Class Notes

Math 017
Introduction to Mathematics

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Chapter 4

Counting Theory

4.1 Introduction

Much of what shall be discussed is, by-and-large, codification of common-sense inductive observations that have been made rigourous with axioms, definitions, etc. The principles involved are casually called the addition principle, multiplication principle, generalised principle of counting, permutations, and combinations.

The addition principle, which is really just a formalization of the common-sense idea of what addition means, to wit:
If a task can be done using either of two different methods (but not both methods simultaneously), the first of which can be done in m ways and the second of which can be done in n ways, then there are $m + n$ different ways of completing the task.

An application of the addition principle is in counting the number of elements in the union of two non-empty disjoint sets, A and B. The "task" to be performed is to choose an element from the union $A \cup B$. This can be done in either of two different methods: either choose an element of A, or choose an element of B. If we let $m = |A|$ ($m \in \mathbb{N}$) and $n = |B|$ ($n \in \mathbb{N}$), then there are m ways to choose an element of A and n ways to choose an element of B, so the total number of ways to choose an element of $A \cup B$ (in other words, the number of elements of $A \cup B$) is $m + n$ which is $|A| + |B|$.

Extending it, the addition principle is in counting the number of elements in the union of two disjoint sets, A and B. Let $m = |A|$ ($m \in \mathbb{N}^*$) and $n = |B|$ ($n \in \mathbb{N}^*$), then there are m ways to choose an element of A and n ways to choose an element of B, so the total number of ways to choose an element of $A \cup B$ (in other words, the number of elements of $A \cup B$) is $m + n$ which is $|A| + |B|$. So:

Theorem 4.1.1. *Let U be a well defined universe and A and B be disjoint sets ($A \cap B = \emptyset$), then*

$$|A \cup B| = |A| + |B|.$$

If we attempt to use the addition principle to find $|A \cup B|$ for two sets A and B that are *not* disjoint, we will have a problem. This is because the elements in the intersection$A \cap B$ are double counted —once for being an element of A and once for being an element of B. To correct for the overcompensation, we need to subtract $|A \cap B|$. This gives the following formula, which is often called the principle of inclusion–exclusion:

Theorem 4.1.2. *Let U be a well defined universe and A and B be sets.*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The theorem also can be stated as:
Let U be a well defined universe and A and B be sets.

$$|A \cap B| = |A| + |B| - |A \cup B|.$$

Like the addition principle, there is a multiplication principle which is a formalization of common sense. In this case, we are formalizing the common-sense idea of what multiplication means. The multiplication principle states:

If an experiment or task can be broken up into two steps, the first of which can be done in m ways and the second of which can be done in n ways (regardless of how the first step is done), then there are $m \cdot n$ different ways of completing the task.

The multiplication principle can easily be extended to tasks that require more than two steps. For example, if a task can be broken up into three steps, the first of which can be done in m ways, the second of which can be done in n ways (regardless of how the first step is done), and the third of which can be done in p ways (regardless of how the first and second steps are done), then there are $m \cdot n \cdot p$ different ways of completing the task.

Suppose we have a set of n objects, and we want to repeatedly choose an element of the set, r times in all, with replacement (meaning that, after we choose an element, we put it back into the set, so the same element may be chosen more than once). Suppose also that the *order* in which we choose the r elements is important (so choosing the element first and the element 2 second should be considered to be different from choosing the element 2 first and the element 1 second).

For example, perhaps we are choosing a six-digit identification number for a new product, where each of the digits can be $0, 1, 2, 3, 4, 5, 6, 7, 8$, or 9 (leading zeroes are allowed). We can view this as a task to be performed, which is broken up into r steps:
making the first choice, making the second choice, and so on. Since there are n objects in the set, each of these choices can be made in n different ways. By the multiplication principle, then, the number of ways to perform the task is $n \cdot n \cdot n \cdot \cdots \cdot n$, with r factors of n; this is n^r. Hence:
The number of ways to choose r elements from a set of n objects, with replacement, when the order of the choices matters, is n^r.

4.1.1 Permutations

Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, where $n \geq k$. A **permutation** of a set of n objects is a particular rearrangement or ordering of the elements of the set. For example, the permutations of the set $\{1, 2, 3\}$ are $1, 2, 3$; $1, 3, 2$; $2, 1, 3$; $2, 3, 1$; $3, 1, 2$; and $3, 2, 1$. To count the number of permutations of a set of n objects, we again view it as a task to be performed and use the multiplication principle. The task can be broken up into n steps: choosing the first element of the permutation, choosing the second element of the permutation, and so on. Note that once we have chosen an element, we cannot choose it again, because each element appears only once in the permutation.

So, let U be a well defined universe and let A be a set such that $|A| = 1 = n$. There is one way to choose it; it is the first and only element. So, the number of permutations of a set with 1 object is $1! = 1$. Let U be a well defined universe and let A be a set such that $|A| = 2 = n$. There are two ways to choose the first element; from that the other element must be chosen second. So, the number of permutations of a set of 2 objects is $2! = 2 \cdot 1$. Continue inductively with this argument and so note where there is the well defined universe U and A a set such that $|A| = n$. There are n different ways to make the first choice, $n - 1$ different ways to make the second choice, $n - 2$ different ways to make the third choice, and so on; there will be only 2 ways to make the second-to-last choice, and only 1 way to make the last choice. Therefore:

The number of permutations of a set of n ($n \in \mathbb{N}$) objects is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

It is called **n factoral**. In addition, we define 0! to be 1.

What of choosing without replacement; but, order is important? Such can be handled; suppose we have a set of n objects, and we want to choose r of them, without replacement (so no element can be chosen twice). Suppose also that the order in which the elements are chosen is important. For example, perhaps we are judging submissions to an essay contest and must choose the first place winner, the second place winner, and the third place winner. Viewing this as a task to be performed, and breaking it up into r steps (making the first choice, making the second choice, and so on), we see that there are n different ways to make the first choice, $n-1$ different ways to make the second choice, and so on, just as before when we were counting permutations. The difference is that here we are only making r choices in all, so there are only r factors in the multiplication. The last factor is $n-r+1$ (not $n-r$). Therefore: The number of ways to choose r elements from a set of $n$objects, without replacement, when the order of the choices matters, is

$$(n-0) \cdot (n-1) \cdot (n-2) \ldots (n-(r-1)) = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

This is called "the number of permutations of n objects taken r at a time."

