Chapter 6

Conditional Probability

6.1 Introduction

The concept of conditional probability is a fascinating one. What if one noticed that an event occurred and we wish to follow that with another event. Does the probability of the second event get influenced or affected by the first event? Furthermore, suppose there is more than one descriptor for an experiment– does one descriptor affect or have some bearing on another?

A couple of examples might be best to illuminate the idea:

Example 6.1.1. *You have a fair standard 52-card bridge deck of cards. The descriptors on each card are rank and suit.*
A. Draw a card. The probability that card is a jack is $\frac{4}{52}$.
B. Draw a card. Assume you know the card is a spade. The probability that spade card is a jack is $\frac{1}{13}$ *(the nomenclature is the probability the card is a jack given the card is a spade).*
C. Draw a card. Assume you know the card is a jack. The probability that jack is a spade is $\frac{1}{4}$.
D. Draw a card. Assume you know the card is a jack. The probability that jack is a 5 is $\frac{0}{4}$.

Notice how the probabilities were not constant. The example just presented was such that the descriptors were temporally concurrent. What if the experiment is such that we have temporally distinct events? How does that work? Another illustration might help.

Example 6.1.2. *You have an urn with 5 red balls, 2 blue balls, and 3 white balls.*
A. Draw a ball from the urn. The probability that ball is red is $\frac{5}{10}$.

B. Draw a ball from the urn; then another. Suppose the first ball was red. The probability that the second ball is red is $\frac{4}{9}$ *(the nomenclature is the probability the second ball is red given the first ball is red).*
C. Draw a ball from the urn; then another. Suppose the first ball was red. The probability that the second ball is blue is $\frac{2}{9}$*.*
D. Draw a ball from the urn; note the colour. Put it back and shake up the urn. Now pick a ball (another). Suppose the first ball was red. The probability that the second ball is blue is $\frac{2}{10}$*.*

The examples suggest the definitions and concepts that follow.

6.2 Definitions and Concepts

Definition 6.2.1. Let S be a well defined sample space; E an event. E is a non-trivial event means $Pr(E) \in (0, 1)$.

Definition 6.2.2. Let S be a well defined sample space; E and F be events where $Pr(F) = 0$. The probability of E given F has occurred is 0. The notation for it is $Pr(E|F)$.

Definition 6.2.3. Let S be a well defined sample space; E and F be events where $Pr(F) \neq 0$. The probability of E given F has occurred is $\frac{Pr(E \cap F)}{Pr(F)}$. The notation for it is $Pr(E|F)$.

Notation 6.2.1. *Let S be a well defined sample space; E and F be events where $Pr(F) \neq 0$. The probability of E given F has occurred is* $Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$.

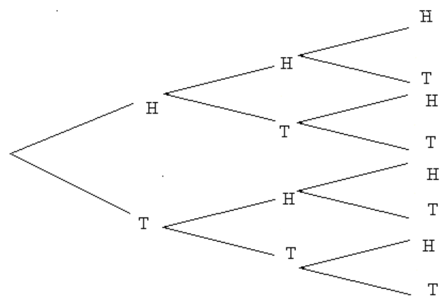
Theorem 6.2.1. *Let S be a well defined sample space; and E a non-trivial event. Therefore it is the case that E^c is a non-trivial event.*

6.3 Understanding Probability: Drawing Diagrammes

In drawing diagrammes there are choices to be made.

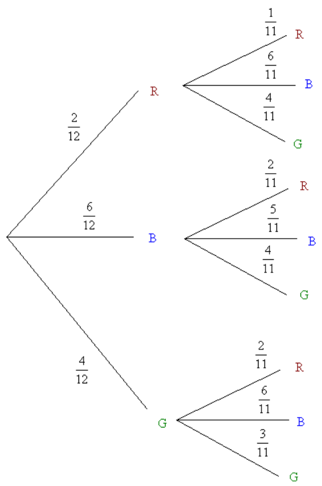
Sometimes, we are going to draw a tree to illustrate the experiment.

For example, flip a fair coins three times.



If there are a lot of things but there aren't many stages of the experiment, then a tree is indeed best drawn with the events and probabilities on the branches:

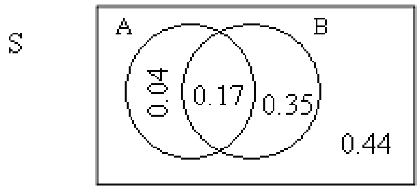
Let us say there is an urn with 2 red, 6 blue, ∧ 4 green balls. We draw a ball then another ball from the urn. We are then going to draw a tree to illustrate the experiment.



