CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1.1 Introduction to Systems of Linear Equations

2. Equations in part (a) do not form a linear system since the second one is not a linear equation.

Equations in parts (b), (c), and (d) form linear systems.

4. **(b)** We can solve this system by inspection since the second equation is a multiple of the first. Letting $x = 4$ solves both equations, therefore the system is consistent.

(c) An easy way to find an example of a solution of this system would be by letting $y = 0$ first, then look for values of x and z that solve the resulting system

$$\begin{aligned} 4x + 2z &= -1 \\ -x - 3z &= 0 \end{aligned}$$

Adding 4 times the second equation to the first will yield a simplified system

$$\begin{aligned} -10z &= -1 \\ -x - 3z &= 0 \end{aligned}$$

which is solved by $z = \frac{1}{10}$ and $x = -\frac{3}{10}$. Together with $y = 0$, this is a solution of the original system, showing it to be consistent.

(d) We add -6 times the first equation to the third, then proceed to add the first equation to the fourth, and the second one to the fourth, obtaining a simplified system

$$\begin{aligned} 3z + x &= -4 \\ y + 5z &= 1 \\ -16z &= 27 \\ 7z &= 1 \end{aligned}$$

Clearly, the last two equations are contradictory, thus the system is inconsistent.

6. For example,

(a)
$$x+y+z=1$$
$$x+y+z=2$$
$$x+y+z=3$$

(b)
$$x=1$$
$$y=2$$
$$z=3$$

(c)
$$x+y+z=1$$
$$x+y+z=1$$
$$x+y+z=1$$

8. The values in (b), (d), and (e) satisfy all three equations – these vectors are solutions of the system.

The vectors in (a) and (c) are not solutions of the system.

10. (a) $x_1=\frac{7+5s-4t}{3}, x_2=s, x_3=t$ where s and t are arbitrary values

(b) $v=\frac{8r-2s+t-4u}{3}, w=r, x=s, y=t, z=u$ where r, s, t, and u are arbitrary values

12. **(a)**
$$\begin{aligned} 2x &= -1 \\ -4x &= -6 \\ x &= -1 \\ 3x &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 3x-y-z &= -1 \\ 5w+2x-3z &= -6 \end{aligned}$$

(c)
$$\begin{aligned} x+2y+3z &= 4 \\ -4x-3y-2z &= -1 \\ 5x-6y+z &= 1 \\ -8x &= 3 \end{aligned}$$

(d)
$$\begin{aligned} 3w+y-4z &= 3 \\ -4w+4y+z &= -3 \\ -w+3x-2z &= -9 \\ -z &= -2 \end{aligned}$$

14. **(a)** $\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 3 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 7 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

1.2 Gaussian Elimination

2. **(a)** This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (2) above it. The matrix is in row echelon form but not reduced row echelon form.

(b) This matrix does not have property 1 since its first nonzero number in the third row (2) is not a 1. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(c) This matrix has properties 1-3 but does not have property 4: the third column contains a leading 1 and a nonzero number (4) above it. The matrix is in row echelon form but not reduced row echelon form.

(d) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (5) above it. The matrix is in row echelon form but not reduced row echelon form.

(e) This matrix does not have property 2 since the row that consists entirely of zeros is not at the bottom of the matrix. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(f) This matrix does not have property 3 since the leading 1 in the second row is directly below the leading 1 in the first (instead of being farther to the right). The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(g) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

4. **(a)** A unique solution: $x = -3,\ y = 0,\ z = 7.$

(b) Infinitely many solutions: $w = 8 + 7t,\ x = 2 - 3t,\ y = -5 - t, z = t$ where t is an arbitrary value.

(c) Infinitely many solutions: $v = -2 + 6s - 3t,\ w = s,\ x = 7 - 4t,\ y = 8 - 5t,\ z = t$ where s and t are arbitrary values.

(d) The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

6.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← 2 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← -8 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{7}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 7 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ← -1 times the second row was added to the first row.

Infinitely many solutions: $x_1 = -\frac{1}{7} - \frac{3}{7}t,\ x_2 = \frac{1}{7} - \frac{4}{7}t,\ x_3 = t$ where t is an arbitrary value.

8.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ⟵ The augmented matrix for the system.

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ⟵ The first and second rows were interchanged.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ⟵ The first row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ⟵ -6 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ⟵ The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}$$ ⟵ 6 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ⟵ The third row was multiplied by $\frac{1}{6}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ⟵ $\frac{1}{2}$ times the third row was added to the second row.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ⟵ $\frac{2}{3}$ times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ⟵ -2 times the second row was added to the first row.

The last row corresponds to the equation

$$0a + 0b + 0c = 1$$

therefore the system is inconsistent.

(Note: this was already evident after the fifth elementary row operation.)