Definition 4.1.1. Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, where $n \geq k$. The **permutations** of n objects taken k at a time is ${}_nP_k = P(n,k) = P_k^n = \dfrac{n!}{(n-k)!}$.

4.1.2 Combinations

Combinations is counting where one is choosing without replacement; but, where order is not important.

Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, where $n \geq k$. A **combination** of a set of n objects is a particular rearrangement or ordering of the elements of the set. For example, the permutations of the set $\{1,2,3\}$ are $1,2,3$; $1,3,2$; $2,1,3$; $2,3,1$; $3,1,2$; and $3,2,1$. To count the number of combinations we then delete the repetitions – note **all are repetitions** so the combinations of a set of size n objects choosing all the objects without regard to order is 1.

Suppose, as with permutations, that we have a set of n objects from which we want to choose k elements without replacement, but this time suppose that the order in which the elements are chosen is **not** important. For example, soppose we choose a subset of size k from a larger set of size n. We know that ${}_nP_k$ is the number of ways to do this if order is important. However, each individual subset of size k has $k!$ permutations, so ${}_nP_k$.will count it $k!$ times. Therefore, since we order of the choices does not matter, we must divide ${}_nP_k$ by $k!$ to correct for the over-count. So:

The number of ways to choose k elements from a set of n objects (where n is a counting natural number, k is a counting natural number and $n \geq k$), without replacement, when the order of the choices not of import, is

$$\frac{{}_nP_k}{k!}$$

This is called "the number of combinations of n objects taken k at a time."

Definition 4.1.2. Let $n \in \mathbb{N}^*$, $w \in \mathbb{N}^*$, where $n \geq w$. The **combinations** of n objects taken w at a time is ${}_nC_w = C(n, w) = C_w^n = \binom{n}{w} = \dfrac{n!}{(n-w)! \cdot w!}$.

The special notation $\binom{n}{w}$ is the most common and it is read as "n choose w." The formula for $\binom{n}{w}$ provides one good reason for the definition that $0! = 1$. Consider the expression $\binom{n}{n}$. This is the number of ways to choose n elements from a set of size n, where the order in which the elements are chosen does not matter. Clearly there is only one way to do this—choose all n elements of the set. So $\binom{n}{n}$ must be 1. Note we get a 0! as part of the denominator; since it is $\binom{n}{n} = \dfrac{n!}{(n-n)! \cdot n!}$ forcing 0! to be one.

What about choosing with replacement and order is not important? We have seen that, when counting the number of ways to make a sequence of choices, it is essential to know whether replacement is or is not allowed and whether the order in which the elements are chosen is or is not important. We have covered three of the possibilities above; the last possibility is that the choices are made with replacement and the order is not important. This is somewhat subtler than the other three cases.

4.1.3 The Pigeonhole Principle

The pigeonhole principle, like the other "principles" described above, is just a formalization of a common-sense idea. However, despite its apparent obviousness, the pigeonhole principle is often a very useful tool, so it is worthwhile to state it explicitly. The pigeonhole principle in essence is:
If we are placing objects in boxes and there are more objects than boxes, then at least one of the boxes must receive at least two objects.

4.1.4 Pascal's Triangle

$$
\begin{array}{lccccccccccc}
n = 0: & & & & & & 1 & & & & & \\
n = 1: & & & & & 1 & & 1 & & & & \\
n = 2: & & & & 1 & & 2 & & 1 & & & \\
n = 3: & & & 1 & & 3 & & 3 & & 1 & & \\
n = 4: & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n = 5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
n = 6 & 1 & & 6 & & 15 & & 20 & & 15 & \vdots \quad \ldots & \ddots
\end{array}
$$

The sum of the numbers on a diagonal of Pascal's triangle equals the number below the last summand. For example, $1 + 2 = 3$, $1 + 2 + 3 = 6$, $1 + 3 = 4$, $1 + 3 + 6 = 10$, etc. Formally we have:

$$\sum_{k=0}^{p} \binom{k+n-1}{n-1} = \binom{p+n}{n}$$

Now note:

$$
\begin{array}{ll}
n=0: & \binom{0}{0} \\
n=1: & \binom{1}{0} \quad \binom{1}{1} \\
n=2: & \binom{2}{0} \quad \binom{0}{1} \quad \binom{2}{2} \\
n=3: & \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
n=4: & \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\vdots & \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{array}
$$

Moreover look at the sum of the rows:

$$
\begin{array}{ll}
2^0: & \binom{0}{0} \\
2^1: & \binom{1}{0} \quad \binom{1}{1} \\
2^2: & \binom{2}{0} \quad \binom{0}{1} \quad \binom{2}{2} \\
2^3: & \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
2^4: & \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\vdots & \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{array}
$$

And look at what the values mean in terms of set theory; for example:

2^4:	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$
$n=4$:	1	4	6	4	1

Example 4.1.1. *Let $U = \mathbb{R}$. Let $A = \{\pi, 5, -2, \frac{7}{2}\}$.*
The $n = 4$ line: consider a well defined universe and a set with four elements.
The number of subsets of the set which has no elements is 1.

$$\emptyset$$

The number of subsets of the set which has one elements is 4.

$$\{\pi\}, \{5\}, \{-2\}, \{\frac{7}{2}\}$$

The number of subsets of the set which has two elements is 6.

$$\{\pi, 5\}, \{\pi, -2\}, \{\pi, \frac{7}{2}\}, \{5, -2\}, \{5, \frac{7}{2}\}, \{-2, \frac{7}{2}\},$$

The number of subsets of the set which has three elements is 4.

$$\{\pi, 5, -2\}, \{\pi, 5, \frac{7}{2}\}, \{\pi, -2, \frac{7}{2}\}, \{5, -2, \frac{7}{2}\}$$

The number of subsets of the set which also has four elements is 1.

