce and use only one of them in this course. It is **the axiom of null** which states there exists a set with no elements, call it $\varnothing$. Note that $\varnothing$ is a special symbol. Do not use $\{\}$ or any other creative use of notation to reference $\varnothing$, just use $\varnothing$.

Let $U=\mathbb{N}$, $A=\{6,7,8,9,10,11\}$; $B=\{6,7,8,9\}$; $C=\{7,8,9,10,11\}$; and, $D=\{6,7,8,9,10,11\}$. Notice $\varnothing\subseteq A$ (indeed $\varnothing\subseteq B$, $\varnothing\subseteq C$, and $\varnothing\subseteq D$)! Note this statement illustrates the reasonableness of the truth table for implication in chapter one of $(F\Rightarrow T)$ is true since the statement every element of null is an element of the set A since there are no elements in null; hence, a counterexample cannot be constructed to challenge the validity of $\varnothing\subseteq A$. It is an example of a **vacuously true statement**.

Next let us define the **complement** of a set A, usually denoted as A^c. It is the set of all elements in the universe that are not in A.[7] So, $A^c=\{x|x\in U$, but $x\notin A\}=\{x|x\notin A\}$ when it is understood the universe has been defined. Other symbols that mean the complement of the set A are A', $\overline{A}$, and $U-A$. The $U-A$ symbol is quite representative since we are subtracting for the universe all elements that were in A. This notation gives rise to the concept of the **relative complement** of two sets. The set B **relative complement** to the set A, denoted as $B-A$ or $B\backslash A$, is the set of all elements of B that are not in A.

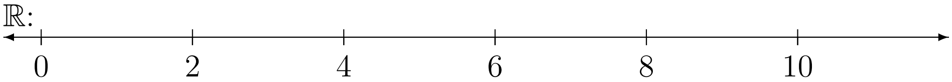
Let $U=\mathbb{N}$, $A=\{6,7,8,9,10\}$; $B=\{6,7,8,9\}$; $C=\{7,8,9,10,11\}$; and, $D=\{6,7,8,9,10,11\}, E=\{7,8,9\}$. It is the case that $B-A=\varnothing$. However, it is the case that $A-B=\{10\}$. Furthermore, it is the case that $C-A=\{11\}$ and $A-C=\{6\}$. Notice $y\in B\Longrightarrow y\in A$ so $B\subseteq A$ and $m\in A\Longrightarrow m\in D$ so $B\subseteq D$ by transitivity. Also, notice $k\in E\Longrightarrow k\in B$ so $E\subseteq B$ and $j\in E\Longrightarrow j\in C$ whist there are not elements in B or D that are in both sets but not in E. Hence, the set E is very interesting - - it

[7]Again note the need for defining the universe first; for if the universe is not defined, then A^c is ambiguous.

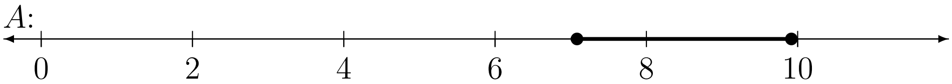
contains all the elements that are contained in both B and D. This set is called the **intersection** of the sets B and D and is denoted as $B \cap D$. Now, suppose the two sets have no elements in common, then we say they are **disjoint**.[8] It is symbolised by $A \cap B = \varnothing$. Note that $z \in A \rightarrow z \in D$ hence $A \subseteq D \ \wedge \ z \in B \rightarrow z \in D$ hence $B \subseteq D$. Further note that D is also quite fascinating - - it contains all the elements that are contained in A or in B whist there are not elements in D that are in not in either set A or B. This set is called the **union** of the sets A and B and is denoted as $A \cup B$.

Let $U = \mathbb{R}$, $A = \{x|7 \leq x \leq 10\}$, $B = \{x|5 < x < 8\}$, $C = \{x|7 \leq x < 8\}$, and $D = \{x|5 < x \leq 10\}$. Note that an alternate way of expressing subsets of the reals is with another form of notation.

$\mathbb{R}$:

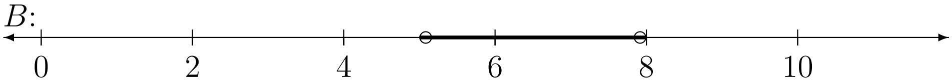


A:



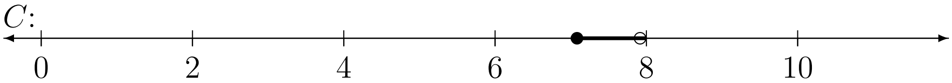
$A = [7, 10]$ and A is called an **interval**.

B:



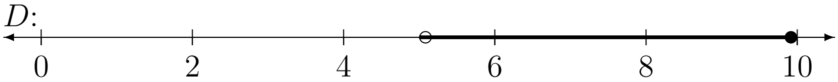
$B = (5, 8)$ and B is called a **segment**.

C:



$C = [7, 8)$ and C is called a **half-interval** or a **half-segment**.

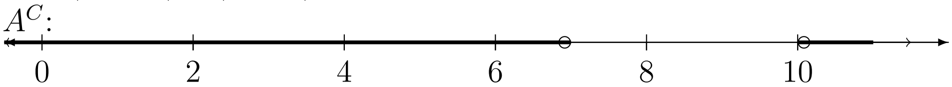
D:



$D = (5, 10]$ and C is called a **half-interval** or a **half-segment**.

Note $A^C = \{x|x < 7 \vee 10 < x\}$. So, in the interval or segment notation $A = (-\infty, 7) \cup (10, \infty)$.

A^C:



Note that $A \backslash C = [8, 10]$; $C \backslash A = \{7\}$ which is expressed in interval notation as $[7, 7]$; and, $A \backslash D = \varnothing$.

Another import kind of set is the **power set of a set** denoted as $\mathcal{P}(A)$ where A is the set. It is the set of all subsets of a set. For a set of sets we call such a **collection**.

[8] Also referred to as mutually exclusive.

Example 2.2.6. *Suppose* $\mathrm{U} = \mathbb{N}$*;* $M = \{1, 2, 3\}$*;* $K = \{4, 5\}$*;* $W = \{2, 3, 4, 5\}$ *whilst* $E = \{9\}$*.*
$\mathcal{P}(E) = \{\varnothing, \{9\}\}$.
$\mathcal{P}(K) = \{\varnothing, \{4\}, \{5\}, \{4, 5\}\}$.
$\mathcal{P}(M) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
$\mathcal{P}(W) = \{\varnothing, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\},$
$\{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{2, 4, 5\}, , \{2, 3, 4, 5\}\}$.

So, for any given set the power set of the set is larger (contains more sets than elements of the original set. Furthermore inductively note that there is a relationship between and betwixt the size of a set and the size of its corresponding power set.

Suppose $U = \mathbb{N}$; $M = \{1, 2, 3\}$
$\mathcal{P}(M) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. One can also consider $\mathcal{P}(\mathcal{P}(M))$. We shall denote **a power set of a power set of a set** denoted as $\mathcal{P}(\mathcal{P}(M))$ where $\mathcal{P}(M)$ is the power set and where M is the set. For a set of sets of sets we call such a **family**. Let $\mathscr{F} = \mathcal{P}(\mathcal{P}(M))$ denote the family of the power set $\mathcal{P}(M)$.
Thus, $\mathscr{F} = \{\varnothing, \boldsymbol{\{}\varnothing\boldsymbol{\}}, \boldsymbol{\{}\{1\}\boldsymbol{\}}, \boldsymbol{\{}\varnothing, \{1\}\boldsymbol{\}}, \ldots, \boldsymbol{\{}\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\boldsymbol{\}}\}$.
So, in toto, M and E are sets of elements in U, the elements of M and E are not. They are elements of another universe, namely, $\mathcal{P}(U)$ since $\mathcal{P}(U)$ contains all the subsets of U. Further, M and E are not elements in $\mathscr{F}$, the elements of $\mathscr{F}$ are sets of sets.
For example, $4 \in \{4, 5\}$; $4 \notin \mathcal{P}(\{4, 5\})$.
$\{4\} \notin \{4, 5\}$; $\{4\} \subseteq \{4, 5\}$; $\{4\} \in \mathcal{P}(\{4, 5\})$.
$\varnothing \notin \{4, 5\}$; $\varnothing \subseteq \{4, 5\}$; $\{\varnothing\} \notin \mathcal{P}(\{4, 5\})$; and $\varnothing \subseteq \mathcal{P}(\{4, 5\})$.
Moreover $\varnothing \notin \{4, 5\}$; $\{\varnothing\} \notin \mathcal{P}(\{4, 5\})$; $\{\varnothing\} \in (\mathcal{P}(\{4, 5\}))$; whilst $\varnothing \subseteq \{4, 5\}$;
$\varnothing \subseteq \mathcal{P}(\{4, 5\})$; $\{\varnothing\} \subseteq (\mathcal{P}(\{4, 5\}))$
Intense.

Finally note the similarity of the nomenclature and notation between sets and logic. Then combining symbols we deduce $(P \vee Q) \wedge \neg(P \wedge Q)$ has as its logical equivalent in set theory of $(A \cup B) \cap (A \cap B)^C$ or $(A \cup B) - (A \cap B)$ and so forth. It is no coincidence that the symbols are quite similar. Note also that both the symbols from set theory and logic are similar to the inequality symbols of comparison of points for the real line; again no coincidence. Noting these similarities should help you remember, learn, and understand the symbols for sets. There are basic, moderate, and complex claims about sets which you will prove or disprove in Math 224 (Set Theory).

Meaning	**Logic**	**Meaning**	**Set Theory**
Not P	$\neg P$	Not A (A complement)	A^C
P or Q	$P \vee Q$	A or B (A union B)	$A \cup B$
P and Q	$P \wedge Q$	A and B (A intersect B)	$A \cap B$
P implies Q	$P \Rightarrow Q$	A is a subset of B	$A \subseteq B$
P is logically equivalent to Q	$P \Leftrightarrow Q$ $P \equiv Q$	A is equal to B	$A = B$

Table 2.1: Notational and Definitional Similarity between Logic and Set Theory

2.2.1 Exercises

Exercise 2.2.1. Some of the following claims are true, some are false. It might be worth your while to try to establish some intuitive feel about sets by considering these claims and determining if they are true or false.
Basic Assumptions: Let U be a well defined universe and A, B, and C be sets.

Claim 2.2.1. $A \subseteq B \quad \wedge \quad B \subseteq C \Longrightarrow A \subseteq C.$

Claim 2.2.2. $A \subset B \ \wedge \ B \subset C \Longrightarrow A \subset C.$

Claim 2.2.3. $A \subset B \ \wedge \ B \supset C \Longrightarrow A = C.$

Claim 2.2.4. $A \subseteq B \ \wedge \ B \supseteq C \Longrightarrow A = C.$

Claim 2.2.5. $\varnothing \subseteq A.$

Claim 2.2.6. $\varnothing \subset A.$

Claim 2.2.7. $\varnothing \subset U.$

Claim 2.2.8. $\varnothing \subset U.$

Claim 2.2.9. $A \subseteq B \ \wedge \ B \subseteq A \Longrightarrow A = B.$

Claim 2.2.10. $A \subseteq \varnothing \Longrightarrow A = \varnothing.$

Claim 2.2.11. $\varnothing \subseteq \varnothing.$

Claim 2.2.12. $\varnothing \subset \varnothing.$

Claim 2.2.13. $U \subseteq U$.

Exercise 2.2.2. Determine which of the following symbols: $<$, $\leq$, $>$, $\geq$, $\Longrightarrow$, $\cup$, $\cap$, $\subset$, $\subseteq$, $\nsubseteq$, $\supset$, $\supseteq$, $\notin$, $\in$, $=$, or $\neq$ should properly be placed between the following in column A and column B:

	Column A	Column B
A.	$\{1, 2, 3, 4\}$	$\{x \mid x \text{ is a divisor of } 24\} \ni U = \mathbb{N}$
B.	6	$\{x \mid x \text{ is a root of } x^2 - 13x + 42 = 0\} \ni U = \mathbb{N}$
C.	$\{6\}$	$\{x \mid x \text{ is a root of } x^2 - 13x + 42 = 0\} \ni U = \mathbb{N}$
D.	$\{6, 7\}$	$\{x \mid x \text{ is a root of } x^2 - 13x + 42 = 0\} \ni U = \mathbb{N}$
E.	0	$\{x \mid x \text{ is a root of } x^3 - 6x^2 + 11x = 6\} \ni U = \mathbb{N}$
F.	0	$\{x \mid \frac{1}{x} \in \mathbb{R}\}$

Exercise 2.2.3. Determine which of the following symbols: $<$, $\leq$, $>$, $\geq$, $\Longrightarrow$, $\cup$, $\cap$, $\subset$, $\subseteq$, $\nsubseteq$, $\supset$, $\supseteq$, $\notin$, $\in$, $=$, or $\neq$ should properly be placed between the following in column A and column B:

	Column A	Column B
A.	$\mathbb{N}$	$\mathbb{N}$
B.	$\mathbb{N}$	$\mathbb{Z}$
C.	$\mathbb{N}$	$\mathbb{Q}$
D.	$\mathbb{N}$	$\mathbb{I}$
E.	$\mathbb{N}$	$\mathbb{R}$
F.	$\mathbb{N}$	$\mathbb{C}$
G.	$\mathbb{Z}$	$\mathbb{Z}$
H.	$\mathbb{Z}$	$\mathbb{Q}$
I.	$\mathbb{Z}$	$\mathbb{I}$
J.	$\mathbb{Z}$	$\mathbb{R}$
K.	$\mathbb{Z}$	$\mathbb{C}$
L.	$\mathbb{Q}$	$\mathbb{Q}$
M.	$\mathbb{Q}$	$\mathbb{I}$
N.	$\mathbb{Q}$	$\mathbb{R}$
O.	$\mathbb{Q}$	$\mathbb{C}$
P.	$\mathbb{I}$	$\mathbb{R}$
Q.	$\mathbb{I}$	$\mathbb{C}$
R.	$\mathbb{R}$	$\mathbb{C}$
S.	$\mathbb{N}$	$\mathcal{P}(\mathbb{N})$
T.	π	3.14159265358979323846264338327950