However, if the experiment is not in stages but the descriptors are multiple descriptions of things (***at the same time***) then it is best to draw a Venn diagramme with the probabilities.

S is a well defined sample space, $Pr(A) = 0.21, Pr(B) = 0.52$, and $Pr(A \cap B) = 0.17$

We get:



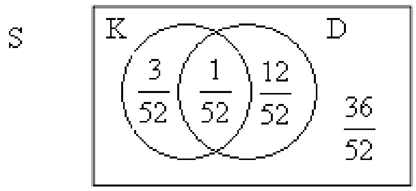
Notice the probability of an element that is in A and not in B is 0.04 ($Pr(A \cap B^C) = 0.04$).

Notice the probability of an element that is not in A and not in B is 0.04 ($Pr(A^C \cap B^C) = 0.44$).

Notice the probability of an element that is in A or in B is $0.04 + 0.17 + 0.35$ ($Pr(A \cup B) = 0.56$).

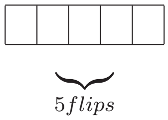
The Venn Diagramme is quite useful.

Pick a card from a standard 52 deck of cards. Finding probability of picking a king, but not a spade: Let K be the event draw a king and D be the event draw a spade. We get:



$Pr(K) = \frac{4}{52}$; $Pr(K \cap D^C) = \frac{3}{52}$; $Pr(K \cap D) = \frac{1}{52}$; $Pr(K^C \cap D^C) = \frac{36}{52}$; $Pr(K \cup D) = \frac{16}{52}$ since it is $Pr(K \cup D) = Pr(K) + Pr(D) - Pr(K \cap D)$.

Example 6.3.1. *Flip a fair coin 5 times – find the probability of flipping exactly 3 heads.*



There are five slots to fill in a heads or tails:

The number of ways to fill in three heads is $\binom{5}{3}$.

The other two slots have to be filled in with tails; the number of ways to fill in the tails is $\binom{2}{2}$.

So, ways to fill in three heads is $\binom{5}{3} \cdot \binom{2}{2} = 10 \cdot 1 = 10$.

Each box can be filled in 2 ways; so, by the generalised principle of counting the number of ways to fill in the boxes is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

So, the probability of flipping a fair coin 5 times and flipping exactly 3 heads is $\frac{10}{32}$.

6.3.1 Understanding Probability: More Diagrammes

There are ways to draw diagrammes, graphs, or charts well so that the graph, chart, or diagramme illustrates the probability in such a manner as to enhance both one's understanding of the concept and portray accurately the scenario. For probability we use either Venn diagrammes or Tree diagrammes usually; but not always.

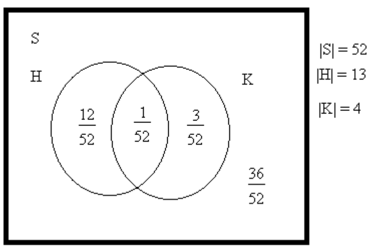
Example 6.3.2. *You roll a pair of fair dice and view the upward faces of the dice such that an event is defined as the sum of the values of the two dice faces.*

first die →	*1*	*2*	*3*	*4*	*5*	*6*
second die ↓						
1	*2*	*3*	*4*	*5*	*6*	*7*
2	*3*	*4*	*5*	*6*	*7*	*8*
3	*4*	*5*	*6*	*7*	*8*	*9*
4	*5*	*6*	*7*	*8*	*9*	*10*
5	*6*	*7*	*8*	*9*	*10*	*11*
6	*7*	*8*	*9*	*10*	*11*	*12*

The probability of rolling a five (call it E) on one of the dice given you rolled a sum that is prime (call it M) is illustrated in the table.There are 36 entries
eleven of which is where a five is rolled on one of the two dice ($Pr(E) = \frac{11}{36}$)*;*
fifteen are where a sum is rolled that is prime ($Pr(M) = \frac{15}{36}$)*;*
and, four where a five is rolled on a die and a prime sum was rolled ($Pr(E \cap M) = \frac{4}{36}$)*.*
So, $(Pr(E|M) = \dfrac{Pr(E \cap M)}{Pr(M)} = \dfrac{4}{36} \div \dfrac{15}{36} = \dfrac{4}{15})$

The probability of rolling a sum of five (call it F) given you rolled a sum that is greater than eight (call it E) is facile, too. Notice if you rolled a sum greater than 8 you can't roll a sum that is five; therefore, the answer is zero ($Pr(F|E) = 0$)*.*