$$\{\pi, 5, -2, \frac{7}{2}\}$$

4.2 Formal Principle of Counting, Permutations, and Combinations

4.2.1 Factoral

Definition 4.2.1. Let $n \in \mathbb{N}^*$ **n factoral** is defined as the integer that is

$$\int_0^\infty t^{(n)} \cdot e^{-t} dt$$

and the notation is $n!$

I do not expect anyone in this class to understand the defintion; but, that is the actual definition. We use:

Definition 4.2.2. Let $n \in \mathbb{N}^*$. **n factoral** is defined as the natural number that is $\prod_{k=1}^{n} k$ when $n \in \mathbb{N}$;
is 1 when $n = 0$; and, the notation is $n!$

4.2.2 The Finite Principle of Counting

Theorem 4.2.1. *Let U be a well-defined universe such that U is finite. Consider the disjoint sets A and B such that the cardinality of A is α and the cardinality of B is β. It is the case that $|A \cup B| = \alpha + \beta$.*

Theorem 4.2.2. *(Ordered Counting of Elements From Two Sets)*
Let U be a well-defined universe such that U is finite. Let the disjoint sets A and B be ordered such that the cardinality of A is α and the cardinality of B is β. It is the case that the number of ways an element from A then B is chosen in order is $\alpha \cdot \beta$.

Theorem 4.2.3. *(The Finite Principle of Counting)*
Let U be a well-defined universe such that U is finite. Let the non-empty sets $A_1, A_2, A_3, \ldots, A_k$ be chosen in order such that the cardinality of A_1 is α_1, the cardinality of A_2 is α_2, ..., the cardinality of A_k is α_k.[1] *It is the case that the number of ways an element from each set is chosen in order is $\prod_{m=1}^{k} \alpha_m = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k$.*

Example 4.2.1. *Let us have an urn containing 7 tokens, a box containing 12 bobby-pins, and a can containing 3 tennis balls. The number of ways a person can choose a token from the urn, a bobby-pin from the box, and then a tennis ball from the can is $7 \cdot 12 \cdot 3$.*

[1] Meaning each set A_i has α_i distinct elements.

Theorem 4.2.4. *(The Generalised Principle of Counting)*
Let U be a well-defined universe. Let the sets $A_1, A_2, A_3, \ldots, A_k$ be chosen in order such that the cardinality of A_1 is α_1, the cardinality of A_2 is α_2, ..., the cardinality of A_k is α_k.[2] *It is the case that the number of ways an element from each set may be chosen in order is*
$\prod_{m=1}^{k} \alpha_m = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k.$

Note if any set is infinite; then the number of ways to do such is infinite (not interesting to us yet); whilst if any set is empty; then the number of ways to do such is zero (also dull).

4.2.3 Permutations

Definition 4.2.3. Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$. The **permutations** n things **ordered** k at a time is defined as the integer

$$\frac{n!}{(n-k)!}$$

and the notation is ${}_nP_k \equiv P(n,k)$

Example 4.2.2. *Let U be a well defined universe and A be a set such that $|A| = m$ where $m \in \mathbb{N}$. Let w be a non-negative integer such that $w \leq m$. The number of ways to order w elements from the set A is $m \cdot (m-1) \cdot (m-2) \ldots \cdot (m-(w-1))$ which is equal to ${}_mP_w = \frac{m!}{(m-w)!}$.*

4.2.4 Combinations

Definition 4.2.4. Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.
The **combinations** n things **chosen** k at a time is defined as the integer

$$\frac{n!}{k! \cdot (n-k)!}$$

and the notation is ${}_nC_k \equiv \binom{n}{k}$

Example 4.2.3. *Let U be a well defined universe and A be a set such that $|A| = m$ where $m \in \mathbb{N}$. Let w be a non-negative integer such that $w \leq m$. The number of ways to choose w elements from the set A is $\frac{m \cdot (m-1) \cdot (m-2) \ldots \cdot (m-(w-1))}{w!}$ which is equal to $\binom{m}{w} = \frac{m!}{(m-w)! \cdot w!}$.*

Theorem 4.2.5. *$n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.*

$$P(n,k) \geq \binom{n}{k}$$

[2] Meaning each set A_i has α_i distinct elements.

Theorem 4.2.6. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n > k$.*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Theorem 4.2.7. *$n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Theorem 4.2.8. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{k} = \binom{n}{n-k}$$

Theorem 4.2.9. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{0} = 1$$

Theorem 4.2.10. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{n} = 1$$

Theorem 4.2.11. *$n \in \mathbb{N}$.*

$$\binom{n}{1} = n$$

Theorem 4.2.12. *$n \in \mathbb{N}$.*

$$\binom{n}{n-1} = n$$

4.3 Exercise Section 1

Exercise 4.3.1. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert –in that order. Find the number of ways a meal consists of a Caesar salad, pork entrée, and then custard).

Exercise 4.3.2. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert –in that order. Find the number of ways to create a meal of a salad, then an entrée, then a dessert –in that order.

Exercise 4.3.3. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert. Find the number of ways a meal consists of a Caesar salad, not a pork entrée, and then cake or pie).

Exercise 4.3.4. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing exactly two heads and then twice exactly no heads.

Exercise 4.3.5. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing exactly two heads amongst the tosses.

Exercise 4.3.6. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing at least two heads.

Exercise 4.3.7. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing no heads.

Exercise 4.3.8. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing at most two heads.

Exercise 4.3.9. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 4 red balls.

Exercise 4.3.10. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 4 red balls amongst the six balls chosen.

Exercise 4.3.11. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing at most two heads.[3]

Exercise 4.3.12. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

Exercise 4.3.13. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; and, finally we draw another ball from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

Exercise 4.3.14. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball from the urn; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; and, finally we draw another ball from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

Exercise 4.3.15. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing a red ball; then a blue ball; then exactly 2 white balls, and finally exactly 2 blue balls.

Exercise 4.3.16. Suppose there is an urn (snore) that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing 2 red balls then at least 3 blue balls; then a green ball.

[3] Yes, this is the question.

Exercise 4.3.17. Suppose there is an urn (ugh) that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing 2 red balls initially and then at least 2 blue balls of the last four remaining.

Exercise 4.3.18. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing at least 2 red balls amongst the six chosen.

Exercise 4.3.19. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out two balls from the urn. Find the number of ways of choosing 2 blue balls.