Exercise 2.2.4. Determine what the relationship is between what is in column A and column B:

	Column A	Column B
A.	$1, 2, 3, 4$	$\{1, 2, 3, 4\} \ni U = \mathbb{N}$
B.	$\{1, 2\}$	$\{1, 2, 3, 4\} \ni U = \mathbb{N}$
C.	$\{1, 2\}$	$\{1, 2, 3, 4\} \ni U = \mathbb{R}$
D.	$\varnothing$	$\{1, 2, 3, 4\} \ni U = \mathbb{N}$
E.	0	$\{x \| x \text{ is a root of } x^3 - 6x^2 + 11x = 0\} \ni U = \mathbb{N}$
F.	$\{1,\ 2,\ 3\}$	$\{x \| x \text{ is a root of } x^3 - 6x^2 + 11x - 6 = 0\} \ni U = \mathbb{N}$

Exercise 2.2.5. Determine what the relationship is between what is in column A and column B:

	Column A	Column B
A.	$\varnothing$	$\{\varnothing, \{1\}\} \ni U = \mathcal{P}(\{1, 2, 3\})$
B.	$\{\varnothing\}$	$\{\varnothing, \{1\}\} \ni U = \mathcal{P}(\{1, 2, 3\})$
C.	$\{\varnothing, \{1\}\}$	$U = \mathcal{P}(\{1, 2, 3\})$
D.	$\{\varnothing, \{1\}, \{2\}, \{3\}\}$	$U = \mathcal{P}(\{1, 2, 3\})$
E.	$\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$	$U = \mathcal{P}(\{1, 2, 3\})$

Exercise 2.2.6. Let U be a well-defined universe and A and B be sets. Determine what the relationship is between what is in column A and column B:

	Column A	Column B
A.	A	A^C
B.	$A \cap B$	A
C.	$A \cup B$	A
D.	$A \cap B$	$A \cup B$
E.	$(A \cup B)^C$	$A^C \cup B^C$
F.	$(A \cup B)^C$	$A^C \cap B^C$
G.	$(A \cap B)^C$	$A^C \cap B^C$
H.	$(A \cap B)^C$	$A^C \cup B^C$
I.	$A - B$	$A \cup B$
J.	$A - B$	$A \cap B$

Exercise 2.2.7. Let U the a well-defined universe $\mathbb{R}$. Determine what the relationship is between what is in column A and column B:

	Column A	Column B
A.	0.33	$\frac{1}{3}$
B.	0.34	$\frac{1}{3}$
C.	$0.\overline{3}$	$\frac{1}{3}$
D.	$0.\overline{9}$	1
E.	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$
F.	$\sqrt{2}$	2
G.	$\sqrt{\frac{1}{2}}$	$\frac{1}{2}$

Exercise 2.2.8. Write the following sets in set-builder form or with standard notation (where $U = \mathbb{R}$):
A. $\{1, 2, 3, 4, 5\}$
B. $\{1, 2, 3, 4, 6, 8, 12, 24\}$
C. $\{-3, -2, -1, 0, 1, 2, 3, 4\}$ D. The set of real numbers less than 4.
E. The set of natural numbers less than 4.
F. The set of integers between -7 and 5.
G. The set of integers between -7 and 5, inclusive.
H. The set of integers that are solutions to the equation $x^2 + 3x - 10 = 0$.
I. The set of natural numbers that are solutions to the equation $x^2 + 3x - 10 = 0$.
J. The set of real numbers that are solutions to the equation $x^2 + 3x - 10 = 0$.
L. The set of integers that are solutions to the equation $x^4 - 16 = 0$.
M. The set of natural numbers that are solutions to the equation $x^4 - 16 = 0$.
N. The set of real numbers that are solutions to the equation $x^4 - 16 = 0$.
O. The set of real numbers that are solutions to the equation $x^4 + 16 = 0$.

Exercise 2.2.9. Let $U = \mathbb{N}_{12}$ and sets $A = \{x : 1 < x \leq 11\}$, $B = \{x : |x - 5| < 3\}$, $C = \{x : \sqrt{x} \in \mathbb{N}_{12}\}$, $D = \{x : x < 5\}$, $E = |x - 5| = 1\}$, and $F = \{1, 2, 3, 4, 6, 8\}$. Find the following:

A. A in list form. B. B in list form. C. C in list form. D. D in list form.
E. E in list form. F. $A \cap B$ G. $B \cap C$ H. $A \cap B \cap C$
I. $A \cup B$ J. $A \cup B \cap C$ K. $(A \cup B) \cap C$ L. $A \cup (B \cap C)$
M. $(A \cup B)^C$ N. $D - C$ O. $C - D$ P. $C \cap D$
Q. $C \cup D$ R. $D \cap E \cup F$ S. $(D \cap E) \cup F$ T. $D \cap (E \cup F)$
U. $(D \cap E) \cup (D \cap F)$ V. $(D \cup E) \cap (D \cup F)$ W. $D \cup E \cup F$ X. $F \cup E \cup D$
Y. $E \cap E$ Z. $E \cap E \cap E$

Exercise 2.2.10. Let $U = \mathbb{R}$ and sets
$A = (1, 11], B = [2, 9), C = (-\infty, 11], D = \{x : 4 < x\}, E = [1, 11]$, and $F = (4, 9)$.
Find the following (express you solution in interval or segment notation)[9] :
A. $A \cap D$ B. $D \cap C$ C. $A \cap D \cap C$ D. $A \cup D$
E. $D \cup C$ F. $A \cup D \cup C$ G. $C^C \cap D$ H. $A \cap B \cap C \cap D$
I. $B \cup E$ J. $B \cap E$ K. $B - E$ L. $E \cap B^C$
M. $E - B$ N. $\varnothing \cup F$ O. $\varnothing \cap F$ P. $\varnothing^C$
Q. $A \cap B \cup E \cap F$ R. $A \cap (B \cup E) \cap F$ S. $(A \cap B) \cup (E \cap F)$ T. $A \cap (B \cup E \cap F)$

Exercise 2.2.11. Let $U = \{-3, -2, -1, 0, 1, 2, 3\}$ and sets $A = \{x : x$ is a solution to the equation $x^2 - 4 = 0\}$, $B = \{x : x$ is in U and $\frac{1}{x}$ is in $U\}$, $C = \{x : |x - 1| \leq 2\}$,
$D = \{-2, 2\}$, $E = \{-2, 0, 1, 2\}$, $F = \{-2, -1, 0, 2\}$, and $G = \{0\}$.
Find the following:
A. $A \cap B$ B. $B \cap C$ C. $C \cap D$ D. $D \cap E$
E. $E \cap F$ F. $A \cup B \cap C$ G. $A \cap B \cup C$ H. $A \cup (B \cap C)$
I. $(A \cup B) \cap C$ J. $\mathcal{P}(G)$ K. $\mathcal{P}(B)$ L. $\mathcal{P}(F)$
M. $D \cup E \cup F$ N. $(D \cup E \cup F)^C$ O. $D^C \cup E^C \cup F^C$ P. $D \cap E \cap F$
Q. $D^C \cap E^C \cap F^C$ R. $(D \cap E \cap F)^C$

[9]Recall that $\mathbb{R}$ expressed in segment notation is $(-\infty, \infty)$.

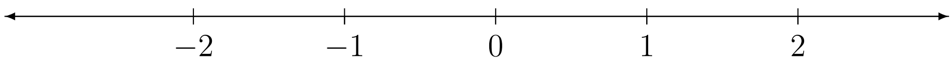
2.3 Venn Diagrammes and Other Illustrations for Sets

Now, let us investigate the potential use of pictures to represent the concepts from the previous section. All of you are familiar with the line, $\mathbb{R}$, and its graphical representation:

$\mathbb{R}$:

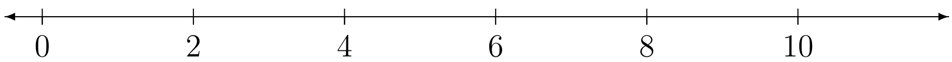
There is no centre so it can be drawn as:

$\mathbb{R}$:



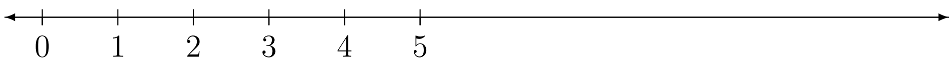
or

$\mathbb{R}$:



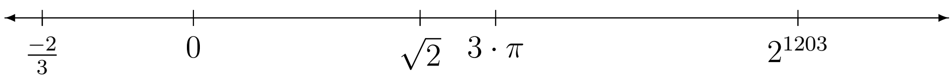
or

$\mathbb{R}$:



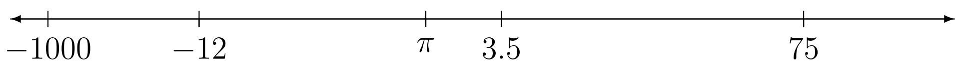
or

$\mathbb{R}$:



or

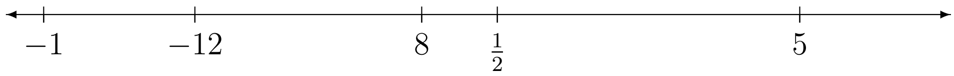
$\mathbb{R}$:



It can be stretched, shrunk, shifted, etc. and it is still infinitely long without centre– there is ***no*** centre (e.g.: the nonsense about + (-) = 0 one may have leant in high school is a fallacy) so one can reasonably represent the line as any of the previous.

However, the points must follow the axioms of the reals so the order must be correct. It is incorrect if one drew:

"ℝ":



In an undergraduate mathematics major, in depth study of the properties of the line occur in Real Analysis I (Math 361), additional topics of the line in Real Analysis II (Math 362), and more generalisations of the line in Real Variables (Math 463), Complex Variables (Math 465), or Topology (Math 481). The line has some very subtle and fascinating properties; but, such is beyond the scope of this course (perhaps someone in this course will be inspired to major or minor in mathematics – which would be swell).

Nonetheless, there are general pictorial representations of sets that are of use to us at this stage of our mathematical development. They are called **Venn diagrammes**, after the mathematician Venn, who adapted them from the mathematician Euler (more on this topic later). Standard representation of the universe is a rectangle and sets of interest as circles within the rectangle. The sets are typically designated with upper case English letters. Elements within the universe are designated with points or little × and marked, usually, lower case letters.

Example 2.3.1. *Let* $\mathrm{U} = \{1, 2, 3, 4, 5\}$*. Let* $A = \{1, 3, 5\}$*. Note* $1 \in A, 2 \notin A, 3 \in A, 4 \notin A$*, and* $5 \in A$*. The Venn Diagramme for this is:*

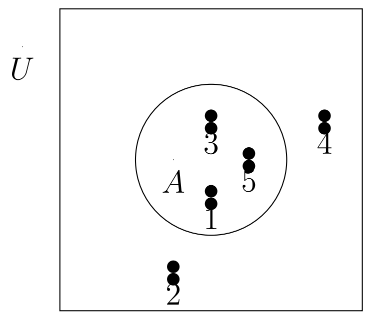


Figure 2.3.1

Indeed specification of the elements is not necessary. So, suppose we simply wanted to represent a situation where there was a well defined universe and two sets. Let U be the universe whilst A and B be the sets. Nothing is stated about $A \cap B$, $A \cup B$, etc. so the way to draw two sets, in general, is to have them overlap and be restricted inside the universe as:[10]

[10]Note that the programme used to draw Venn Diagrammes in LaTeX does not label the universe - that is an error, it should be labelled U.

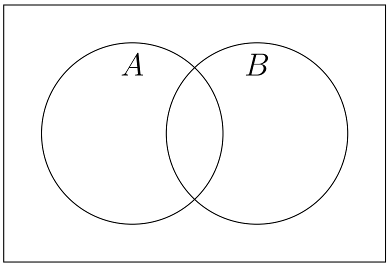


Figure 2.3.2

Usage of the Venn Diagramme is for illustration of different aspects of sets; for example, let U be the universe whilst A and B are sets and we wish to illustrate $A \cap B$. To do so one shades the overlaps as:

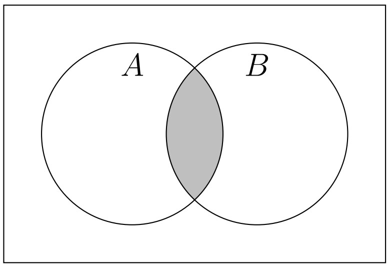


Figure 2.3.3

Let U be the universe whilst A and B are sets and we wish to illustrate $(A \cap B)^C$. To do so one shades all the parts of the diagramme except the overlap of A and B (which is $A \cap B$)as:

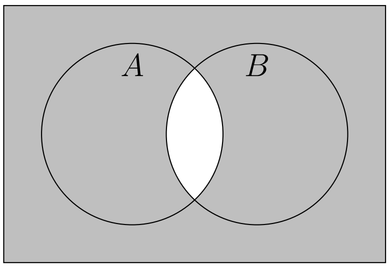


Figure 2.3.4

Let U be the universe whilst A and B are sets and we wish to illustrate $A \cup B$. To do so one shades all the parts of the diagramme that are in A or B as:

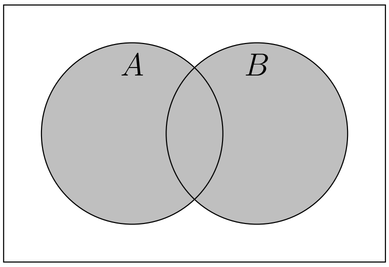


Figure 2.3.5

Usage of the Venn Diagramme is for illustration of different aspects of not just two sets – let U be the universe whilst A, B, and C are sets. To do so one draws three overlapping circles within the rectangle illustrating a well defined universe. So:

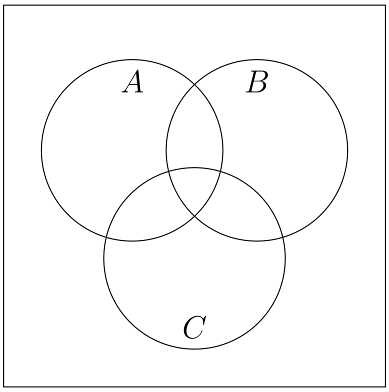


Figure 2.3.6

Usage of the Venn Diagramme is for illustration of different aspects of sets; for example, let U be the universe whilst A, B, and C are sets and we wish to illustrate $A \cap B \cap C$. To do so one shades the overlaps as:

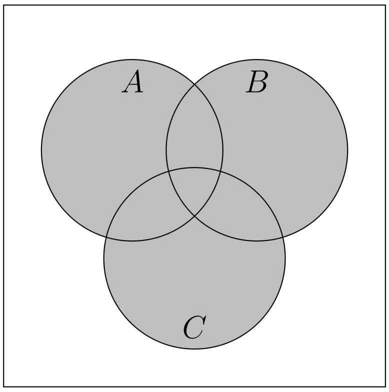


Figure 2.3.7

Now, note that for a Venn diagramme for a generalised one set scenario, there are but two (2) distinct 'regions' that compose the illustration. Also note that for a Venn diagramme for a generalised two set scenario, there are but four (4) distinct 'regions' that compose the illustration and that for a Venn diagramme for a generalised three set scenario, there are eight (8) distinct 'regions' that compose the illustration. Let us number them in no particular order with Roman numerals (to distinguish this idea from actual elements):

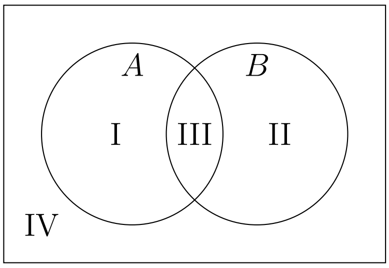


Figure 2.3.8

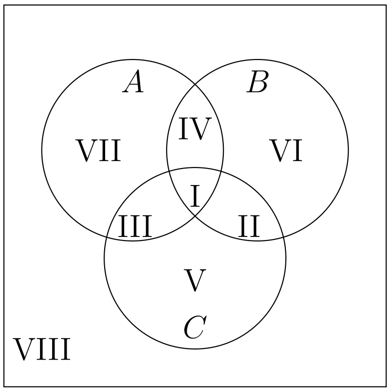


Figure 2.3.9

The use of 'regions' helps with the shading in more complex instances; for example:

Example 2.3.2. *Let* U *be the universe whilst* A*,* B*, and* C *are sets. Shade the region to illustrate* $(A \cup B)^C \cup (C \cap A)$*.*

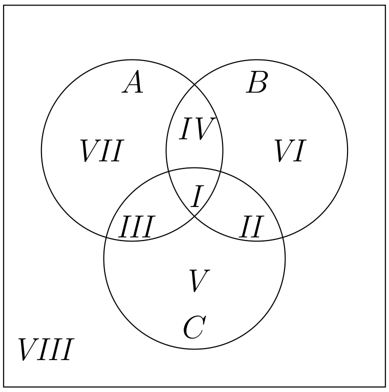


Figure 2.3.10

Notice $A \cup B$ *are 'regions' I, II, III, IV, VI, and VII. Thus,* $(A \cup B)^C$ *are 'regions' V and VIII.* $C \cup A$ *are 'regions' I and III. So,* $(A \cup B)^C \cup (C \cap A)$ *is denoted by shading 'regions' V, VIII, I, and III.*

Now, one might naturally opine that the pictorial representations generalise in an intuitively 'pleasant' manner. Hence, one might think that a Venn diagramme for a generalised four set scenario would be composed as follows.

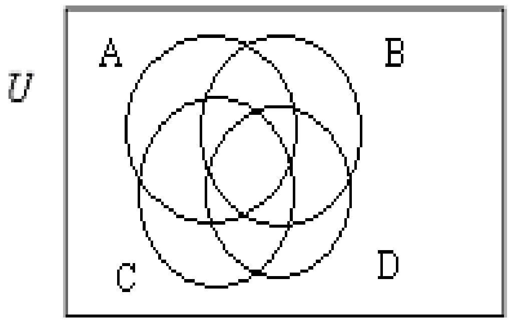


Figure 2.3.11: Incorrect Venn Diagramme for Four Sets

It is *not* for several reasons (the most clear is there are not 16 'regions'). The four set generalised Venn diagramme does not illustrate easily; indeed I had to look it up in another text[11] to remind myself the precise manner of drawing it since I have a vague recollection of it. Hence, we shall not bother with it (unless provided with the illustration). Hence, the student should not waste time learning how to draw it.

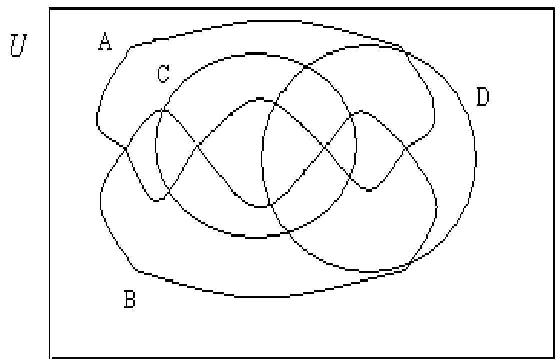


Figure 2.3.12: Correct Venn Diagramme for Four Sets

[11] Louis M. Rotando, Finite Mathematics, page 67, New York: D. Van Nostrand, 1980 (originally published by Litton Publishers). The generalised four set Venn diagramme was conceived by Carol Guadagni in 1974 who was a freshman at Nassau Community College in New York.

An appealing exercise is to try to determine why figure 2.3.11 is incorrect and figure 2.3.12 is correct (see the exercise set).

One can create generalised Venn diagrammes for five or more sets; but, it is beyond practicality. Sometimes picturing the situation is at best useful whilst other times it is an exercise in frustration and might not aide in solving a problem. Venn diagrammes are by design of use in order to assist a person in visualising a problem; but there are some problems that might not be so easily visualised. Hence, caution is advised. Determine whether or not it is of use to you, the student, to spend time attempting to draw a graph of a problem, claim, etc. before attempting to solve the problem, construct an argument or counter-argument, etc.

Nonetheless, there are instances where drawing a Venn diagramme with more than three sets is if use: that is, when there are added hypotheses which yield an easier illustration. Such complex Venn diagrammes *might* be of use (it is up to you to decide).

Example 2.3.3. *Let* U = *(note that is a well defined universe) such that the sets* $A, B, C, D, E,$ *and* F *are defined as follows:*
$A = \{x|x \text{ is prime }\}$,
$B = \{x|$ *there is an element* j *in the universe so that* $x = 2j\}$,
$C = \{x|$ *there is an element* k *in the universe so that* $x = 4k + 1\}$,
$D = \{2\}$,
$E = \{x|$*there is an element* m *in the universe so that* $x = 6m + 1\}$, *and,*
$F = \{1\}$.
The Venn diagramme to illustrate these sets in the universe is:

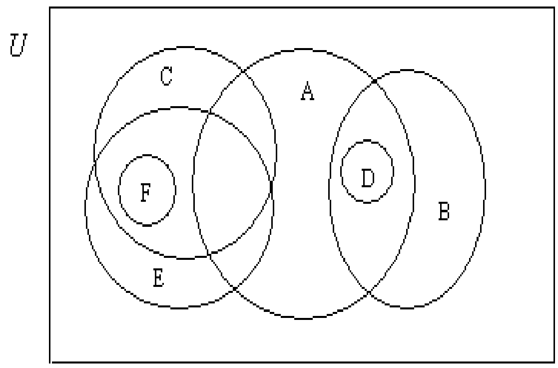


Figure 2.3.13: Example 2.3.3

One of the most useful applications of Venn diagrammes is in the visualisation it allows for operations between sets. This use will be especially important for the student if he takes Set Theory (Math 224) and subsequent courses since he will be asked to prove or disprove certain assertions about sets.

Example 2.3.4. *Suppose* U *is a well defined universe such that A and B are sets. Is it the case that $A^C \cup B^C = (A \cup B)^C$? One may opine it is true or it is false - - the Venn diagramme provides visual evidence to suggest a proper course of action (to prove the claim or to provide a counterexample).*
So, let us consider a generalised Venn diagramme for two sets:

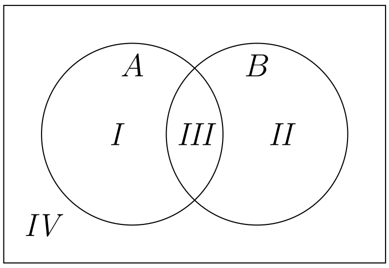


Figure 2.3.14

Note that A is regions I and III so, therefore A^C is regions II and IV. Note that B is regions II and III so, therefore B^C is regions I and IV. The operation is union; thus, we choose any that are mentioned– so, $A^C \cup B^C$ is regions I, II, and IV.

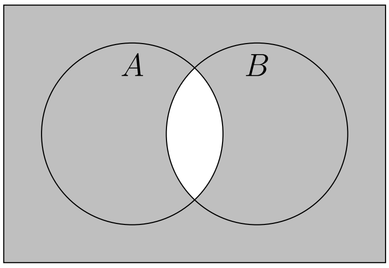


Figure 2.3.15

Note that A is regions I and III and B is regions II and III so, since we are consider a union, this means that $A \cup B$ is composed of regions I, II, and III. But we must further note that we are considering $(A \cup B)^C$, so we realise that the complement of A union B is composed of region IV. So, we would shade only region IV to illustrate the set $(A \cup B)^C$.

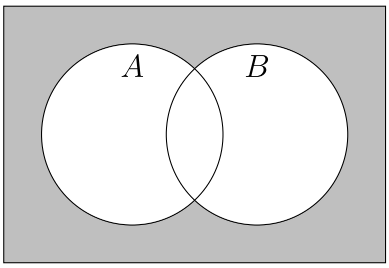


Figure 2.3.16

So, the question would be answered negatively (is it the case that $A^C \cup B^C = (A \cup B)^C$ *?) and we would need to present a counterexample to the claim.*

But wait! We have a counterexample - - all we have to do is present it properly:
Claim: Suppose U is a well defined universe such that A and B are sets. It is case that $A^C \cup B^C = (A \cup B)^C$.
Counterexample: Let $U = \mathbb{N}_4$. Let $A = \{1, 3\}$. Let $B = \{2, 3\}$. Note that $A^C = \{2, 4\}$ and $B^C = \{1, 4\}$. So, $A^C \cup B^C = \{1, 3, 2\}$. But, $A \cup B = \{1, 2, 3\}$. So, $(A \cup B)^C = \{4\}$.

EEF.

Note that we had to construct an example [define a universe, construct sets A and B within that universe and demonstrate that there was an element in one of the sets that was not in the other so that the claim of equality of sets must be false] which shows the claim false (hence, the term counterexample) rather than just record a suggestive illustration of the case that the claim is false.

Example 2.3.5. *Suppose* U *is a well defined universe such that* A, B, *and* C *are sets. Is it the case that* $A^C \cup B^C - (C \cup (A^C \cap B^C \cap C^C) \subseteq (A \cap B \cap C)^C$ *? One may opine it is true or it is false - the Venn diagramme provides visual evidence to suggest a proper course of action (to prove the claim or to provide a counterexample). So, let us consider a generalised Venn diagramme for three sets:*

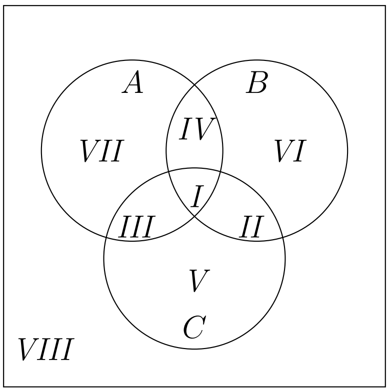


Figure 2.3.17

Note that A^C *is composed of regions II, V, VI, and VIII*[12]*;* B^C *is composed of regions VII, V, III, and VIII; and* C *is composed of regions I, II, III, and V.*
So, $A^C \cup B^C$ *is composed of regions II, III, V, VI, VII, and VIII. Notice* $(A^C \cap B^C \cap C^C)$ *is region VIII; so,* $C \cup (A^C \cap B^C \cap C^C)$ *is I, II, III, V, and VIII. Now the regions in that set that overlap* $C \cup (A^C \cap B^C \cap C^C)$ *are II, III, V, and VIII. Hence,* $A^C \cup B^C \setminus C \cup (A^C \cap B^C \cap C^C)$ *is denoted by shading regions VI and VII.*

[12]Since A is composed of VII, IV, III, and I.

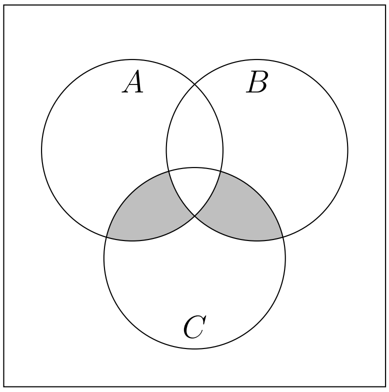


Figure 2.3.18

Now consider A is composed of regions I, III, IV, and VII; B is composed of regions I, II, IV, and VI. C is composed of regions I, II, III, and V. So, $A \cap B \cap C$ is region I (the only common to all three). Thus, $(A \cap B \cap C)^C$ is regions II through VIII.

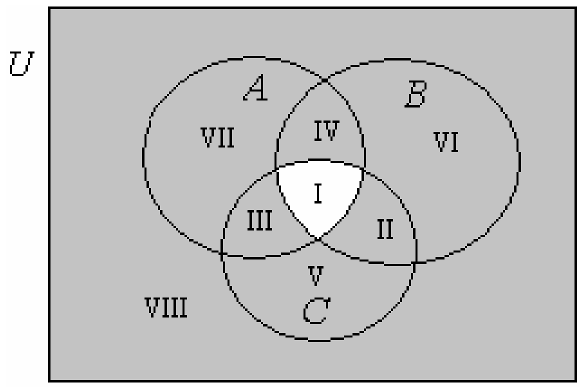


Figure 2.3.19

So, we deduce that it is the case that $(A^C \cup B^C - (C \cup (A^C \cap B^C \cap C^C))) \subseteq (A \cap B \cap C)^C$ is true.