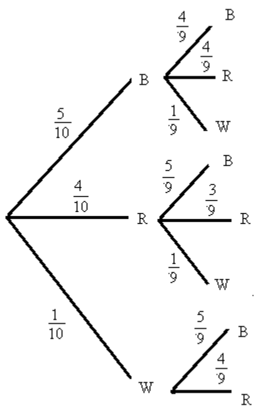
Example 6.3.3. *You draw a card from a fair bridge deck of cards. The probability of drawing a king (call it K) given you drew a heart (call it H) is illustrated in the Venn Diagramme.*



Note $Pr(K) = \dfrac{4}{52}$*;* $Pr(H) = \dfrac{13}{52}$*; and,* $Pr(K \cap H) = \dfrac{1}{52}$.

So, $Pr(K|H) = \frac{1}{4}$ *which can be seen in the Venn diagramme or through the algebra of* $Pr(K|H) = \frac{Pr(K \cap H)}{Pr(H)} = \frac{1}{52} \div \frac{4}{52} = 0.25$

Example 6.3.4. *There is an urn with 5 blue, 4 red, and a white ball. You draw a ball from the urn and note the colour. You draw a second ball from the urn and note its colour. The probability of drawing a red ball (call it R) second given you drew a blue ball first (call it B) is illustrated in the Tree Diagramme.*



Note $Pr(R_{second}|B_{first}) = \frac{4}{9}$ *which can be seen in the Tree diagramme as the second branch labelled R off of the top primary branch (B).*

Also, $Pr(R_{second}|R_{first}) = \frac{3}{9}$ *which can be seen in the Tree diagramme as the second branch labelled R off of the middle primary branch (R).*

To get probabilities for multi-stage events we multiply the probabilities from the branches, $Pr(R_{first} \wedge W_{second}) = Pr(R_{first}) \cdot Pr(W_{second}|R_{first}) = \frac{4}{10} \cdot \frac{1}{9} = \frac{4}{90}$ *which can be seen in the Tree diagramme as the middle primary branch (R) times the third branch labelled W off of the middle primary branch (R).*

What about general questions of multi-stage events? Then we multiply the probabilities from the branches, and add

$$Pr(Red \wedge White) =$$
$$Pr(R_{first} \wedge Pr(W_{second}) + Pr(W_{first} \wedge R_{second}) =$$
$$Pr(R_{first}) \cdot Pr(W_{second}|R_{first}) + Pr(W_{first}) \cdot Pr(R_{second}|W_{first}) =$$
$$\frac{4}{10} \cdot \frac{1}{9} + \frac{1}{10} \cdot \frac{4}{9} = \frac{4}{90} + \frac{4}{90} = \frac{8}{90}$$

which can be seen in the Tree diagramme as the middle primary branch (R) times the third branch labelled W off of the middle primary branch (R) along with the bottom primary branch (W) times the second branch labelled R off of the bottom primary branch (W).

Notice that the illustrations sometimes have the conditional probabilities labelled within and other times the conditional probability must be derived after the illustration. Nonetheless, for computational problems the diagrammes are, I opine, most helpful. In drawing diagrammes there are choices to be made.

6.4 Exercises

For computational exercises: Perhaps use a Venn diagramme or Tree diagramme (where apropos); use the definition of the probability of W given M when $Pr(M) \neq 0$.

Exercise 6.4.1. Suppose a pair of dice is tossed. You view the sum of the two faces up-turned.

A. Find the probability that the sum of the sides facing up is more than 5 given the sum of the sides facing up is 7 (from here on out we will term this 'Find the probability that the you rolled more than 5 given that you rolled a 7.'

B. Find the probability that you rolled a seven given you rolled more than 5.

C. Find the probability that you rolled a less than 10 given you rolled more than 6.

Exercise 6.4.2. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.

A. Two balls are drawn from the urn. Find the probability that both balls drawn are red.

B. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is green given the first ball is green.

C. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the first ball is red and the second ball is green.

D. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that of the balls one is green and one is red.

E. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is green given the first ball is not red.

F. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is not green given the first ball is red.

G. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the first ball drawn is green and the second is not green.

Exercise 6.4.3. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.

A. A ball is drawn from the urn. Note its colour; put it back and shake up the urn. Draw a ball from the urn. Find the probability that both balls drawn are red.

B. Put all the balls back, shake up the urn. A ball is drawn from the urn. Note its colour; put it back and shake up the urn. Draw a ball from the urn. Find the probability that the first ball is red and the second ball is green.

Exercise 6.4.4. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.