Exercise 4.3.20. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball then another ball from the urn. Find the number of ways of choosing a red ball and then a white ball.

Exercise 4.3.21. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball look at its colour. Put it back shake up the urn and then draw another ball from the urn. Find the number of ways of choosing a red ball and then not drawing a white ball as the second ball.

Exercise 4.3.22. Suppose there is an urn (Lord help us all – McLoughlin is sure adept at copying and pasting) that contains 5 red, 4 white, and 11 blue balls. We draw a ball. We replace the ball shake up the urn and then draw a ball from the urn. Find the number of ways of choosing a blue ball and a white ball.

Exercise 4.3.23. Suppose there is an urn (Please enough!) that contains 5 red, 4 white, and 11 blue balls. We draw a ball. We replace the ball shake up the urn and then draw a ball from the urn. Find the number of ways of choosing at least one red ball.

With dice (unless otherwise stated) we toss a pair of dice and sum up the number of dots of the up-turned faces.

Exercise 4.3.24. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a sum of seven or an eleven.

Exercise 4.3.25. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a seven or an eleven.

Exercise 4.3.26. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a sum of five or better.

Exercise 4.3.27. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a product of the upturned faces of a twelve.

4.4 Exercise Section 2

Exercise 4.4.1. Compute (where such exists):

1. $\binom{7}{4}$
2. $\binom{7}{5}$
3. $\binom{8}{4}$
4. $\binom{8}{5}$
5. $\binom{4}{0}$
6. $\binom{4}{1}$
7. $\binom{4}{2}$
8. $\binom{4}{3}$
9. $\binom{4}{4}$
10. $\binom{4}{5}$
11. $\binom{-4}{-2}$
12. $\binom{14}{13}$

4.4.1 Temporally Distinct vs. Concurrent Exercises

So, hopefully having completed the exercises from the previous section you opined that there was an (or some) important point about the wording of a problem. It, loosely, has to do with the timing of what is being done in an experiment: concurrent timing or temporally distinct.

Example 4.4.1. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw a ball; then another; then another. The number of ways this can be done such that we draw a red ball; then a blue ball; and, then a red ball is* $5 \cdot 11 \cdot 4$.

Example 4.4.2. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw a red ball; put it back; shake up the urn. We then a blue ball; put it back; shake up the urn. We finally draw a red ball. The number of ways this can be done is* $5 \cdot 11 \cdot 5$.

Example 4.4.3. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done such that we draw two red balls and a blue ball is* $\binom{5}{2} \cdot \binom{11}{1}$

Example 4.4.4. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done is* $\binom{20}{3}$

Example 4.4.5. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done such that we draw a non-blue ball and two blue ball is* $\binom{9}{1} \cdot \binom{11}{2}$

4.4.2 Exercise Section 3

Exercise 4.4.2. A survey was done of 304 people regarding programmes they like. 123 people liked *Westworld* , 111 liked *Game of Thrones* , 104 like *Star Trek* , 33 liked all 3 programmes, 45 liked *Game of Thrones* and *Star Trek* , 76 liked *Westworld* and *Star Trek* , while 42 liked *Westworld* and *Game of Thrones* .
A. How many people liked none of the three shows?
B. How many liked *Westworld* only?
C. How many liked exactly two programmes?
D. How many did not like *Game of Thrones*?
E. How many liked neither *Westworld* nor *Star Trek*?
F. How many people liked one and only one programme?

Exercise 4.4.3. Compute (where possible). If not computable; explain why it is not computable. Which are the same; why?

A. $P(7,4)$ B. $\binom{20}{3}$ C. $\binom{20}{17}$ D. $\binom{20}{1}$

E. $P(7,3)$ F. $\binom{-3}{0}$ G. $\binom{-3}{-2}$ H. $\binom{7}{8}$

I. $\binom{3}{2} - \binom{3}{1}$ J. $\binom{4}{3} + \binom{3}{2}$ K. $\binom{7}{5}$ L. $\binom{8}{5}$

M. $\binom{5}{2} + \binom{5}{3}$ N. $\binom{6}{3}$ O. $\binom{10}{5}$ P. $\binom{6}{2}$

Q. $\binom{6}{0}$ R. $\binom{6}{1}$ S. $\binom{6}{4}$ T. $\binom{6}{5}$

U. $\binom{6}{6}$ V. C_4^7 W. P_4^7

For the following exercises: Show all work (where appropriate). If an answer DNE, explain why it does not exist. Note: balanced coin (fair two sided); balanced die (fair six side); and, no one can hold more than one office at a time.

Exercise 4.4.4. You flip a balanced coin and then you roll a balanced die. Find the number of ways to do this.

Exercise 4.4.5. You roll a balanced die and the flip a balanced coin. Find the number of ways to do this.

Exercise 4.4.6. You have 10 Yankee fans create a club. They elect a president and a treasurer. Find the number of ways to do this.

Exercise 4.4.7. You have 10 Philly fans create a club. They pick 2 members to go to the Yankee club and hurl insults at the Yankee fans cheer. Find the number of ways to do this.

Exercise 4.4.8. You have 10 Yankee fans and 10 Philly fans in a barrel. Pick 17 of them to go to Fenway Park. Find the number of ways to do this.

Exercise 4.4.9. You have 10 Yankee fans and 10 Philly fans in a barrel. 3 of them win the lottery. How many ways are there for the 3 of 20 people to win the lottery?

Chapter 5

On the Naïve or Intuitive Idea of Probability

5.1 Introduction

Informal understanding of probability pre-dated the formal theory of probability. In this chapter we will simply look at the ideas that first were encountered.

5.1.1 Finite Sample Space

Definition 5.1.1. Let U be a well defined universe. A **sample space** is a subset of U.

Definition 5.1.2. Let U be a well defined universe. Let S be a well defined sample space. An **event** is a subset of S.

Definition 5.1.3. Let U be a well defined universe. Let S be a well defined sample space. Let A be an event. An **outcome** is an element of A.

The intuitive idea of probability of an event, E, from a sample space, S, is $\frac{|E|}{|S|}$ where S is finite (meaning $|S| < \aleph_0$).