10.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← 2 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← -8 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{7}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 7 times the second row was added to the third row.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + \frac{4}{7}x_3 &= \frac{1}{7} \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$x_1 = -x_2 - x_3$$
$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

then substitute the second equation into the first

$$x_1 = -\frac{1}{7} - \frac{3}{7}x_3$$
$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$$x_1 = -\frac{1}{7} - \frac{3}{7}t, \qquad x_2 = \frac{1}{7} - \frac{4}{7}t, \qquad x_3 = t$$

12.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first and second rows were interchanged.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← -6 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}$$ ← 6 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{6}$.

The system of equations corresponding to this augmented matrix in row echelon form

$$\begin{aligned} a + 2b - c &= -\frac{2}{3} \\ b - \frac{3}{2}c &= -\frac{1}{2} \\ 0 &= 1 \end{aligned}$$

is clearly inconsistent.

14. The system does not have nontrivial solutions.

(The third equation requires $x_3 = 0$, which substituted into the second equation yields $x_2 = 0$. Both of these substituted into the first equation result in $x_1 = 0$.)

16. The second equation is a multiple of the first one.

This system has nontrivial solutions (e.g., $x_1 = 2$ and $x_2 = 3$).

18. We present two different solutions.

Solution I uses Gauss-Jordan elimination

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$ ← The first row was added to the second row.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix}$$ ← -1 times the first row was added to the third row.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix}$$ ← The second row was multiplied by $\frac{2}{3}$.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$ ← $-\frac{3}{2}$ times the second row was added to the third row.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{10}$.

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← 3 times the third row was added to the second row and $\frac{3}{2}$ times the third row was added to the first row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← $\frac{1}{2}$ times the second row was added to the first row.

Unique solution: $x = 0,\ y = 0,\ z = 0$.

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion "Some Facts About Echelon Forms" on p. 21)

$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ -1 & 2 & -3 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$ ⟵ The first and third rows were interchanged (to avoid introducing fractions into the first row).

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$ ⟵ The first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & -11 & 0 \end{bmatrix}$ ⟵ -2 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{bmatrix}$ ⟵ The second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by $-\frac{1}{10}$.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -1 times the third row was added to the second row.

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -4 times the third row was added to the first row.

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{1}{3}$.

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -1 times the second row was added to the first row.

Unique solution: $x = 0,\ y = 0,\ z = 0$.

20. $\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ⟵ The first and second rows were interchanged.

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{bmatrix}$$ ← -2 times the first row was added to the third row and 4 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← -2 times the second row was added to the third row and the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← $-\frac{1}{2}$ times the second row was added to the first row.

If we assign w and x the arbitrary values s and t, respectively, the general solution is given by the formulas

$$u = \frac{7}{2}s - \frac{5}{2}t, \quad v = -3s + 2t, \quad w = s, \quad x = t$$

22.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix}$$ ← -1 times the first row was added to the second row, -2 times the first row was added to the fourth row, and -1 times the first row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix}$$ ← 2 times the second row was added to the third row, 10 times the second row was added to the fourth row, and 5 times the second row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{41}{2} & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← -21 times the third row was added to the fourth row and -9 times the third row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← The fourth row was multiplied by $\frac{2}{41}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← $-\frac{17}{2}$ times the fourth row was added to the fifth row.

The augmented matrix in row echelon form corresponds to the system

$$\begin{aligned} x_1 + 3x_2 \quad\quad + x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \\ x_3 - \frac{3}{2}x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

Using back-substitution, we obtain the unique solution of this system

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0$$

24. $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$ ← The first and third rows were interchanged.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix}$ ← The first row was added to the second row and -2 times the first row was added to the last row.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix}$ ← The second and third rows were interchanged.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$ ← -3 times the second row was added to the fourth row.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$ ← The third row was multiplied by $-\frac{1}{3}$.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← 3 times the third row was added to the fourth row.

$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← -1 times the third row was added to the second row.

$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← 2 times the second row was added to the first row.

If we assign Z_2 and Z_5 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$Z_1 = -s - t, \quad Z_2 = s, \quad Z_3 = -t, \quad Z_4 = 0, \quad Z_5 = t$$

26. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & -(a^2-3) & a \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix}$ ← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix}$ ← The second row was multiplied by $-\frac{1}{6}$.

The system has no solutions when $a = \sqrt{2}$ or $a = -\sqrt{2}$ (since the third row of our last matrix would then correspond to a contradictory equation).

For all remaining values of a (i.e., $a \neq \sqrt{2}$ and $a \neq -\sqrt{2}$) the system has exactly one solution.

There is no value of a for which this system has infinitely many solutions.

28. $\begin{bmatrix} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & a^2+1 & 3a \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & a^2-6 & 3a+7 \end{bmatrix}$ ← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & a^2-9 & 3a+9 \end{bmatrix}$ ← -1 times the second row was added to the third row.

The system has no solutions when $a = 3$ (since the third row of our last matrix would then correspond to a contradictory equation).

The system has exactly one solution when $a \neq 3$ and $a \neq -3$.

The system has infinitely many solutions when $a = -3$.