A. Three balls are drawn from the urn. Find the probability that all of the balls drawn are red.

B. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that one of the balls is red and the other two are not white.

C. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that the at least two of the balls are blue.

Exercise 6.4.5. You have a fair deck of 52 standard bridge cards.

A. Draw a card. Find the probability that it is a ten.

B. Put the card back. Shuffle the cards. Draw a card. Find the probability that the card drawn is a ten or a queen.

C. Put the card back. Shuffle the cards. Draw a card. Find the probability that the card drawn is a ten or a heart.

D. Put the card back. Shuffle the cards. Draw a card. Find the probability that the card drawn is a ten given it is a heart.

E. Put the card back. Shuffle the cards. Draw a card. Find the probability that the card drawn is a heart given it is a 10.

F. Put the card back. Shuffle the cards. Draw a card. Find the probability that the card drawn is a club given it is a heart.

6.5 Independence

Definition 6.5.1. Let U be a well defined universe and S be a well defined sample space (a subset of U). Let E and F be two events. E and F are **independent** if and only if it is the case that $Pr\left(F|E\right) = Pr\left(F\right)$ and $Pr\left(E|F\right) = Pr\left(E\right)$.

Alternately, the definition can be stated as:

Definition 6.5.2. Let U be a well defined universe and S be a well defined sample space (a subset of U). Let E and F be events. E and F are **independent** if and only if it is the case that $Pr\left(F \cap E\right) = Pr\left(F\right) \cdot Pr\left(E\right)$

Definition 6.5.3. Let U be a well defined universe and S be a well defined sample space (a subset of U). Let E, F, and G be three events. E, F, and G are **independent** if and only if it is the case that
(1) $Pr(G \cap F \cap E) = Pr(G) \cdot Pr(F) \cdot Pr(E)$
and
(2A) $Pr(F \cap E) = Pr(F) \cdot Pr(E)$
(2B) $Pr(G \cap E) = Pr(G) \cdot Pr(E)$
(2C) $Pr(F \cap G) = Pr(F) \cdot Pr(G)$

Definition 6.5.4. Let U be a well defined universe and S be a well defined sample space (a subset of U). Let E, F, G, and H be four events. E, F, G, and H are **independent** if and only if it is the case that
(1) $Pr(H \cap G \cap F \cap E) = Pr(H) \cdot Pr(G) \cdot Pr(F) \cdot Pr(E)$
and
(2) all triple-wise collections of events is satisfied such as (1) in the previous definition
(3) all pair-wise collections of events is satisfied such as (2) in the previous definition.

The alternate definition of two events being independent, "let U be a well defined universe and S be a well defined sample space (a subset of U). Let E and F be events. E and F are independent if and only if it is the case that $Pr(F \cap E) = Pr(F) \cdot Pr(E)$" is the most useful one for theoretical work (proofs, counterexamples, etc. I opine) but the original is of use, too. Independence is a more generalised idea of 'reset.' Indeed consider:

Example 6.5.1. *There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.*

Non-independence exercise: *Shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is red is given the first ball is green.*

Independence exercise: *Put all the balls back, shake up the urn and draw a ball from the urn. Note its colour. Put the ball back and shake up the urn. Draw a ball from the urn. Find the probability that the second ball is red given the first ball is green.*

The idea is that when we have a well defined sample space and events A and B that are independent A does not effect B and B does not effect A (with regard to probability of one occurring given the other occurs or had occurred).

Example 6.5.2. *There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls. Also, you have a fair six sided standard die. One chooses a ball from the urn and then rolls the die. One wishes to find the probability that he rolls an odd natural number on the die given he chose a blue ball. It is* $\frac{3}{6}$. *Note the probability of rolling an odd natural number on the die given he chose a blue ball is the same as the probability he rolls an odd natural number on the die.*

One rolls the die and then chooses a ball from the urn. One wishes to find the probability that he chooses a blue ball given he rolled an odd natural number on the die. It is $\frac{6}{21}$. *Note the the probability that he chooses a blue ball given he rolled an odd natural number on the die is the same as the probability he chooses a blue ball from the urn.*
So, the events are independent (and understood practically to be such). This is an example of practical and stochastic independence – both what you deduce practically agrees with the mathematical understanding.

Independence is a very critical property for applied probability and statistics; thus, I opine it is quite important to understand the concept for future course-work in this area of mathematics and for applied mathematics (sociology, sociometrics, anthropology, education, physics, chemistry, computer science, actuarial science, technometrics, edumetrics, econometrics, biometrics, data analysis, financial mathematics, etc.).