Definition 5.1.4. Let S be a well defined sample space. An outcome is defined as **fair** *iff* the probability that an outcome in the space is picked is equal to the probability that any other outcome in the space is picked (another way to express this is that all outcomes are equiprobable of being selected).

Notation 5.1.1. *Let S be a well defined sample space. Let E be an event. The probability of E occurring is denoted as* $\mathbf{Pr(E)}$ *or* $\mathbf{P(E)}$.

5.1.2 Exercises

Exercise 5.1.1. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing exactly two heads and then twice exactly no heads.

Exercise 5.1.2. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing two heads.

Exercise 5.1.3. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing at least two heads.

Exercise 5.1.4. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing no heads.

Exercise 5.1.5. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing at most two heads.

5.1.3 Non-finite Sample Space

An intuitive idea of probability of an event, E, from a sample space, S, is NOT $\frac{|E|}{|S|}$ because the space we have is **not** finite.

Example 5.1.1. *Let $U = \mathbb{R}$ and $S = \mathbb{N}$ whilst $E = \{x | x = 2 \cdot a, a \in \mathbb{N}\}$. $Pr(E) = \frac{|E|}{|S|}$ fails.*

Example 5.1.2. *Let $U = S = \mathbb{R}$ and $E = \mathbb{N}$. $Pr(E) = \frac{|E|}{|S|}$ fails.*

For a one-dimensional sample space we use length. For a two-dimensional sample space we use area. For a three-dimensional sample space we use volume.

A key is to realise that an outcome is **in** an event (an element belongs to a set) and the probability of the event is a real **number** restricted to the interval of zero to one, inclusive.

Example 5.1.3. *Suppose $U = \mathbb{R}$ and $S = [0, 10]$. Let $E = [0, 5]$. Let $F = [0, 5)$. Let $H = (0, 4)$. Let $J = \{x | x \in S \wedge \exists m \in \mathbb{N} \ni x = 2 \cdot m\}$.*

$$Pr(J) = \frac{\ell(J)}{\ell(S)} = \frac{0}{10}$$
$$Pr(F) = \frac{\ell(F)}{\ell(S)} = \frac{5}{10}$$
$$Pr(E) = \frac{\ell(E)}{\ell(S)} = \frac{5}{10}$$
$$Pr(H) = \frac{\ell(H)}{\ell(S)} = \frac{4}{10}$$
$$Pr(E \cap F^c) = \frac{\ell(E \cap F^c)}{\ell(S)} = \frac{0}{10}$$

5.1.4 Exercises

Exercise 5.1.6. Suppose $U = \mathbb{R}$ and $S = [0, 8]$. Let $E = [1, 3]$. Let $F = [0, 3)$. Let $H = (1, 3)$. Let $J = \{x | x \in S \wedge \exists m \in \mathbb{N} \ni x = 2 \cdot m\}$.
A. Find $Pr(E)$. B. Let $E = [1, 3]$ C. Find $Pr(E^c)$. D. Find $Pr(F)$.
E. Let $G = E - F$ Find $Pr(G)$. F. Find $Pr(H)$. G. Find $Pr(J)$.

Exercise 5.1.7. Suppose $U = \mathbb{R}$ and $S = [1, \pi)$. Let $E = [2, 3]$. Let $F = (2, 3)$. Let $G = (3, 3)$. Let $H = (e, \pi)$. Let $J = (1, \frac{\pi}{2})$. Let $K = (1, \frac{\pi - 1}{2}]$. Let $M = [3, 3]$.
A. Find $Pr(E)$. B. Find $Pr(E^c)$. C. Find $Pr(F)$. D. Find $Pr(G)$.
E. Find $Pr(H)$. F. Find $Pr(J)$. G. Find $Pr(M)$. H. Find $Pr(K)$.

Exercise 5.1.8. Suppose $U = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and let $S = \big\{(x, y)|\ \ (x-1)^2 + (y+2)^2 = 9\big\}$.
A. Let k be the point $(2, 0)$. Find $Pr(k \in S)$. B. Let p be the point $(0, 2)$. Find $Pr(p \notin S)$.
C. Let C be the set of all points in S such that $x > 0$. Find $Pr(C)$.
D. Let D be the set of all points in S such that $x \geqslant 0 \wedge y \geqslant 0$. Find $Pr(D)$.
E. Let E be the set of all points in S such that $(x-1)^2 + (y+2)^2 = 9$. Find $Pr(E)$.
F. Let $F = \big\{(x, y)|\ \ (x-1)^2 + (y-4)^2 < \frac{1}{2}\big\} \bigcap S$. Find $Pr(F)$.
G. Let $G = \{(x, y)|\ \ y = -2x\} \bigcap S$. Find $Pr(G)$.
H. Let $J = \{(x, y)|\ \ y = -2x\} \bigcap S$. Find $Pr(H)$.
J. Let $J = \{(x, y)|\ \ y \leq -2x\} \bigcap S$. Find $Pr(J)$.

Exercise 5.1.9. Suppose $U = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and let $S = \big\{(x, y)|\ \ (x-1)^2 + (y+2)^2 \leqslant 9\big\}$.
With this problem realise that we are referencing the set-theoretic universe for definitions in the exercises and the probabilities computed are those sets intersect the sample space. This is not a 'good' way to state a problem; but, as we shall see it is a way we will sometimes encounter problems (e.g.: see any book).
A. Let k be the point $(2, 0)$. Find $Pr(k \in S)$. B. Let p be the point $(0, 2)$. Find $Pr(p \notin S)$.
C. Let C be the set of all points such that $x > 0$. Find $Pr(C)$.
D. Let D be the set of all points such that $x \geqslant 0 \wedge y \geqslant 0$. Find $Pr(D)$.
E. Let E be the set of all points such that $(x-1)^2 + (y+2)^2 = 9$. Find $Pr(E)$.

Chapter 6

Introduction to Probability Theory

6.1 Introduction

Mathematics is predicated on logic (for the rules of inference) and on Set Theory. From those foundations we build geometry, real analysis, abstract algebra, the theory of differential equations and difference equations, probability theory, point-set topology, general topology, numerical analysis, number theory, combinatorics, complex variables, dynamical systems, graph theory, game theory, statistics, cryptology, etc.

