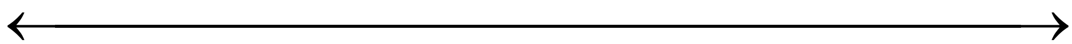
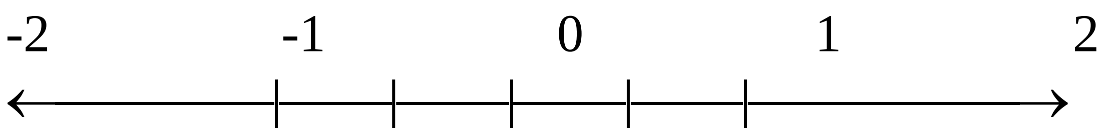
HANDOUT 6
THE AXIOMS OF THE REALS

M. P. M. M. MCLOUGHLIN

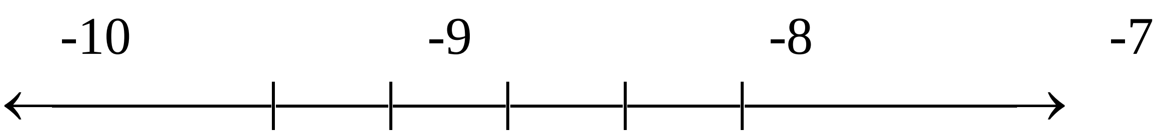
Some basics on the real line ($\mathbb{R}$) :

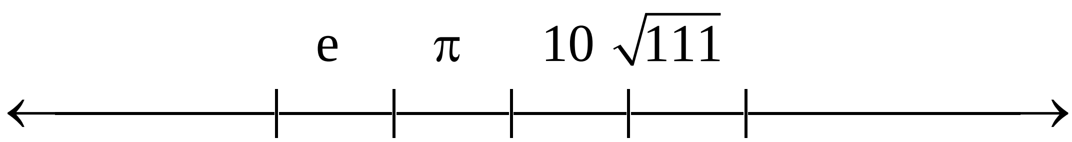


There is ***no*** centre (e.g.: the nonsense about $\infty + (-\infty) = 0$ one may have learnt in high school is a fallacy) so one can reasonably represent the line as:



or





The reals (as all of math) begins with axioms:

The Field Axioms of $\mathbb{R}$

<u>Axiom 1</u> (closure of addition): $\forall$ x, y $\in \mathbb{R}$,

$x + y \in \mathbb{R}$ and $(x = w \wedge y = v) \Rightarrow (x + y = w + v)$

<u>Axiom 2</u> (commutative of addition): $\forall$ x, y $\in \mathbb{R}$,

$x + y = y + x$.

<u>Axiom 3</u> (associative of addition): $\forall$ x, y, z $\in \mathbb{R}$,

$(x + y) + z = x + (y + z)$

<u>Axiom 4</u> (existence of identity of addition):

$\exists$ a unique number 0 $\ni x + 0 = x \quad \forall x \in \mathbb{R}$

<u>Axiom 5</u> (existence of additive inverse):

$\forall x \in \mathbb{R}$ $\exists$ a unique number -x $\ni x + (-x) = 0$

<u>Axiom 6</u> (closure of multiplication): $\forall$ x, y $\in \mathbb{R}$,

$x \cdot y \in \mathbb{R}$ and $(x = w \wedge y = v) \Rightarrow (x \cdot y = w \cdot v)$

<u>Axiom 7</u> (commutative of multiplication):

$\forall$ x, y $\in \mathbb{R}$, $x \cdot y = y \cdot x$.

<u>Axiom 8</u> (associative of multiplication):

$\forall$ x, y, z $\in \mathbb{R}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

<u>Axiom 9</u> (existence of identity of multiplication):

$\exists$ a unique number 1 $\ni x \cdot 1 = x \quad \forall x \in \mathbb{R}$

$(1 \neq 0)$.

<u>Axiom 10</u> (existence of multiplicative inverse):

$\forall x \in \mathbb{R} \ni x \neq 0 \; \exists$ a unique number $x^{-1} \ni$

$x \cdot (x^{-1}) = 1$

<u>Axiom 11</u> (distributive of multiplication over addition): $\forall x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

The Order Axioms of $\mathbb{R}$

<u>Axiom 12</u> (trichotomy): $\forall x, y \in \mathbb{R}$, exactly one of the following relationships exists between x and y : $x < y$, $x = y$, $\vee$ $x > y$.

$[(x < y)$ exor $(x = y)$ exor $(x > y)]$

<u>Axiom 13</u> (transitive): $\forall x, y, z \in \mathbb{R}$,

$[(x < y) \wedge (y < z)] \Rightarrow (x < z)$

<u>Axiom 14</u> (preservation of order under addition):

$\forall x, y, z \in \mathbb{R}$, $(x < y) \Rightarrow (x + z < y + z)$

<u>Axiom 15</u> (preservation of order for positive multiplier): $\forall x, y \in \mathbb{R}$, $[(x < y) \wedge (0 < z)] \Rightarrow$

$(x \cdot z < y \cdot z)$

The Completeness Axiom of $\mathbb{R}$

Axiom 16 (completeness): $\forall$ A $\subseteq \mathbb{R}$ ∍ A is bounded above $\exists$ a number m which is the supremum of the set