6.6 Theorems About Independence

Theorem 6.6.1. *Let* S *be a well defined sample space with* $E \wedge F$ *independent events. Therefore it is the case that* $E^c \wedge F$ *are independent events.*

Theorem 6.6.2. *Let* S *be a well defined sample space with* $E \wedge F$ *independent events. Therefore it is the case that* $E^c \wedge F^c$ *are independent events.*

So. not only do the events turn out independent but so too the complements of the events (with the original events but the complements regarding each other).

6.6.1 Exercises

Exercise 6.6.1. There is an urn. It contains 8 white, 3 red, and 6 blue balls. One chooses a ball from the urn and looks at the colour. Let event E be he chooses a white ball and F be he chooses a blue ball. Determine if events E and F are independent.

Exercise 6.6.2. A standard deck of 52 fair cards is shuffled. Draw a card. Let event H be draw a heart and J be the event draw a jack. Determine if events H and J are independent.

Exercise 6.6.3. A standard deck of 52 fair cards is shuffled. Draw a card. Let event Q be draw a queen and J be the event draw a jack. Determine if events Q and J are independent.

Exercise 6.6.4. There is an urn. It contains 8 white, 3 red, and 6 blue balls. One chooses a ball from the urn then another ball from the urn and looks at the colour. Let event W be he chooses a white ball and B be he chooses a blue ball. Find $Pr(W_{first})$, $Pr(B_{first})$, $Pr(W_{second}|B_{first})$, and $Pr(B_{second}|W_{first})$. Determine if events W and B are independent.

Exercise 6.6.5. There is an urn. It contains 8 white, 3 red, and 6 blue balls. One chooses a ball from the urn then another ball from the urn and looks at the colour. Let event W be he chooses a white ball. Find $Pr(W_{first})$ and $Pr(W_{second}|W_{first})$. Determine if event the W is independent of itself.

Exercise 6.6.6. Let $U = S = \mathbb{N}_{12}$. Let event D be the event one chooses an odd natural number. Let M be the event one chooses a prime natural number. Determine if events D and M are independent.

Exercise 6.6.7. Let $U = S = \mathbb{N}_{12}$. Let event R be the event one chooses a natural number that is a multiple of 3. Let F be the event one chooses a natural number that is a multiple of 4. Determine if events R and F are independent.

Exercise 6.6.8. Suppose a fair six-sided die is tossed. One views the up-turned face. The fair six-sided die is tossed again and the up-turned face is viewed. Let event E be defined as the first roll was a 3 and event F is defined as the second roll is a 1, 2, or 4. Determine if events E and F are independent.

Exercise 6.6.9. Suppose a fair six-sided die is tossed. One views the up-turned face. The fair six-sided die is tossed again and the up-turned face is viewed. Let event E be defined as the sum of the two rolls is odd and event F is defined as the first roll is a 2. Determine if events E and F are independent.

Exercise 6.6.10. Suppose a fair six-sided die is tossed. One views the up-turned face. The fair six-sided die is tossed again and the up-turned face is viewed. Let event E be defined as the sum of the two rolls is odd and event G is defined as the second roll is a 2 or a 3. Determine if events E and G are independent.

Exercise 6.6.11. Suppose there is a well defined sample space with two events A and B such that $Pr(A) = \frac{2}{5}$ and $Pr(B) = \frac{9}{10}$. Further, $Pr(A \cap B) = 0.36$. Determine if events A and B are independent.

Exercise 6.6.12. Suppose there is a well defined sample space with two events C and D such that $Pr(C) = \frac{3}{4}$ and $Pr(D) = \frac{8}{10}$. Suppose we also know $Pr(C \cap D^C) = \frac{3}{20}$. Determine if events C and D are independent.

Exercise 6.6.13. Suppose there is a well defined sample space with two events E and F such that $Pr(E) = 0.30$ and $Pr(F) = 0.35$. Suppose we also know $Pr(E \cap F) = 0.40$. Determine if events E and F are independent.

Exercise 6.6.14. Suppose there is a well defined sample space with two events G and H such that they are independent. Suppose $Pr(G) = \frac{2}{7}$ and we know $Pr(G \cap H) = \frac{5}{42}$.
A. Determine the probability of the event H.
B. Determine the probability of the event H^c.
C. Determine the probability of the event $G \cup H$.

Exercise 6.6.15. Suppose one flips a fair coin five times and notes whether the toss was a heads or a tails. Determine the probability of flipping all heads. Were the tosses independent? If so, how did it influence your solution. If not, why not?