Now, the extremely important point for this part of the discussion: **this is not a proof** that when A, B, and C are sets it the case that $(A^C \cup B^C - (C \cup (A^C \cap B^C \cap C^C))) \subseteq (A \cap B \cap C)^C$. It is merely a pictorial representation that leads one to the realisation that he needs to prove that A, B, and C are sets. Is it the case that $(A^C \cup B^C - (C \cup (A^C \cap B^C \cap C^C))) \subseteq (A \cap B \cap C)^C$. We shall not 'cover' this in this class; it is a matter for Math 224 (Set Theory (Foundation of Mathematics)). Suffice it to say, however, that one could prove this using an indirect or a direct method. To reiterate, the important lesson is that a Venn diagramme is neither a proof nor a counterexample to a claim. It is a tool that suggests a course of action (do a proof or construct a counterexample) that one would follow to successfully prove or disprove a claim.

Venn diagrammes do not have to be drawn such that there are circles within a rectangle. Given a particular arrangement of premises there are other ways to draw a Venn. We shall, though, adhere to standard drawing of Venn diagrammes and will follow the convention.

But we do not have to draw them overlapping if the premises do not support such. Consider drawing a Venn diagramme to illustrate given a well defined universe U such that A, B, and C are sets let us draw a situation where A is disjoint from C and B is also disjoint from C. The Venn diagramme is best drawn as follows:

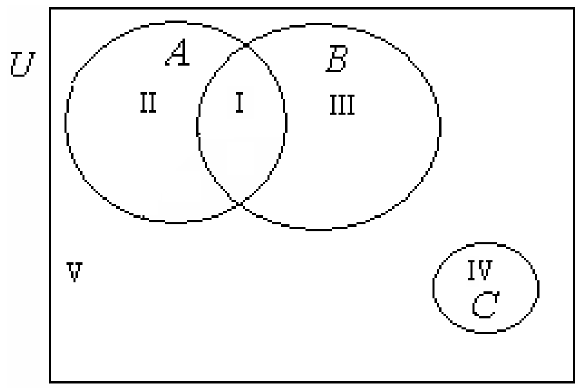


Figure 2.3.20

This is because $A \cap C = \varnothing$ and $B \cap C = \varnothing$ but $A \cap B$ may be $\varnothing$ or not so we have to allow for the 'overlap' between A and B but should not draw an overlap for $A \cup B$ and C. Further, region 'V' also must be drawn so that we allow for something to exist that is not in $A \cup B \cup C$ (they comprise much of Ubut not necessarily all of U. When sets 'fill up' the universe in such a manner ($A \cup B \cup C = U$) we say the sets **span** the universe or we say the sets are **exhaustive** of the universe.
The Venn diagramme is best drawn as follows:

Another useful Venn diagramme illustrates sets as follows: two 'overlapping' (say $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$), but a pair are 'non-overlapping' ($A \cap C = \varnothing$). The Venn diagramme is best drawn as follows:

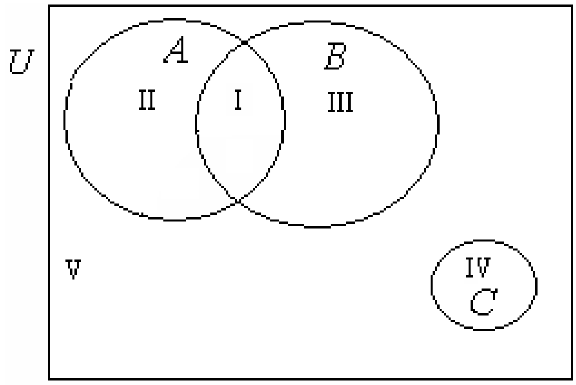


Figure 2.3.21

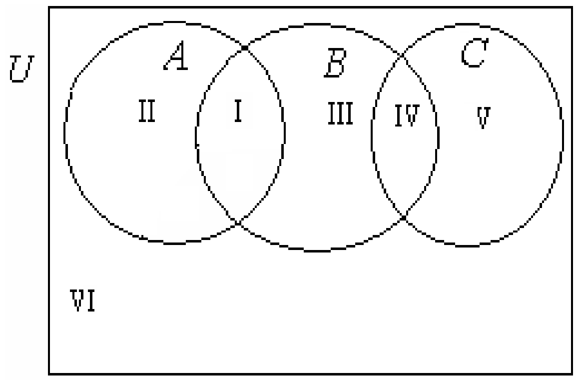


Figure 2.3.22

Another interesting use of Venn diagrammes is in solving problems in elementary descriptive statistical analysis (survey, questionnaire, population amalgams). It requires that we accept without proof that certain properties of sets are so (these are proven in Math 224):
For all the theorems assume U is a well defined universe and A, B, and C are sets.

Theorem 2.3.1. $A \subseteq \text{U}$.

Theorem 2.3.2. $A \subseteq B \wedge B \subseteq C \implies A \subseteq C$.

Theorem 2.3.3. $\varnothing \subseteq A$.

Theorem 2.3.4. $A \cap B \subseteq A \wedge A \cap B \subseteq B$.

Theorem 2.3.5. $A \subseteq A \cup B \wedge B \subseteq A \cup B$.

Theorem 2.3.6. $A \setminus B \subseteq A \wedge B \setminus A \subseteq B$.

Theorem 2.3.7. $A \cup B = B \cup A$.

Theorem 2.3.8. $A \cap B = B \cap A$.

Theorem 2.3.9. $A \subseteq A$.

Theorem 2.3.10. $A \cap A = A$, $A \cup A = A$.

It also requires we have a definition for the term 'finite' set. That is beyond the scope of the course[13]; but we can have an intuitive understanding of a finite set to be able to solve the problem.

The intuitive sense is based on the intuitive sense of the word 'infinite' (notice the indirect reasoning since the concept of infinite is not finite). We naively understand 'infinite' to mean 'goes on and on;' so, 'finite' would intuitively mean 'stops.' We believe that a finite set stops (which does match the definition of finite). That will suffice for surveys, etc.[14]

Notation 2.3.1. *Let* U *is a well defined universe and A be a set. $|A|$ means the number of elements in A. It is read as the* ***cardinality*** *of the set A.*

[13] The set A is said to be finite if and only if either A is null or there exists a bijective function, f, from A to $\mathbb{N}_p$ for some $p \in \mathbb{N}$

[14] Indeed it serves to illustrate the note that one need not really understand something in order to use it. There seem to be many more people who use computers than people who understand computers and there seem to be many more people who use statistics than people who understand statistics, alas.

Example 2.3.6. *Let* $\mathrm{U} = \mathbb{N}_{60}^{*}$ *(note that is a well defined universe) such that the sets* $A, B, C, D, E, F,$ *and* G *are defined as follows:* $A = \{x | x \text{ is prime}\}$,
$B = \{x |$ *there is an element* j *in the universe so that* $x = 2j\}$,
$C = \{x :$ *there is an element* k *in the universe so that* $x = 4k + 1\}$,
$D = \{2\}$,
$E = \{x |$ *there is an element* m *in the universe so that* $x = 6m + 1\}$,
$F = \{1\}$, *and,*
$G = \{x | x < x\}$. *It is the case that:*
$|U| = 61. |A| = 18. |B| = 31. |C| = 15. |D| = 1. |E| = 10. |F| = 1. |G| = 0.$ *These values seem to come from nowhere. Let us delineate the sets in list form (other than the universe). The two easiest are* D *and* F *since they were in list form:* $D = \{2\}$. $F = \{1\}$.
Next G. *It is the empty set since no real number is less than itself. Then:*
$A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 57, 59\}$.
$B =$
$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60\}$.
$C = \{1, 5, 9, 17, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57\}$.
$E = \{1, 7, 13, 19, 25, 31, 37, 43, 49, 55\}$. *The Venn diagramme for this example is:*

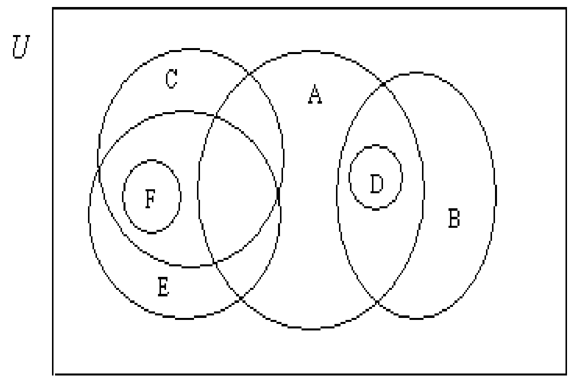


Figure 2.3.23

Notice that G *was not drawn as a circle since it is null.*

Example 2.3.7. *A survey was administered to 500 Kutztown students regarding movies they had seen. 204 saw* Thunderball*, 137 saw* On Her Majesty's Secret Service*, 180 saw* Goldfinger*, 33 saw all 3, 41 saw* Goldfinger *and* Thunderball*, 78 saw* On Her Majesty's Secret Service *and* Thunderball*, while 33 saw* On Her Majesty's Secret Service *and* Goldfinger*. Draw a Venn Diagramme illustrating the survey. Answer the following questions: How many saw* On Her Majesty's Secret Service *only? How many saw exactly* *two movies? How many did not see* Goldfinger*? How many saw neither* On Her Majesty's Secret Service *nor* Thunderball*?*

One begins by drawing a Venn and defining the sets. Let B represent the students who saw *Thunderball*, M represent the students who saw *On Her Majesty's Secret Service*, and G represent the students who saw *Goldfinger*. Note $|U| = 500$; $|B| = 204$; $|M| = 137$; $|G| = 180$; $|B \cap M \cap G| = 33$; $|G \cap B| = 41$; $|M \cap B| = 78$; and, $|M \cap G| = 33$. Start with a set that represents only *one* region- that is $B \cap M \cap G$ and it has 33 elements.

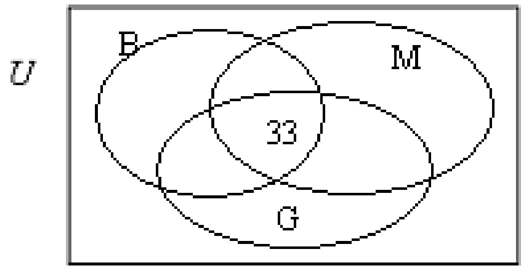


Figure 2.3.24: $B \cap M \cap G$

$B \cap M$ has 78 elements 33 of which have already been accounted for in $B \cap M \cap G$. Thus, $|B \cap M \setminus (B \cap M \cap G)| = 45$. So, we have at this point:

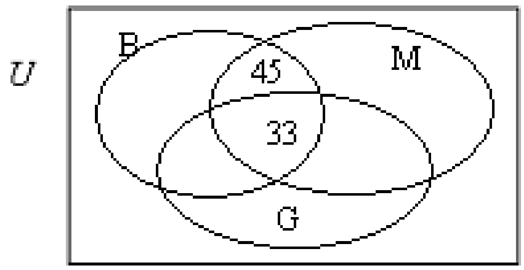


Figure 2.3.25

$M \cap G$ has 33 elements 33 of which have already been accounted for in $B \cap M \cap G$; so $|G \cap M \setminus (B \cap M \cap G)| = 0$.

Continuing, $B \cap G$ has 41 elements 33 of which have already been accounted for in $B \cap M \cap G$; so $|B \cap G \setminus (B \cap M \cap G)| = 8$.

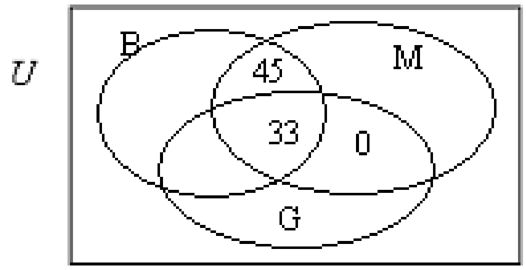


Figure 2.3.26

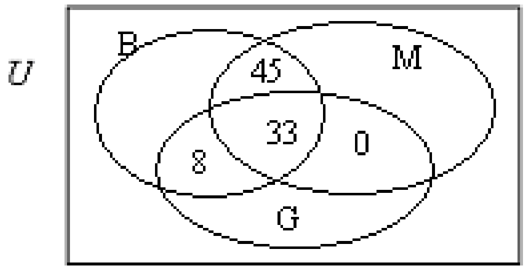


Figure 2.3.27

Recall, $|B| = 204$ and we have accounted for 8, 33, and 45 of the elements so far; hence, $|B - (M \cap G)| = 204 - (8 + 33 + 45) = 204 - 86 = 118$.

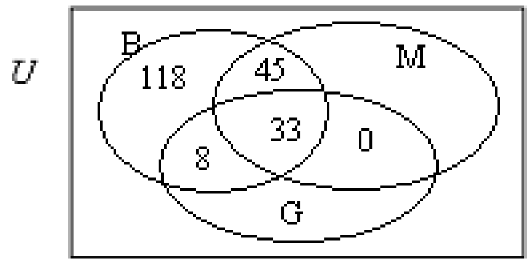


Figure 2.3.28

Also, $|M| = 137$ and we have accounted for $0, 33$ and 45 of the elements so far; hence, $|M - (B \cap G)| = 137 - (33 + 45) = 137 - 78 = 59$.
$|G| = 180$ and we have accounted for 8 and 33 of the elements so far; hence, $|G - (M \cap B)| = 180 - (8 + 33) = 180 - 41 = 139$. So, we have:

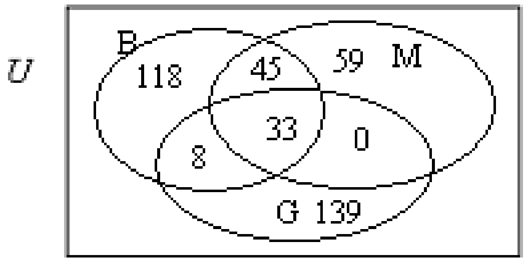


Figure 2.3.29

But, we are not done! There is still $(M \cup B \cup G)^C$ to which to account.
Recall $|U| = 500$ and $|(M \cup B \cup G)| = 118 + 45 + 59 + 8 + 33 + 0 + 139 = 402$. This implies that $|(M \cup B \cup G)^C|$ must be 98. So, we are done with the Venn diagramme when we draw:

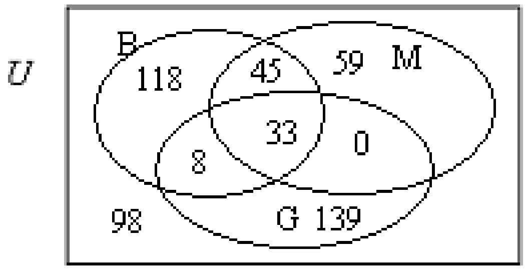


Figure 2.3.30

Now, the questions are facile to answer.
Question: how many saw *On Her Majesty's Secret Service* only? Since $|M - (B \cup G)| = 59$, the answer is 59.
Question: how many saw exactly two movies?
Exactly two movies means $((B \cap G) - M) \cup ((M \cap B) - G) \cup ((G \cap M) - B)$. So, that is $45 + 8 + 0 = 53$. The answer is 53.
Question: how many did not see *Goldfinger*? For this we need $|G^C| = 98 + 118 + 45 + 59 = 320$. So, the answer is 320.
Question: how many saw neither*On Her Majesty's Secret Service* nor *Thunderball*? That corresponds with $|(M \cup B)^C|$. As we can see that is $98 + 139 = 237$.
We could go on and on with this example; but, suffice it to say the Venn diagramme allows for a succinct reference to answer such questions.