Recall from Set Theory that every set, A, is a subset of some well defined universe, U. So, for probability theory we rename the universe as a sample space [the space from whence a sample may be chosen]; an arbitrary set is called an event; and an element of that set an outcome.

Logic	**Set Theory**	**Probability Theory**
Domain of definition	Universe, U	Sample space, S
	example $U = \{1, 2, 3, 4, 5\}$	example $S = \{1, 2, 3, 4, 5\}$
Statements p, q, and r	Sets A, B, and C.	Events E, F, and G
	example $A = \{1, 2\}$ $B = \{2, 3, 4\}$ $C = \{1, 4\}$	example $E = \{1, 2\}$ $F = \{2, 3, 4\}$ $G = \{1, 4\}$
Simple statements	Elements example	Outcomes example
true statements like 1 is in A	$1 \in A$ and $1 \notin B$	$\{2, 4\} \subseteq F$
false statements like 3 is in A	$1 \in B$, $\{2\} \subseteq C$	$\{2, 4\} \subseteq E$, $4 \in E$

etcetera

What is 'tricky' in doing basic probability proofs is you are working with sets; functions; and real numbers. So, before tackling these claims think about what is the claim. Is it a numeric, set-theoretic, or both? How so? And do you opine it is true or not?

6.2 The Kolmogorov Axioms of Probability

[1]

Let S denote the sample space, E, E_i, F, etc. events and the notation $Pr(\cdot)$ the probability of whatever.

Axiom 6.2.1. *(The Space Axiom) S is the space* $\Rightarrow Pr(S) = 1$.

Axiom 6.2.2. *(The Event Axiom) E is an event* $\Rightarrow 0 \leq Pr(E) \leq 1$.

Axiom 6.2.3. *(The Collection of M. E. E.*[2] *Axiom) Let I be an index set. The collection* $\{E_i\}_{i\in I}$ *being mutually exclusive* $\Rightarrow \Pr(\bigcup_{i\in I} E_i) = \sum_{i\in I} \Pr(E_i)$.

6.2.1 Some Theorems, Lemmas, or Corollaries of Probability

Lemma 6.2.1. *Let S be a well defined sample space; whilst E and F are events.*

1. *E is an event* $\Rightarrow E^c$ *is an event.*
2. *E and F are events* $\Rightarrow E \cap F$ *is an event.*
3. *E and F are events* $\Rightarrow E \cup F$ *is an event.*
4. *E and F are events* $\Rightarrow E - F$ *is an event.*

Corollary 6.2.1. $\mathbb{B}$ $\mathbb{D}1$ *(The Null Corollary) Let S be a well defined sample space whilst* $E = \emptyset$. *It is the case that* $Pr(E) = 0$.

Corollary 6.2.2. $\mathbb{B}$ $\mathbb{D}1$ *(The Complement Corollary) E is an event* $\Rightarrow Pr(E^c) = 1 - Pr(E)$.

Corollary 6.2.3. $\mathbb{B}$ $\mathbb{D}1$ *(The Subset Corollary) E and F are events* $\ni E \subseteq F \Rightarrow Pr(E) \leq Pr(F)$.

Corollary 6.2.4. $\mathbb{B}$ $\mathbb{D}5$ *(The Spanning Corollary) Let I be an index set. The collection* $\{E_i\}_{i\in I}$ *being mutually exclusive and exhaustive* $\Rightarrow \Pr(\bigcup_{i\in I} E_i) = \sum_{i\in I} \Pr(E_i) = 1$.

Corollary 6.2.5. $\mathbb{B}$ $\mathbb{D}1$ *(The Same Event Corollary) E and F are events* $\ni E = F \Rightarrow Pr(E) = Pr(F)$.

Theorem 6.2.1. $\mathbb{B}$ $\mathbb{D}2$ *(The Union Theorem) E and F are events.* $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$.

Note some of the other axioms lists from other classes that are of great use to us in this class.

[1] The Kolmogorov Axioms of Probability are named after the creator of the axioms: the Russian mathematician Kolmogorov. These are so named since he created them as an answer to one of the famous Hilbert 20^{th} century 10 questions. The axioms of probability are one of the shortest lists of axioms I can recall for an area of mathematics.

[2] M. E. E. : Mutually Exclusive Events

6.2.2 The Axioms of Set Theory

Axiom 6.2.4. *(The Axiom of Extension) Two sets are equal iff they have the same elements.*

Axiom 6.2.5. *(The Axiom of Null) There exists a set with no elements, call it* $\emptyset$

Axiom 6.2.6. *(The Axiom of Pairing) Given any sets A and B, there exists a set C whose elements are A and B.*

Axiom 6.2.7. *(The Axiom of Union) Given any set A, the union of all elements in A is a set.*

Axiom 6.2.8. *(The Axiom of Power Set) Given any set A, there exists a set B consisting of all the subsets of A.*

Axiom 6.2.9. *(The Axiom of Separation) Given any set A and a sentence* $p(a)$ *that is a statement for all* $a \in A$*, then there exists a set* $B = \{a \in A : p(a) \text{ is true}\}$.

Axiom 6.2.10. *(The Axiom of Replacement) Given any set A and a function f defined on A, the image* $f[A]$ *is a set.*

Axiom 6.2.11. *(The Axiom of Infinity) There exists a set A such that* $\emptyset \in A$*, and whenever* $a \in A$*, it follows that* $a \cup \{a\} \in A$

Axiom 6.2.12. *(The Axiom of Regularity) Given any non-empty set A, there exists an ainA such that* $a \cap A = \emptyset$.

Axiom 6.2.13. *(The Axiom of Choice) Given any non-empty set A whose members are pair-wise disjoint non-empty sets, there exists a set B consisting of exactly one element taken from each set belonging to A.*

6.2.3 The Peano Axioms for $\mathbb{N}$

The properties of addition of natural numbers are be derived from a short set of axioms which also form the basis for mathematical induction. The axioms are called **the Peano Axioms**: There exists a set, P, which is defined by the following four axioms.