2.3.1 Exercises

Exercise 2.3.1. Draw a Venn diagramme to illustrate the following. Assume U is a well defined universe and A, B, and C are sets (use only as many sets as is necessary per part):

A. $A \cap C = \varnothing$ and B is a set. B. A is a set and $B \cup C = U$
C. $A \cap C = \varnothing$, $A \cap B = \varnothing$ and $B \cap C = \varnothing$
D. $A \cap B \cap C = \varnothing$

Exercise 2.3.2. Shade in the corresponding region or regions associated with the following sets in a Venn diagramme. Assume U is a well defined universe and A and B are sets:

A. $A \cap B^C$ B. $(A \cap B)^C$
C. $A \cup B^C$ D. $(A \cup B)^C$
E. $A^C \cap B^C$ F. $A^C \cup B^C$
G. $A - B^C$ H. $(A - B)^C$

Exercise 2.3.3. Shade in the corresponding region or regions associated with the following sets in a Venn diagramme. Assume U is a well defined universe and A, B, and C are sets:

A. $A \cap B^C \cap C$ B. $(A \cap B \cap C)^C$
C. $A \cup (B - C)$ D. $(A \cup B) \setminus C$
E. $A \cup B - C$ F. $A \cap (B^C \cap C)$
G. $A \cup (B \cap C)$ H. $A \cap (B \cup C)$
I. $(A \cap B) \cup (A \cap C)$ J. $(A \cup B) \cap (A \cup C)$
K. $A \cup (B \cap C^C)$ L. $A \cup (B \cap C)^C$

Exercise 2.3.4. Suppose $U = \mathbb{N}_{10}, A = \{x|x = 3p \text{ for some } p \in U\}, B = \{1, 4, 6, 9\}$, and $C = \{x\colon \text{there is not a } q \in U \text{ where } x = 4q\}$. Draw a Venn diagramme and place the elements in their corresponding sets.

Exercise 2.3.5. Suppose $U = \mathbb{N}_{10}, A = \{x|x = 2p \text{ for some } p \in U\}, B = \{1, 4, 6, 9\}$, and $C = \{x|x = 2q - 1 \text{ for some } q \in U\}$. Draw a Venn diagramme and place the elements in their corresponding sets.

Exercise 2.3.6. Suppose the CDC had records of 189 patients from a cardiac care unit at Elmhurst Medical Centre and the records reflected the fact that 69 patients had cancer, 72 had pneumonia, 89 had emphysema, 22 had cancer and emphysema, 14 had pneumonia and emphysema, 23 had cancer and pneumonia, and 8 had all three of the aforementioned diseases. Draw a Venn Diagramme illustrating the records.
Answer the following questions:
How many patients had only cancer?
How many had emphysema, but not cancer nor pneumonia?
How many had pneumonia or emphysema but not cancer?

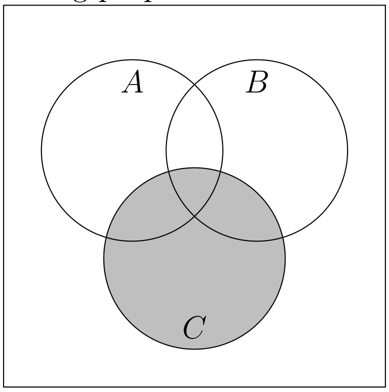
How many had cancer or emphysema but not pneumonia?
How many had pneumonia and emphysema but not cancer?
How many had none of the three diseases?

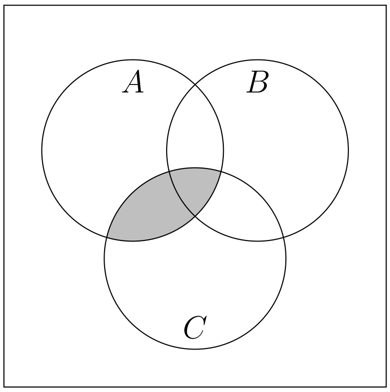
Exercise 2.3.7. A survey was done of finitely many children (obviously, there are not infinitely many children) regarding programmes they like. 44 kids liked *Barney*, 37 liked *Sesame Street*, 40 like *Mr. Rogers*, 18 liked all 3, 16 did not like any of the three, 23 likes *Sesame Street* and *Mr. Rogers*, 28 liked*Barney* and *Mr. Rogers*, while 21 liked *Barney* and *Sesame Street*. Draw a Venn Diagramme illustrating the survey.
Answer the following questions:
How many children were surveyed?
How many liked *Barney* only?
How many liked exactly two programmes?
How many did not like *Sesame Street*?
How many liked neither *Barney* nor *Mr. Rogers*?

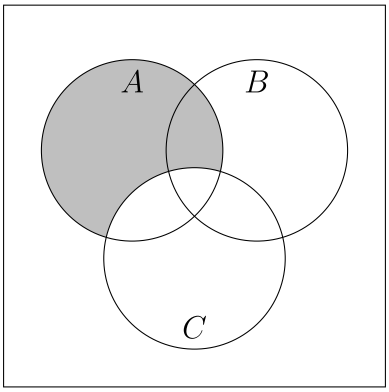
Exercise 2.3.8. A survey was done of 81 Kutztown students whose declared major was housed in the College of Liberal Arts and of Sciences. The survey statistics are as follows. 44 students chose their major based on potential earnings (i.e.: they are 'going for the Benjamins'), 35 students majored in an area that did not interest them but had potential earnings, 23 students were in a major that interested them. Draw a Venn Diagramme illustrating the survey.
Answer the following questions:
How many students were majoring in an area that din not interest them and interested them?
How many students were majoring in an area that din not interest them or interested them?
How many students were majoring in an area that interested them and did not choose it based on potential earnings?
How many students were majoring in an area that did not interest them and did not choose it based on potential earnings (i.e.: seem clueless)?

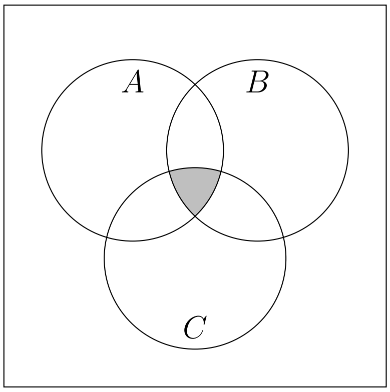
Exercise 2.3.9. Consider the problem of drawing a generalised Venn diagramme for four sets. Why is figure 2.3.11 incorrect whilst figure 2.3.12 which was based on the figure drawn by Carol Guadagnicorrect? Can you opine as to what set theoretic construct (for example: $A \cap B \cap C \cap D$ exists for one and not the other for four sets but for 3 sets $A \cap B \cap C$ exists and $A \cap B$ exists for two sets) held for the generalised one, two, and three set Venn diagrammes that does not hold for figure 2.3.11 but does for figure 2.3.12?

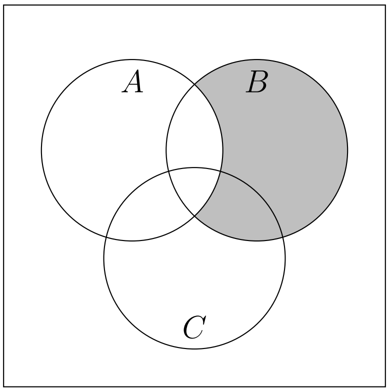
Exercise 2.3.10. Consider the following Venn Diagrammes shaded. Identify the shaded area using proper set theoretic notation:

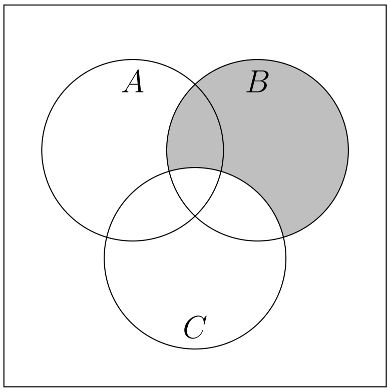
1. 

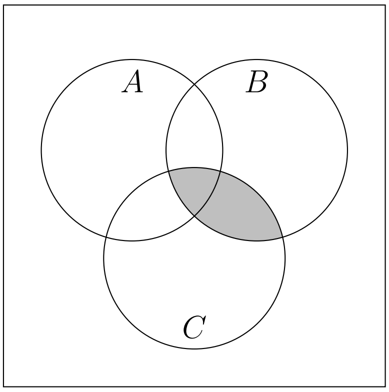
2. 

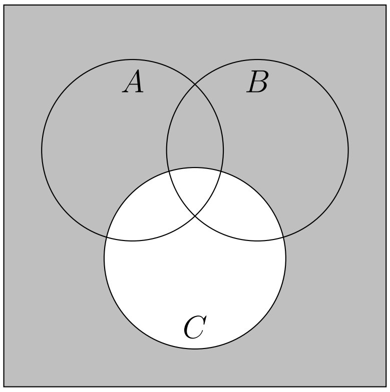
3. 

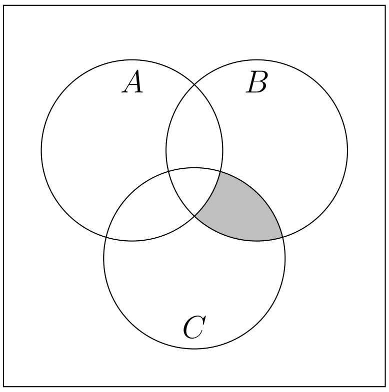
4. 

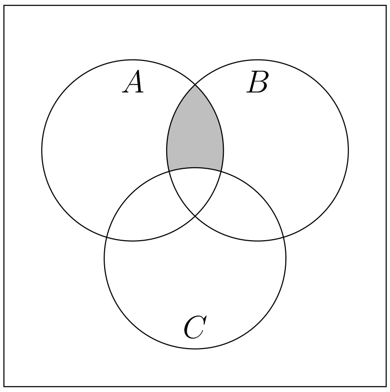
5. 

6. 

7. 

8. 

9. 

10. 

2.4 Syllogistic Logic and Quantification

Not every argument is of the form or type (i.e.: given the premises $A \implies \neg B$, $\neg A \vee C$, C, $D \implies B$, it is the case that D follows as a conclusion) we studied in chapter one. Some argument types are claims of quantification. Now this seems especially apropos for a discussion in a mathematics class. Many mathematical arguments hinge on an understanding of sets and involve questions of what is in a set or not, how many elements are in the set, etc. Thus, as this section illustrates we use terms such as 'not every,' 'some,' 'many,' 'none,', 'all,' for every,' 'few,' etc. in our discussions, arguments, and applications. Each of the aforementioned words are examples of quantifiers and are part of the area of logic known as syllogistic logic.

Arguments of the type in chapter one were examples of *propositional or symbolic arguments*; that is, arguments that use the conditional, biconditional, conjunction, and disjunction. However, there are other types of arguments known as *syllogistic arguments* which differ from symbolic argument in that syllogistic arguments use quantifying words such as all, none, or some.

Note the difference in the following two arguments:

Example 2.4.1. *All frogs have warts.*
All creatures that have warts are blue.
Therefore, all frogs are blue.

If Malcolm is talking, then people listen.
Malcolm is not talking.
Therefore, no one is listening.

The former is a syllogistic argument (or syllogisms); the latter is a symbolic argument.[15] To determine evidence to suggest that a syllogistic argument is valid or invalid one may use a diagramming technique known as Euler's diagrammes (named after the mathematician Leonhard Euler). Euler's Diagrams are really an application of set theory to logic for they are really just Venn diagrammes set to logic problems.[16]

Let us consider the example from above. Since all frogs have warts, the set of frogs is a subset of the set of creatures that have warts. Since all creatures that have warts are blue, the set of creatures that have warts is a subset of blue things. Ergo, the of frogs is a subset of the set of creatures that have warts. Since the set of frogs is a subset of blue

[15] There is a translation from syllogistic argumentation to symbolic argumentation that we will discuss later.

[16] For historical purposes, the Euler diagrammes or circles came first; Venn adapted them to the (then) newly emerging study in a rigorous sense of set theory. We shall adopt the more rigorous requirements of the Venn diagramme convention and insist on specification (presumed) of a well defined universe (which in Syllogism terms means there exists a well defined domain of discourse).

things. Hence, all frogs have warts. It makes sense.[17]

We need to assume there is a well defined universe, D, defined that in this context we shall reference as the **domain of discourse**. We need to also the sets we shall use (i.e.: the symbols we shall use). Let F denote the set of frogs, W denote the set of creatures that have warts, and B be the set of blue things. The diagramme (Euler diagramme solution) illustrates this:

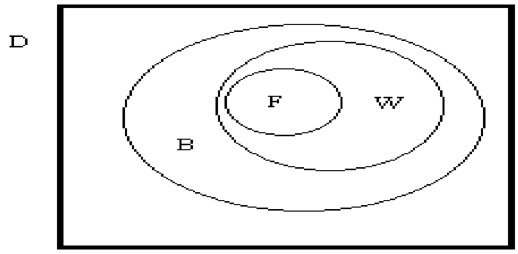


Figure 2.4.1

Note that we are simply showing a Venn diagramme such that the universe is D, $F \subseteq W$, and $W \subseteq B$. Therefore, $F \subseteq B$ (Reasoning by Transitivity (the Hypothetical Syllogism)). It must be the case since there is not another way to illustrate this in another manner. Hence, the Euler diagramme illustrates that the syllogistic argument is valid.[18]

In terms of quantification, the symbol to represent each element of the set F is an element of the set W is $\forall x \in F, x \in W$. The translation of this symbol ($\forall$) is 'for all,' 'for each,' 'every,' etc.

Notation 2.4.1. *The* **universal** *symbol is* $\forall$. *It is read as 'for all,' 'for each,' 'every,' etc.*

So, the following statements are symbolized as:
All frogs have warts.
All creatures that have warts are blue.
Therefore, all frogs are blue.
Let F be frogs; W be creatures that have warts; and B be creatures that are blue.

All frogs have warts.	$\forall x \in F, x \in W$
All creatures that have warts are blue.	$\forall y \in W, y \in B$
$\Longrightarrow$, All frogs are blue.	$\forall z \in F, z \in B$

[17]Again the contention that I bring to you - logic and set theory are codifications of common sense.

[18]But, remember that this *does not prove* the argument is true - it simply demonstrates in a meaningful and illustrative manner that *strong* evidence supports the veracity of the claim that the syllogism is valid.

The lower case letters represent the elements in each of the sets (notice I changed them line - by - line so one can realise that the element symbol is a 'dummy variable' to simply represent a generalised element of a set. The set symbols do not change since the set symbols represent specific sets.

The traditional way to represent therefore in syllogistic argument for is the "$\because$" rather than the "$\longrightarrow$" but either is (of course) fine.
Consider the following example:

Example 2.4.2. *No Kutztown students are Moravian students.*
Some Moravian students work. Therefore, no Kutztown students work.

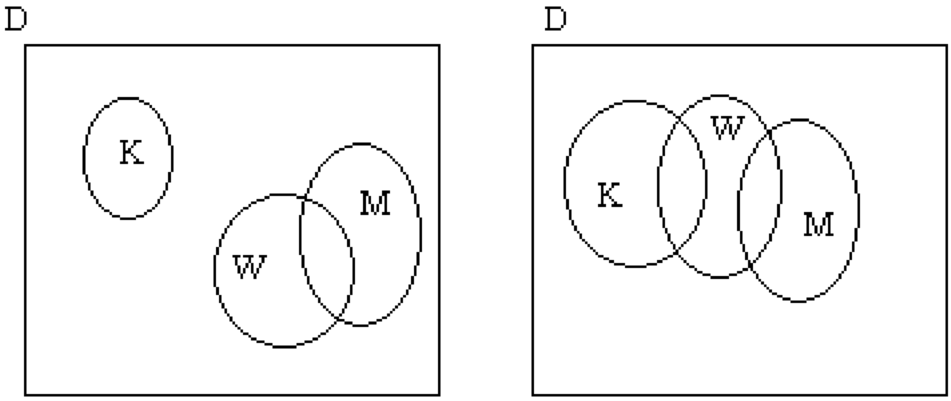
Before defining some new symbols, let us not this is a syllogistic claim since it is in terms of sets, elements, and quantification. Let us represent the domain of discourse as D, the set of Kutztown students as K, the set of Moravian students as M, and the set of people who work as W.

The first sentence means that $K \cap M = \varnothing$ because if there was an element of K that was in the set M, then there would be a Kutztown student who was a Moravian student. Further, if there was an element of M that was in K, then there would be a Moravian student who was a Kutztown students which implies that this person is a Kutztown student who is a Moravian student (which cannot be).

The second sentence is a tad more challenging. We need a new symbol to represent the statement. In terms of quantification, the symbol to represent, 'some,' $\exists$. The translation of this symbol, $\exists$ is 'some,' 'there exists,' 'there is at least one,' etc. So, the second sentence quantified symbolically is $\exists m \in M \ni m \in W$. [19]

The third sentence symbolized is: $K \cap W = \varnothing$.
Now, do you think the claim is true or false? Let us reflect on the claim.
No Kutztown students are Moravian students. We represent the set of Kutztown students as K and the set of Moravian students as M. $K \cap M = \varnothing$. Thus, the sets are disjoint. Some Moravian students work. Notice that $W \cap M \neq \varnothing$ (since $\exists x \in M \ni x \in W$). So, S and W intersect. The two statements are the suppositions (the premises). We are now faced with drawing any possible Euler diagramme that illustrates these two suppositions. Note that for the argument to be valid, it **must** be the case that $K \cap W = \varnothing$. Let us consider the following two diagrammes.
Figure 2.4.2 & Figure 2.4.3 Both*could be* true for the premises. There is nothing in the premises that forces either to be the only possibility. However, note the second diagramme represents an instance such that the premises hold and is more generalised. The first represents a diagramme where one is assuming the conclusion (a logical fallacy from chapter one, recall). Therefore, there is evidence to conclude that the syllogism is invalid. The justification for that is the second Euler diagramme (thus, the first one need not be drawn).

[19]The $\ni$ means such that - it is by me quite often.



Notation 2.4.2. *The existential symbol is* $\exists$*. It is read as 'some,' 'there exists,' 'there is at least one,' etc.*

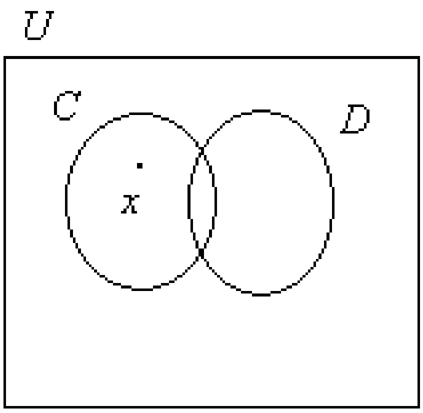
Example 2.4.3. *Consider the following statements:*
Each student is attentive. Let S *represent the set of students and* A *represent the set of attentive people. So, translated into symbols we have* $\forall s \in S, s \in A$*.*
Not every student is attentive. This is the negation so we write $\neg(\forall s \in S, s \in A)$*.*[20]
It is not the case that all students are attentive. This is the negation of the first statement but is the same as the second; so, we write $\neg(\forall s \in S, s \in A)$*.*
It is not the case that some students are attentive. This is the negation of an existential statement; so, we write $\neg(\exists s \in S, s \in A)$*.*

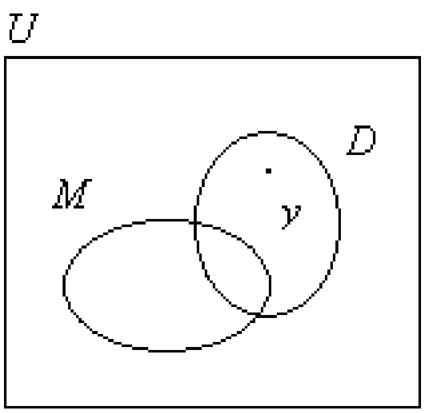
When a universal is negated it becomes an existential. When an existential is negated it becomes a universal. Consider:
$\neg(\forall s \in S, s \in A) \equiv \exists s \in S, s \ni A$ and
$\neg(\exists s \in S, s \in A) \equiv \forall s \in S, s \ni A$. When properly negated, the second part (after the comma) of the statement is negated. We can generalise this concept as follows. Let $\Omega(x)$ be some statement about x. $\neg(\forall x, \Omega(x))$ is $\exists x, \neg(\Omega(x))$ and
$\neg(\exists x, \Omega(x))$ is $\forall x, \neg(\Omega(x))$.
It ties into the conditional quite nicely, notice, since the logical negation of $p \implies q$ is $p \wedge \neq q$.

One might believe the next example is obvious. Well, let's consider it.

Example 2.4.4. *Some cats are not dogs.*
Some dogs are not mice.
Therefore, some cats are not mice.

[20]The parentheses matter; more on that later.





This is a syllogistic argument with premises, "Some cats are not dogs," and "Some dogs are not mice." The conclusion is "Some cats are not mice."

Let us define our sets. Let U be the domain of discourse, C be the set of cats, D be the set of dogs, and M be the set of mice. Note we are not going to use D as the domain of discourse since it might cause confusion with the set of dogs. So, we will use our old friend, U, for that.

The first statement means $C \cap D^C \neq \varnothing$ because $\exists x \in C \ni x \in D^C$ which is logically equivalent to $\exists x \in C \ni x \ni D$.

The second statement means $D \cap M^C \neq \varnothing$ because $\exists y \in D \ni y \in M^C$ which is logically equivalent to $\exists y \in D \ni y \ni M$.
The two premises in Euler diagramme form are:

Figure 2.4.4 & Figure 2.4.5

Must the conclusion follow from the premises? Do you think the claim is true or false? Consider that **if** the conclusion must follow the Euler diagramme would be:

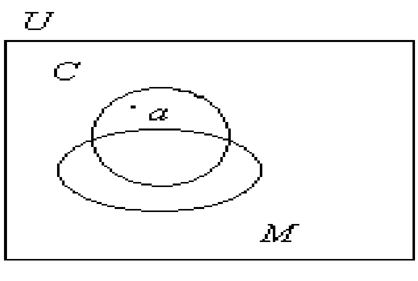


Figure 2.4.6

Does this have to be the case? Decide.

Notice we can draw the Euler diagramme to fulfill the premises and deny the conclusion:

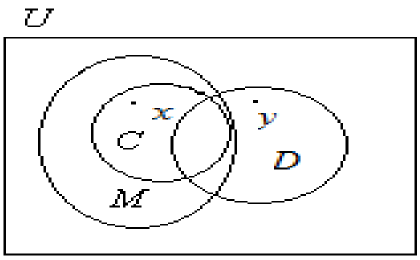


Figure 2.4.7

Many people would decide on the first Euler based on empirical evidence. They would be *wrong* since we are trying to judge the veracity of the syllogism, not the state of nature. Indeed, they added conditions to the syllogism that are not stated; hence, erred. So, one can reasonably draw an Euler diagramme such that the premises hold, but the conclusion does not. Thus, the syllogistic argument is invalid.

The difficulty for many with syllogisms such as the previous example is that experience dictates in the 'real world' the claim seems true (indeed can be strengthened). However, we are not interested in the existential question of life and mice, cats, and dogs! We are interested in the argument form because it should be clear from the Euler diagramme that one can change the wording of the example to:
'Some students are not men. Some men are not tall. Therefore, some students are not tall,' and the argument is still invalid.
'Some rocks are not mammals. Some mammals are not fish. Therefore, some rocks are not fish,' and the argument is still invalid! Suffice it to say, there are some important principles to consider in this section.

Assume D is a well defined domain of discourse, A is a set and B is a set. Let it be so that $\forall x \in A, x \in B$. Does it mean $\exists w \in A \ni w \in B$? The answer is , 'no,' since a counterexample exists to the claim.

Claim 2.4.1. $\forall x \in A, x \in B \implies \exists w \in A \ni w \in B$.

Counterexample:
Let $U = \mathbb{N}$. Let $A = \{x | x < x\}$. Let $B = \{1, 2, 3\}$. It is true (by default) that all elements of A are in B since there are no elements in A!
E. E. F.

Indeed note theorem 2.3.3. So, we say $\forall x \in A, x \in B$ does NOT necessarily imply$\exists w \in A \ni w \in B$.

Thus, in mathematics note the following two claims are approached differently:

Claim 2.4.2. Let U be a well defined universe and A be a set. It is the case that $A \subseteq U$.

In this instance we are trying to prove $\forall \ x \in A, x \in U$. We must consider the cases where A is null and the case where A is not null.

Claim 2.4.3. Let U be a well defined universe and A is a non-empty set whilst B is a set. It is the case that $A \subseteq A \cup B$.

In this instance we are trying to prove $\forall \ x \in A, x \in A \cup B$. Moreover we have as a premise $\exists x \in A$ so we do not have to consider a case where A is null. Notice further $\exists x \in A \implies \exists x \in A \cup B$ (why?). However we do NOT know $\exists p \in B$ (why?).

Finally suppose we have the following scenario. Let U be a well defined universe whilst A and B are sets. Let us assume $\exists \ x \in A$ such that $x \in B$. Does that mean also that $\exists \ y \in A$ such that $y \ni B$? The answer is, "no." since a positive existential does not necessarily imply a negative existential.

Claim 2.4.4. $\exists x \in A, x \in B \implies \exists w \in A \ni w \ni B$.

Counterexample:
Let $U = \mathbb{N}$. Let $A = \{1\}$. Let $B = \{1, 2, 3\}$. It is true that there is an element of A that is in B since $1 \in A \ \wedge \ 1 \in B$. But there are no elements of A that are at the same time not elements of B!
E. E. F.

So, it is the case that $\exists x \in A, x \in B$ does NOT necessarily imply$\exists w \in A \ni w \ni B$.

2.4.1 Negation of Quantifiers

Another principle that is of import is negation of quantifiers. Let U be a well defined universe and A be a set. What does it mean to say $\neg(\forall \ x \in A$, some property about x holds)? Think about it. There has to be a point in A where that property does not hold.

Therefore, $\neg(\forall \ x \in A$, some property about x holds) is logically equivalent to $\exists \ p \in A$, where that property about x does not hold).

Example 2.4.5. *Let $U = \mathbb{N}$ and $A = \{2, 3, 4\}$. Suppose someone claims that $\forall \ x \in A$, x is a prime natural number. The claim is false because $\exists \ p \in A \ni p$ is not prime. Namely, 4 since 4 has factors of $1, 2, 4$ (three factors and recall a prime has exactly two factors itself and one under the definition of a prime natural number).*

So, in symbolic logic the standard general way to write such is:

Definition 2.4.1. Let $\Phi(x)$ be a statement about x. $\neg(\forall \ x, \Phi(x)) \equiv \exists \ x, \neg(\Phi(x))$.

How about negating the existential quantifier? Let U be a well defined universe and A be a set. What does it mean to say $\neg(\exists \ x \in A$, some property about x holds)? Think about this concept. It has to be the case that there a no points in A where that property holds.

Therefore, $\neg(\exists \ x \in A$, some property about x holds) is logically equivalent to $\forall \ p \in A$, that property about x does not hold).