Axiom 6.2.14. *There exists a natural number, call it 1, that is not the successor of any other natural number.*

Axiom 6.2.15. *Every natural number has a unique successor. If* $k \in P$ *, then let* $k\prime$ *denote the successor of k.*

Axiom 6.2.16. *Every natural number except one is the successor of exactly one natural number.*

Axiom 6.2.17. *If M is* ***a*** *set of natural numbers such that*

1. $1 \in M$ *and*
2. *for each* $k \in P$*, if* $k \in M$*, then* $k' \in P$*,*

then $P = M$.

P, of course is $\mathbb{N}$.

So, the Peano axioms assert the uniqueness of the naturals that this successor property along with the element 1 creates the entirety of the natural numbers. No matter how you name the set (you can call it Ray, or you can call it Jay, . . .) if it has these properties then it really is the naturals. From these axioms arise the natural numbers by defining what addition by one means.

Definition 6.2.1. For every $k \in \mathbb{N}$, define $k + 1 = k'$.

Then, note inductively, the entire understanding of addition flows from this definition (likewise multiplication, etc.). Also we note the Archimedean Principle of $\mathbb{N}$ states that $\mathbb{N}$ is unbounded above; in other words, there does not exist a greatest natural number. It is worth repeating that the Peano axioms and the Archimedean principle provide a justification for the concept of mathematical induction.

6.2.4 The Axioms of the Reals: Field, Order, and Completeness

Axiom 6.2.18. *(Closure of Addition)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x + y) \in \mathbb{R}$ *and* $(x = w \wedge y = v) \Rightarrow (x + y = w + v)$

Axiom 6.2.19. *(Commutativity of Addition)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x + y) = (y + x)$

Axiom 6.2.20. *(Associativity of Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x + y) + z = x + (y + z)$

Axiom 6.2.21. *(Existence of Identity of Addition)* $\exists$ *a unique number* $0 \ni x + 0 = 0 + x \forall\ x \in \mathbb{R}$.[3]

Axiom 6.2.22. *(Existence of Additive Inverse)* $\forall\ x \in \mathbb{R}$ *it is the case that there exists a unique number* $-x$ *such that* $x + (-x) = (-x) + x = 0$.

Axiom 6.2.23. *(Closure of Multiplication)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x \cdot y) \in \mathbb{R}$ *and* $(x = w \wedge y = v) \Rightarrow (x \cdot y = w \cdot v)$

Axiom 6.2.24. *(Commutativity of Multiplication)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x \cdot y) = (y \cdot x)$

Axiom 6.2.25. *(Associativity of Multiplication)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Axiom 6.2.26. *(Existence of Identity of Multiplication)* $\exists$ *a unique number* $1 \ni x \cdot 1 = 1 \cdot x \forall\ x \in \mathbb{R}$. *Moreover,* $0 \neq 1$.

Axiom 6.2.27. *(Existence of Multiplicative Inverse)* $\forall\ x \in \mathbb{R} \ni x \neq 0$ *it is the case that there exists a unique number* x^{-1} *such that* $x \cdot (x^{-1}) = (x^{-1}) \cdot x = 1$.

Axiom 6.2.28. *(Axiom of the Distribution of Multiplication over Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Axiom 6.2.29. *(Trichotomy)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x < y) \vee (x = y) \vee (x > y)$ *and moreover it is not the case that any two of these conditions can simultaneously hold.*

[3] The existence of a unique element is also denoted as $\exists!$.

Axiom 6.2.30. *(Transitivity)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x < y) \wedge (y < z) \implies (x < z)$.

Axiom 6.2.31. *(Preservation of Order under Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x < y) \implies (x + z) < (y + z)$.

Axiom 6.2.32. *(Preservation of Order for a Positive Multiplier)* $\forall\ x, y \in \mathbb{R}\ \wedge z \in \mathbb{R}\ \ni 0 < z$ *it is the case that* $(x < y) \implies (x \cdot z) < (y \cdot z)$.

Axiom 6.2.33. *(Completeness)* $\forall A \subseteq \mathbb{R} \ni A$ *is bounded above*[4] $\exists\ m \in \mathbb{R}$ *such that it is the supremum*[5] *of the set A.*

6.2.5 Important Lemmas of Use

We need certain basic properties of sets and real numbers so we may assume:
Let our universe be $\mathbb{R}$.

Lemma 1: $0 < 1$.

Lemma 2: Let $x \in \mathbb{R}$ It is the case that $x \cdot 0 = 0$.

Lemma 3: $(-1) \cdot (-1) = 1$.

Lemma 4: Let $x \in \mathbb{R}$. It is the case that $(-1) \cdot x = -x$

Lemma 5: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x - y = x + (-y) = x + -y$.

Definition 6.2.2. Let $x \in \mathbb{R}$. It is the case that $x \cdot x = x^2$.

Definition 6.2.3. Let $x \in \mathbb{R} \wedge n \in \mathbb{N}$. It is the case that $x \cdot x = x^2$ whilst $\underbrace{x \cdot \ldots \cdot x}_{n}$ is x^n

Definition 6.2.4. (Law of Exponents) Let $x \in \mathbb{R}$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$.

1. $x^a \cdot x^b = x^{(a+b)}$
2. $x^a \cdot b^a = (x \cdot b)^a$
3. $x^a \div x^b = x^{(a-b)}$ when $x^b \neq 0$
4. $(x^a)^b = x^{a \cdot b}$

Definition 6.2.5. Let $x \in \mathbb{R}$. It is the case that x is **positive** if and only if $x > 0$.

Definition 6.2.6. Let $x \in \mathbb{R}$. It is the case that x is **non-negative** if and only if $x \geq 0$.

Definition 6.2.7. Let $x \in \mathbb{R}$. It is the case that x is **non-positive** if and only if $x \leq 0$.

Definition 6.2.8. Let $x \in \mathbb{R}$. It is the case that x is **negative** if and only if $x < 0$.

[4] A set being bounded above is to be defined later. The inclusion of this axiom is due to my desire to put the axioms together at the beginning of the tome.

[5] A set having a supremum is to be defined later.

6.2.6 For Number Theoretic Claims

Properties of Natural, Integers, or Rational Numbers that you may assume:

Definition 6.2.9. (Closure of addition in $\mathbb{N}$) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m + n) \in \mathbb{N}$.