Example 2.4.6. *Let $U = \mathbb{N}$ and $C = \{2, 3, 5\}$. Suppose someone claims that $\exists \ x \in A$, x is a composite natural number. The claim is false because $\forall \ p \in A \ni p$ is prime. Namely,* 2 *since* 2 *has factors of* $1, 2$*;* 3 *since* 1 *has factors of* $1, 3$*; and,* 5 *since* 5 *has factors of* $1, 5$ *(two factors for each of* $2, 3,$ *&* 5 *and recall a composite number has more than two factors).*

So, in symbolic logic the standard general way to write such is:

Definition 2.4.2. Let $\Phi(x)$ be a statement about x. $\neg(\exists \ x, \Phi(x)) \equiv \forall \ x, \neg(\Phi(x))$.

From these two principles and the law of double negation, it should be clear to the reader that:
$(\neg(\forall x \in A, x \in B)) \equiv (\exists x \in A, x \ni B)$ and
$(\neg(\exists x \in A, x \in B)) \equiv (\forall x \in A, x \ni B)$ being true also force:
$(\neg(\forall x \in A, x \ni B)) \equiv (\exists x \in A, x \in B)$ and
$(\neg(\exists x \in A, x \ni B)) \equiv (\forall x \in A, x \in B)$ to be true also.

So, from a vernacular standpoint one must be quite attentive to what is being said so that we can understand a statement. For example, what is the opposite of the statement, "All Kutztowners are citizens of Berks County?" If you were to say, "no Kutztowner is a citizen of Berks County," you would be wrong. The correct statement would be, "not all Kutztowners are citizens of Berks County;" which is also stated as, "some Kutztowners are not citizens of Berks County." This is because some can still be citizens of Berks, but *at least one* is not.

Likewise, the opposite of "no people wear hearing aids" is "some people wear hearing aids." It is incorrect to say, "all people wear hearing aids." That kind of extremism, it should be noted, is incorrect, and many times, creates many real problems. For example, to negate the statement, "all of us are poor," does not mean all of us will be rich! It simply means some of us will not be poor any more.

Also, the four basic quantification statements that we have studied in this section lead to some interesting relationships (when viewed in functional form). Let $\Phi(x)$ be a statement where x is some element of the domain of discourse.

Figure 2.4.8

Note that $\forall x, \Phi(x)$; $\forall x, \neg\Phi(x)$; $\exists x, \Phi(x)$; and, $\exists x, \neg\Phi(x)$ have interesting properties.

$\forall x, \Phi(x)$ and $\forall x, \neg\Phi(x)$ are **contraries**; that is to say they *might* both be false; one be true the other false; or, both be true. A person employing such is a *contrarian.*

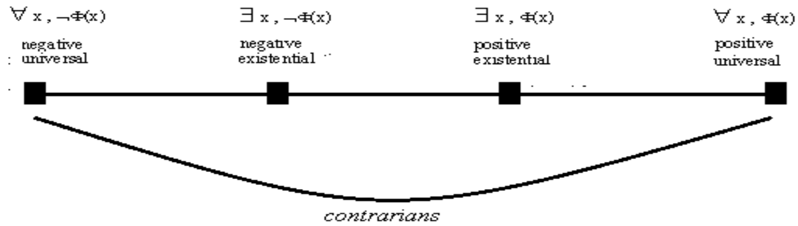


Figure 2.4.9

$\exists x, \Phi(x)$ *and* $\exists x, \neg\Phi(x)$ *are also* ***contraries****. Again, that is to say they* might *both be false; one be true the other false; or, both be true. A person employing such is also a contrarian.*

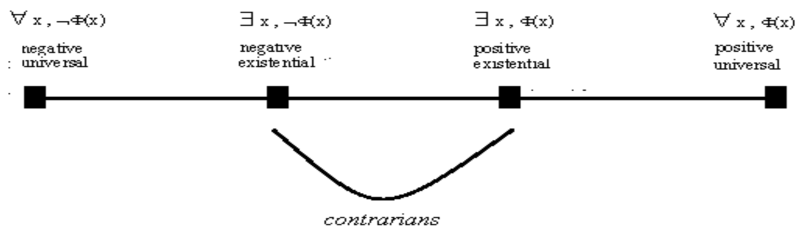


Figure 2.4.10

$\exists x, \Phi(x)$ *and* $\forall x, \neg\Phi(x)$ *are* ***contradictories****; that is to say one* must *false and the other true.*

$\forall x, \Phi(x)$ *and* $\exists x, \neg\Phi(x)$ *are also****contradictories*** *– again one* must *false and the other true.*

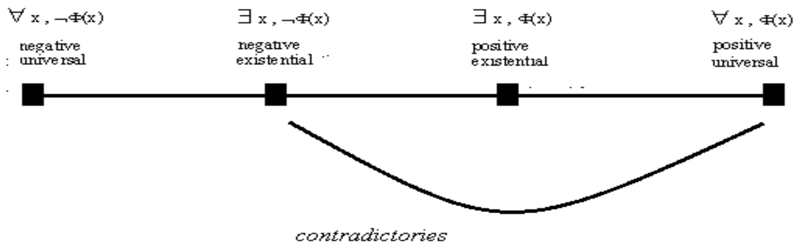


Figure 2.4.11

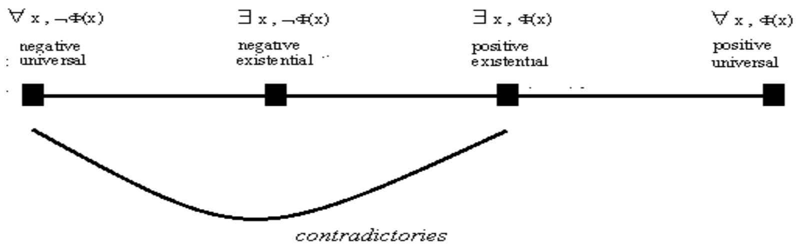


Figure 2.4.12

So, you can see we have extremes: $\forall x, \Phi(x)$ *and* $\forall x, \neg\Phi(x)$ *(the contraries) and a middle (the contradictories to the extremes) which may be viewed as a "continuum."*

In mathematics, contrarians are not of use; but, contradictories are.

2.4.2 Exercises

Exercise 2.4.1. Symbolise the following statements using quantifiers or propositional functions:

A. *Snakes are reptiles.*
B. *Snakes are all not poisonous.*
C. *It is not the case that all snakes are long.*
D. *Not even one visitor stayed for supper.*
E. *All vegetables and fruits are nutritious.*
F. *All vegetables are nutritious and all fruits are nutritious.*
G. *Firemen are both indispensable and under-appreciated.*
H. *Some horses are gentle.*
I. *All badgers are not gentle.*
J. *All opossums are gentle.*
K. *Some people are not gentle.*
L. *There is at least one New Yorker who is rude or arrogant.*
M. *Not all New Yorker are rude and arrogant.*

Exercise 2.4.2. Write the negation in vernacular English of the following statements (suggestion: translate into symbols, then negate, then translate that which you opine is the correct negation – it could be set-theoretic, classical logic, or quantification):

A. *Jermaine is studious or Helen is an economist.*
B. *All English majors are poets.*
C. *Some engineers are mathematicians.*
D. *Jeremiah is a bullfrog.*
E. *Howitson can fly and Marissa can sing.*
F. *Paul is fatuous and Edward is not an architect.*
G. *Some English majors are not poets.*
H. *All engineers are not mathematicians.*
I. *No snake is by its nature evil.*

Exercise 2.4.3. Construct an example to show $\forall x, \Phi(x)$ and $\forall x, \neg\Phi(x)$ are contraries where both are true.

Exercise 2.4.4. Construct an example to show $\exists x, \Phi(x)$ and $\exists x, \neg\Phi(x)$ are contraries where both are true.

Exercise 2.4.5. Construct an example to show $\forall x, \Phi(x)$ and $\forall x, \neg\Phi(x)$ are contraries where both are false.

Exercise 2.4.6. Construct an example to show $\exists x, \Phi(x)$ and $\exists x, \neg\Phi(x)$ are contraries where both are false.

Exercise 2.4.7. Determine if each of the statements below is true or false. Justify why you opine it is true; if you opine it is false please show an example of when it is false.

A. $\exists x \in \mathbb{N} \ni (x-1) \in \mathbb{N}$.
B. $\forall x \in \mathbb{N}$ *it is the case that* $(x-1) \in \mathbb{N}$.
C. $\forall x \in \mathbb{N}$ *it is the case that* $(x+5) \in \mathbb{N}$.
D. $\exists x \in \mathbb{N} \ni (x-1) \ni \mathbb{N}$.
E. $\forall x \in \mathbb{N}$ *it is the case that* $(x-1) \ni \mathbb{N}$.
F. $\exists x \in \mathbb{N}$ *it is the case that* $(x+5) \ni \mathbb{N}$.
G. $\forall x \in \mathbb{N}$ *it is the case that* $(x+5) \ni \mathbb{N}$.
H. $\exists x \in \mathbb{N}^*$ *such that* $x \ni \mathbb{N}$.
I. $\exists x \in \mathbb{N}$ *such that* $x \ni \mathbb{N}^*$.

Exercise 2.4.8. Consider the argument A and argument B:

A. *If Allen shopping, then his brother Bob is driving.*
If Bob is driving, then his sister Mary is imbibing.
Therefore, if Allen goes shopping, then his sister Mary is imbibing.

B. *All brothers of shopping people drive.*
All sisters of driving people imbibe.
Therefore, All sisters of shopping people imbibe.

What type of argument is A and what type of argument is B? For each, to show the argument valid what would one do, construct a truth table to establish the argument is valid or invalid or draw an Euler Diagramme to establish the argument is valid or invalid.

Exercise 2.4.9. First, for each statement define all symbols and translate the argument into symbolic form. Then second, use Euler diagrammes to illustrate whether each argument is valid or not.

A. *Everybody plays the fool.*
All those who play the fool are hurt.
Therefore, everybody is hurt.

B. *Washington is north of California.*
Oregon is north of California.
Therefore, Washington is north of California.

C. *Some math majors study Spanish.*
All people who study history also study Spanish.
Therefore, some math majors study history.

D. *No Arkansans go to Kutztown.*
All Arkansans are Southerners.
Therefore, no Southerners go to Kutztown.

E. *Some Floridians are beach lovers.*
All Floridians are Americans.
Therefore, some Americans are beach lovers.

F. *All students are intelligent.*
All dolphins are intelligent.
Therefore, some students are dolphins.

Exercise 2.4.10. Use either Euler diagrammes or truth tables to illustrate whether the following arguments are valid or invalid:

A. *If Kokayi is sick, then Rachel gives him aspirin.*
Rachel gives Kokayi aspirin.
Therefore, Kokayi is sick.

B. *All Irish are Europeans.*
All Ugandans are Africans.
Some Africans are also Europeans.
Therefore, some Ugandans are Irish.

C. *All Irish are Europeans.*
All Ugandans are Africans.
Some Ugandans are Irish.
Therefore, some Africans are Europeans.

D. *Judith is young or Edith is fair.*
Judith is not young.
If Valeska is speaking Urdi, then Edith is not fair.
Therefore, Valeska is not speaking Urdi.

E. *Beulah is a good cook.*
All good cooks are happy.
Some happy people are rotund.
Therefore, Beulah is rotund.

F. *If Constance goes shopping, then Hildegard is singing.*
If Grace is not cleaning, then Hildegard is not singing.
Therefore, if Constance goes shopping, then Grace is cleaning.

2.5 More Syllogistic Logic and Two-place Quantification

Amended - we shall not discuss this in Math 017.

2.6 Logic and Deduction

Logic as a free-standing subject is interesting. However, one does not oft find it discussed in detail in a high school level or below mathematics text. So, one may reasonably question why it is part of the canon in college. Why do we bother to study logic? Why do mathematicians opine it to be so important a subject that it warrants our attention?

Logic is an instrument or organ for appraising the correctness of an argument. The study of logic is the study of methods and principles used to distinguish correct reasoning from incorrect reasoning. It also helps one to construct arguments which are correct rather than incorrect and avoid the pitfalls of rhetoric noted in chapter one. Charles Pierce, Alfred North Whitehead, George Boole, Bertrand Russell, etc. gave formal structure to logic as we previously noted.

Corresponding to every possible inference is an argument and it is with those that we are concerned. We are concerned that you, the student, be able to construct valid arguments to justify your conclusions. Indeed, it is important that the conclusions be derived in such a way as to be apparent and transparent rather than opaque. It is not the case that one wishes to be awarded credit for a problem attempted on a test that was actually incorrect, but the instructor mistakenly didn't notice. It is not the case that one wishes to gain entry into employment through less than honourable means. It is not the case that one wishes for hedonistic riches without gaining them through credit and hard work. Likewise, the mathematician must demand that his argument be correct and open to inspection. The mathematician desires arguments that are declarative rather than interspersed with interrogatives, pejoratives, etc. So, logic is a system that the mathematician values *for its clarity, and reasoned path to conclusions that are valid rather than just possible. That is the great difference between deductive and inductive reasoning.*

All arguments propose the claim that the premises provide evidence for the truth of the conclusions. Inductive arguments[21] *provide evidence to suggest the conclusion is true. The evidence presented is oft anecdotal and no matter how numerous do not provide positive evidence of the veracity of a claim unless the claim is quantifiable as a finite proposal. However, only deductive arguments provide the evidence such that provided the premises are so, then the conclusion absolutely follows.*
Consider the following example:

Example 2.6.1. [22] *Let* $\mathrm{U} = \mathbb{N}^*$. *Let us consider* $f(n) = n^2 + n + 5$. .
Note when $n = 0$, $f(n) = f(0)$ *which is 5. It is a prime number.*
Note when $n = 1$, $f(n) = f(1)$ *which is 7. It is a prime number.*

[21] Inductive arguments in the general sense rather than the rigorous method of proof called mathematical induction which we will see in chapter three is a valid method of proof.

[22] From the out-of-print text, Volker & Wargo. Fundamentals of Finite Mathematics, (Scranton, PA: Intext, 1972), page 2.