Definition 6.2.10. (Closure of multiplication in $\mathbb{N}$) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m \cdot n) \in \mathbb{N}$.

Definition 6.2.11. (Closure of addition in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a + b) \in \mathbb{Z}$.

Definition 6.2.12. (Closure of subtraction in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a - b) \in \mathbb{Z}$.

Definition 6.2.13. (Closure of multiplication in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(m \cdot n) \in \mathbb{Z}$.

Definition 6.2.14. (Closure of addition in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p + q) \in \mathbb{Q}$.

Definition 6.2.15. (Closure of subtraction in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p - q) \in \mathbb{Q}$.

Definition 6.2.16. (Closure of multiplication in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p \cdot q) \in \mathbb{Q}$.

Definition 6.2.17. (Closure of non-zero division in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ where $q \neq 0$, then $\frac{p}{q} \in \mathbb{Q}$.

6.2.7 For Claims About Odd or Even Natural Numbers

Definition 6.2.18. Let $m \in \mathbb{N}$. m is even if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j$.

Definition 6.2.19. Let $m \in \mathbb{N}$. m is odd if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j - 1$.

6.2.8 For Claims About Odd or Even Integers

Definition 6.2.20. Let $w \in \mathbb{Z}$. w is even if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p$.

Definition 6.2.21. Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p + 1$.
Version 2 of 5.2.21: Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer q (meaning $q \in \mathbb{Z}$) such that $w = 2 \cdot q - 1$.

If there are **any other** seemingly 'obvious' definitions, lemmas, theorems, corollaries, laws, etc. *you wish to cite for a proof for class, please ask about it as soon as possible.*

6.3 Basic Claims To Prove or Disprove

Note: These claims are to be proven or disproven **individually** - there is to be no conferring on a claim until **someone presents a particular claim**.[6] Therefore there is to be no discussing problems on this with other people, a professor other than me, using a tutor (good luck with that), or working with another person.

All claims in this section are board-worthy. They are of a varying degree of difficulty that shan't be defined (yet).

Claim 6.3.1. Let S be a well defined sample space (w.d.s.s.) whilst $E = \emptyset$. So, $Pr(E) = 0$.

Claim 6.3.2. Let S be a well defined sample space (w.d.s.s.) whilst $Pr(E) = 0$.
It is the case that $E = \emptyset$.

Claim 6.3.3. Let S be a w.d.s.s. whilst E is an event. It is the case that $Pr(E^c) = 1 - Pr(E)$.

Claim 6.3.4. Let S be a w.d.s.s. whilst E is an event. $Pr(E) > \frac{1}{2} \implies Pr(E^c) < \frac{1}{2}$

Claim 6.3.5. Let S be a w.d.s.s. whilst E is an event. $Pr(E) > \frac{1}{2} \implies Pr(E^c) \leqslant \frac{1}{2}$

Claim 6.3.6. Let S be a w.d.s.s. whilst E is an event. $Pr(E) \geqslant \frac{1}{2} \implies Pr(E^c) < \frac{1}{2}$

Claim 6.3.7. Let S be a w.d.s.s. whilst E, F are events such that $E \subseteq F$.
It is the case that $Pr(E) \leqslant Pr(F)$

Claim 6.3.8. Let S be a w.d.s.s. whilst E, F are events $\ni E \subset F$.
It is the case that $Pr(E) < Pr(F)$.

Claim 6.3.9. Let S be a w.d.s.s. whilst E, F are events $\ni E = F$.
It is the case that $Pr(E) = Pr(F)$.

Claim 6.3.10. Let S be a w.d.s.s. whilst E, F are events $\ni Pr(E) = Pr(F)$.
It is the case that $E = F$.

Claim 6.3.11. Let S be a w.d.s.s. whilst E, F are events. $Pr(E) > \frac{1}{2} \implies Pr(F) < \frac{1}{2}$

Claim 6.3.12. Let S be a w.d.s.s. whilst E, F are mutually exclusive events.
$Pr(E) > \frac{1}{2} \implies Pr(F) < \frac{1}{2}$

Claim 6.3.13. Let S be a w.d.s.s. whilst E, F are events. Let $G = E - F$.
It is the case that $Pr(G) = Pr(E) - Pr(F)$.

Claim 6.3.14. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cap F) = Pr(E) + Pr(F)$

Claim 6.3.15. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Claim 6.3.16. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

Claim 6.3.17. Let S be a w.d.s.s. whilst $E, F, \wedge G$ are events.
It is the case that $Pr(E \cup F \cup G) = Pr(E) + Pr(F) + Pr(G) - Pr(E \cap F \cap G)$

Claim 6.3.18. Let S be a w.d.s.s. whilst $E, F, \wedge G$ are events where E and F are mutually exclusive and E and G are mutually exclusive.
It is the case that $Pr(E \cup F \cup G) = Pr(E) + Pr(F) + Pr(G) - Pr(F \cap G)$

[6] Pretend it is day 5 and someone presents claim 3.12 on the board and 3.6 has not been presented on this day or any day previous. So, you may discuss claim 3.12 **since it is after** it has been presented. But you can't discuss, let us say, 3.6 since it has not been presented yet.

6.4 More Claims To Prove or Disprove

Claim 6.4.1. There exists a w.d.s.s. S and events E, F such that $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Claim 6.4.2. There exists a w.d.s.s. S and events E, F such that $Pr(E \cup F) \geqslant 1$

Claim 6.4.3. There exists a w.d.s.s. S and events E, F such that $Pr(E \cup F) > 1$

Claim 6.4.4. $\exists$ a w.d.s.s. S and event E such that $Pr(E) \in \mathbb{I}$

Claim 6.4.5. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E) + Pr(F) > 1$

Claim 6.4.6. $\exists$ a w.d.s.s. S and event E such that $E \neq S$ but $Pr(E) = Pr(S)$

Claim 6.4.7. $\exists$ a w.d.s.s. S and event E such that $Pr(E) > Pr(S)$

Claim 6.4.8. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E \cup F) > Pr(E) + Pr(F)$

Claim 6.4.9. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E \cup F) < Pr(E) + Pr(F)$

References:

Einführung in die Theorie Gammafunktion, 1931, Artin, E.
Advanced Calculus, 1956, Buck, R. C.

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