Note when $n = 2$, $f(n) = f(2)$ *which is 11. It is a prime number.*
Note when $n = 3$, $f(n) = f(3)$ *which is 17. It is a prime number.*

Some people would be content to look at those examples and say, " Yes, would you look at that! The function always yields a prime!" Then they would move onto other more 'pressing' concerns and abandon the exercise.
However, let us observe that when $n = 4$, $f(n) = f(4)$ *and it is 25. It is a not prime number since* 25 *has factors* $1, 5,$ *and* 25.

The important point is not that the 'educated guess' broke down when $n = 4$ *(it could have just as easily continued till* $n = 2,678,352,431$*); it is that an 'educated guess' is better than an uneducated guess (a 'stab in the dark') but a guess in any form does not make a claim true. Guessing in all its forms is a very crucial part of the scientific community's modus operandi. We need to hypothesise, conjecture, etc. However,* mathematicians *follow that with proof.*

So, logic assists us in this endeavour and makes us demand accuracy. Now, let us consider some interesting logic exercises proposed by the Reverend Charles Dodgson of whom many of you are familiar but as referenced by his pen-name, Lewis Carroll. He was a mathematical lecturer at Christ Church and he supposedly created these little 'ditties' to train his children in logic then collected them and published them as the text Symbolic Logic and the Game of Logic. *Of them he was reported to have said, "It will give you clearness of thought – the ability to* see your way *through a puzzle. Try it. That is all I ask of you."*

The game involves the quest for a 'best' conclusion from a set of conclusions to a set of given premises (best being defined as those conclusions which are derived from all of the premises).

Example 2.6.2. [23] *Consider the following:*
No ducks waltz.
No officers decline to waltz.
All my poultry are ducks.
Let us denote the domain of discourse as U. Let the set of ducks be D. Let the set of things that waltz be W. Let the set of officers be O. Let the set of my poultry be P.
We could solve this using either symbolic logic or by Euler diagrammes.
Let us consider that the quantifiers all, some, there exists, none, etc. can be translated into propositional statements. For example note the first statement, "no ducks waltz," means $D \cap W = \varnothing$ *which, in turn, means, "if it is a duck, then it does not waltz" which we can symbolise (liberally) as* $D \longrightarrow \neg W$.
Now let us note the second statement, "no officers decline to waltz," means $O \subseteq W$ *which, in turn, means, "if it is an officer, then it waltzes," which we can symbolise as* $O \longrightarrow W$.

[23] The examples of the Dodgson syllogisms were taken from the out-of-print text, Rotando. Finite Mathematics.

Finally, let us note that the statement, "all my poultry are ducks," means $P \subseteq D$ which, in turn, means, "if it is one of my poultry, then it is a duck," which we can symbolise as $P \longrightarrow D$.
Now let us look at the premises in symbolic form and we see:
$D \longrightarrow \neg W$ *and* $O \longrightarrow W$ *and* $P \longrightarrow D$. *The second sentence is logically equivalent to* $\neg W \longrightarrow \neg O$. *So we have:*
$D \longrightarrow \neg W$ *and* $\neg W \longrightarrow \neg O$ *and* $P \longrightarrow D$.
By the law of commutativity and associativity of and we reorder it as:
$P \longrightarrow D$ *and* $D \longrightarrow \neg W$ *and* $\neg W \longrightarrow \neg O$
So, by the Hypothetical Syllogism applied twice we conclude: $P \longrightarrow \neg O$.
So, a reasonable *(logical; correct) conclusion is that, "if it is one of my poultry, then it is not an officer," or "all my poultry are not officers." Nonetheless, it is not the only conclusion for there are many ways to say this that are logically congruent to this conclusion. For example, we could say, "it is not my poultry or it is not an officer." We could say, "It is not the case that it is my poultry and an officer."*

Take the same set of premises and use Euler Diagrammes to deduce a correct, compleat conclusion (first exercise in the exercise set).

It stands to reason that lawyers, politicians, educators, etc. should be well-versed in logic. In government, industry, political science, and law - even mathematics – precise use of language may not be evident. Indeed, as we move on through the centuries, language itself changes; which may cause confusion.

Nonetheless, it is your responsibility to understand that reasonable conclusions may be gleaned from a set of premises but also that often (though hopefully **not** *in your mathematics courses) someone may infer a premise or a conclusion. A two-premise one conclusion argument such that one of the premises or the conclusion is tacitly present but instated is called an* ***enthymeme****. Note that (logically) it* **must** *be the case that at least one of the premises is stated.*

Example 2.6.3. *Consider the following:*
Crooks end up behind bars.
So, John will end up behind bars.

This enthymeme is a typical example of the mis(use) of logic. The first sentence does state that the first premise is "crooks end up behind bars" meaning all crooks end up behind bars (which is not as clear as it should be by using the 'all').
Let U be the domain of discourse, B be the set of people who will end up behind bars, and C be the set of crooks. Note that John is (supposedly) an individual; so, we will denote his existence with a point or an 'x' in the set C.
Clearly (deliberate? misused?), the missing premise is "John is a crook."

Example 2.6.4. *Consider the following:*

Prilosec was shown in clinical trials to heal ulcers in the stomachs of mice. So, prilosec will heal my ulcer.

The underlying premise is that anything that heals ulcers in samples of mice in a laboratory will also work equally well on humans. The argument form may be valid; but, as you will note upon further study of mathematics, the inferred or implied premise is dubious at best.

2.6.1 Exercises

Exercise 2.6.1. Use Euler Diagrammes to deduce a correct conclusion using all of the premises.
No ducks waltz.
No officers decline to waltz.
All my poultry are ducks.

Exercise 2.6.2. Determine ***a*** 'best' valid conclusion assuming as premises the following Dodgson syllogisms (best being defined as those conclusions which are derived from all of the premises).

A. *Babies are illogical.*
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

B. *No one takes in the* Times, *unless he is well-educated.*
No hedge-hogs can read.
Those who cannot read are not well-educated.

C. *All members of the House of Commons have perfect self-command.*
No M. P. who wears a coronet should ride in a donkey race.
All members of the House of Lords wear coronets.

D. *All unripe fruit is unwholesome.*
All these apples are unwholesome.
No fruit, grown in the shade, is ripe.

E. *Showy talkers are conceited people.*
No well-informed people are bad company.
Conceited people are bad company.

F. *No kitten that loves fish is ineducable.*
No kitten without a tail will play with a gorilla.
Kittens with whiskers always love fish.
No educable kitten has green eyes.
No kittens have tails unless they have whiskers.

G. *When I work a logic example without grumbling, you may be certain it is one I can understand. These examples are not arranged in regular order, like the examples I am used to.*
No easy example ever makes my head ache.
I can't understand examples that are not arranged in regular order, like those I'm used to.
I never grumble at an example, unless it gives me a headache.

H. *Promise-breakers are untrustworthy.*
Alcohol drinkers are very verbose.
A person who keeps a promise is honest.
No teetotallers are pawnbrokers.
One can trust a very loquacious person.

Exercise 2.6.3. For each of the following enthymemes, determine an implied premise or conclusion that is missing. Determine whether the enthymeme is valid or invalid.

A. You stole a wheelchair from an old lady. No one but a born thief would steal a wheelchair from an old lady.

B. Communism is immoral, and this is immoral.

C. Republicans always cause wars. So, Trump will get us into a war.

D. When you had no sleep, you made and A on a test. So, I'm not going to sleep.

2.7 A Treatise on Deductive Logic, Sets, and Mathematics

It should become by this time apparent to the reader that mathematics is not only a science but uses a systematic language and is in many ways an art. It is clearly a science by simply considering its many uses and applications. Imagine your life without the consequences of mathematics: it might be a world without technology, a world without much of what the modern man considers necessities (but are in fact comforts). Mathematics, we have shown, clearly uses a language unto itself and is quite insistent on formal arguments and uses symbols which to the outsider are considerably odd if not confusing.[24]

That mathematics is an art form is, perhaps, the most 'controversial' opinion that the author has proposed in the previous paragraph. As we know, philosophy is concerned with ontology, epistemology, and axiology (ethics and aesthetics). The ontological is left to the philosopher and theologians, epistemology is of concern to mathematicians, and axiology is oft thought the realm of poets, painters, sculptors, and musicians. However, aesthetics also plays a part in mathematics.

Consider the Pieta by Michelangelo and The Magic Flute by Mozart. Supposedly Michelangelo 'saw' the sculpture in a stoned and Mozart 'heard' the music before it was written. There is not argument that each was a genius. There is (I hope) no argument that each created a work of art. Why then do so many not question the genius of Cantor when he 'saw' his famous proof on the cardinality of the reals (which one can study in Math 224) or Dedekind's proof of the irrationality of (one can study in Math Math 351 (Real Variables) or Math 330 (Number Theory)) but do question the art intrinsic in mathematics?

Let us suppose there is a creative spark in each of us. Let us further suppose that each of us has a particular talent (which we may or may not be aware of). Is it so far a stretch to think that it takes more than memorisation, a calculator, and a book to really understand and create mathematics? Just as a sculptor who is taught the rudimentary principles of sculpting does not necessarily blossom into a master sculptor and a musician who is taught the rudimentary principles of music does not necessarily develop into a master composer, so too are we left with the rather humbling proposition that a student who is taught the rudimentary principles of mathematics does not necessarily mature into a master mathematician.

Indeed this course is designed to teach the student the principles of mathematics but the course, your professor, this book, nor the school can ***make*** *you do the work, live up to your potential, nor become successful. That is up to you, the individual, to make happen.*

[24] These symbols to the neophyte are of confusing, too, but take heart - - I am living 'proof' that a Philistine can learn to use the symbols, understand the symbols, and at the very least 'get by.'

That we value an education and that we value mathematics is an axiological decision that we make but is not necessarily universal.

When one is making a value judgement (let us say that 'x' is "better" than 'y') one is exercising his aesthetic; he is stating not a fact *but* a perception *that is codified by the statement he values 'x' over 'y.' Nonetheless,* there must be a reason behind said value judgement; it must be justified.

In that same manner, mathematicians oft search for the most 'elegant' proof. A proof that is short, succinct, logically sound, subtle, or bring either new insight or unifies or refines a theory is oft valued by mathematicians and is deemed 'best,' 'elegant,' or termed by some other subjective descriptor. Our exercise in this chapter (indeed in this book) is not to search for the most elegant proof, or the most subtle, etc. ours is simply and succinctly to find one that is right. *I do not give a 'blue blaze' about elegance, sophistication, etc. - - I am simply struggling day in and day out to avoid being wrong. So too, I argue, should you.*

Nonetheless, the beauty of mathematics cannot be denied by one who claims to wish to study it, make it his life's work, dedicate himself to teaching it to others, or who claims to need it for another career (like an engineer, physicist, econometrician, etc.). Let us consider from chapter one the following claim:

Claim 2.7.1. Given the premises $\neg A \wedge C$, $B \rightarrow D$, and $\neg(D \wedge \neg A)$. The conclusion $\neg B$ follows.

Proof:

1.	$\neg A \wedge C$	*1. Premise.*
2.	$\neg A$	*2. Law of Simplification of 'and' (line1)*
3.	$\neg(D \wedge \neg A)$	*3. Premise.*
4.	$\neg D \vee \neg(\neg A)$	*4. DeMorgan's Law (line 3).*
5.	$\neg D \vee A$	*5. Law of Double Negation (line 4).*
6.	$\neg D$	*6. Disjunctive Syllogism (lines 2 and 5).*
7.	$B \rightarrow D$	*7. Premise.*
8.	$\neg B$	*8. Modus Tollens (lines 6 and 7).*
		Q. E. D.

If you reflect upon it notice the beauty of the logic; how the lines flow one to the other; how each is dependent on the truth value of the previous. Note too that if any of the lines are deleted then, at best, we would have an enthymeme which is not what we mathematicians desire.

Let us consider the claim again:

Claim 2.7.2. Given the premises $\neg A \wedge C$, $B \to D$, and $\neg(D \wedge \neg A)$. The conclusion $\neg B$ follows.

Proof:

1.	$\neg(\neg B))$	*1. Negation of the conclusion*
2.	B	*2. Law of Double Negation (line 1)*
3.	$B \to D$	*3. Premise.*
4.	D	*4. Modus Ponens (lines 2 and 3).*
5.	$\neg(D \wedge \neg A)$	*5. Premise.*
6.	$\neg D \vee \neg(\neg A)$	*6. DeMorgan's Law (line 5).*
7.	$\neg(\neg A)$	*7. Disjunctive Syllogism (lines 4 and 6).*
8.	A	*8. Law of Double Negation (line 7)*
9.	$\neg A \wedge C$	*9. Premise.*
10.	$\neg A$	*10. Law of Simplification of 'and' (line 9)*
11.	$A \wedge \neg A$	*11. Law of Adjunction (lines 8 and 10)*
12.	$\neg B$	*12. Contradiction (lines 11, 1)*
		Q. E. D.

If you reflect upon this proof notice the beauty of the logic; how the lines flow one to the other; how each is dependent on the truth value of the previous. Note too that it is longer than the first one. So, some mathematicians claim the first proof is better than the second. Others would claim the first is better than the second because it was done directly. Still other mathematicians prefer the second proof (I amongst them) since it was done indirectly and they value the indirect argument over the direct argument. Now are any of these positions more right than another*?*

No, of course said positions are value judgements; what one may term 'splitting hairs.' Again, we return to the central point of this section: subjective value judgements have a place in society and in our lives but do not constitute truth (epistemological) and really should not be of concern to you as you pursue your education.

By a similar stance, one can see that those who claim 'real world' applications are better than 'pure' mathematics are not right; nor are those who claim 'pure' mathematics is better than 'applied' mathematics. Our concern will simply be on comprehension and understanding. Indeed, if I recall correctly, Albert Einstein was reported to have said, "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain they do not refer to reality." His words are certainly food for thought (pun intended).

Hence, study these basics about deductive logic - - you will use them in the rest of your coursework (hopefully). You can be assured you will need to think clearly and rationally in Math courses as well as in other courses. Study these basics about sets for they are the tools that will assist you in subsequent coursework. You are studying *mathematics not following taking an 'appreciation course' where you marvel at the ingenuity, depth, intelligence, wit, etc. of the professor; hence,* you must do it so that you learn *the material! Consider the course; what is it called? Appreciation of Math 017 for example? Love of Math 017? Let's watch Math 017? No! It is Math 017. So, get to work and study!*