30. $\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2a+b \\ 0 & 3 & 3 & c \end{bmatrix}$ ← -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 3 & 3 & c \end{bmatrix}$ ← The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 3 & -3a+\frac{3}{2}b+c \end{bmatrix}$$ ← -3 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 1 & 0 & 2a-\frac{b}{2}-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$$ ← -1 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & a-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$$ ← -1 times the second row was added to the first row.

The system has exactly one solution: $x_1 = a-\frac{c}{3}$, $x_2 = a-\frac{b}{2}$, and $x_3 = -a+\frac{b}{2}+\frac{c}{3}$.

32.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix}$$ ← -1 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 2 & 1 & 3 \end{bmatrix}$$ ← The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & -5 & -1 \end{bmatrix}$$ ← -2 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & 1 & 86 \end{bmatrix}$$ ← -3 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & -2 & -29 \end{bmatrix}$$ ← The second and third rows were interchanged.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & 0 & 143 \end{bmatrix}$$ ← 2 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & 0 & 1 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{143}$.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← -86 times the third row was added to the second row and -2 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← -3 times the second row was added to the first row.

34. We begin by substituting $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$ so that the system becomes

$$\begin{aligned} 2x &- y &+ 3z &= 3 \\ 4x &+ 2y &- 2z &= 2 \\ 6x &- 3y &+ z &= 9 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$ ← -2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The third row was multiplied by $-\frac{1}{8}$.

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← 8 times the third row was added to the second row and -3 times the third row was added to the first row.

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{4}$.

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The second row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

This system has exactly one solution $x = 1$, $y = -1$, $z = 0$.

The only angles α, β, and γ that satisfy the inequalities $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, $0 \le \gamma < \pi$ and the equations

$$\sin\alpha = 1, \qquad \cos\beta = -1, \qquad \tan\gamma = 0$$

are $\alpha = \frac{\pi}{2}$, $\beta = \pi$, and $\gamma = 0$.

36. We begin by substituting $a = \frac{1}{x}$, $b = \frac{1}{y}$, and $c = \frac{1}{z}$ so that the system becomes

$$\begin{array}{rcrcrcl} a & + & 2b & - & 4c & = & 1 \\ 2a & + & 3b & + & 8c & = & 0 \\ -a & + & 9b & + & 10c & = & 5 \end{array}$$

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row and the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{bmatrix}$ ⟵ The second row was multiplied by -1.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{bmatrix}$ ⟵ -11 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & -\frac{8}{91} \end{bmatrix}$ ⟵ The third row was multiplied by $\frac{1}{182}$.

Using back-substitution, we obtain

$$c = -\frac{8}{91} \qquad \Rightarrow \quad z = \frac{1}{c} = -\frac{91}{8}$$

$$b = 2 + 16c = \frac{54}{91} \qquad \Rightarrow \quad y = \frac{1}{b} = \frac{91}{54}$$

$$a = 1 - 2b + 4c = -\frac{7}{13} \quad \Rightarrow \quad x = \frac{1}{a} = -\frac{13}{7}$$

38. Each point on the curve yields an equation, therefore we have a system of three equations

$$\begin{array}{lrcrcrcrcl} \text{equation corresponding to } (-2,7): & 53a & - & 2b & + & 7c & + & d & = & 0 \\ \text{equation corresponding to } (-4,5): & 41a & - & 4b & + & 5c & + & d & = & 0 \\ \text{equation corresponding to } (4,-3): & 25a & + & 4b & - & 3c & + & d & = & 0 \end{array}$$

The augmented matrix of this system $\begin{bmatrix} 53 & -2 & 7 & 1 & 0 \\ 41 & -4 & 5 & 1 & 0 \\ 25 & 4 & -3 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{29} & 0 \\ 0 & 1 & 0 & -\frac{2}{29} & 0 \\ 0 & 0 & 1 & -\frac{4}{29} & 0 \end{bmatrix}$$

If we assign d an arbitrary value t, the general solution is given by the formulas

$$a = -\frac{1}{29}t, \quad b = \frac{2}{29}t, \quad c = \frac{4}{29}t, \quad d = t$$

(For instance, letting the free variable d have the value -29 yields $a = 1,\ b = -2$, and $c = -4$.)

40. (a) 3 (this will be the number of leading 1's if the matrix has no rows of zeros)

(b) 5 (if all entries in B are 0)

(c) 2 (this will be the number of rows of zeros if each column contains a leading 1)

42. (a) Either the three lines properly intersect at the origin, or two of them completely overlap and the third one intersects them.

(b) All three lines completely overlap one another.

(Since the exercise specifically mentions three lines, we should assume that in each equation at least one of the line coefficients is nonzero.)

1.3 Matrices and Matrix Operations

2. (a) Defined; 1×1 matrix

(b) Undefined (A 3×1 matrix A cannot be multiplied by a 6×3 matrix B^T)

(c) Defined; 6×1 matrix

(d) Undefined ($2A$ is a 3×1 matrix, which cannot be added to a 6×2 matrix C)

(e) Defined; 2×3 matrix

(f) Undefined (CD is a 6×6 matrix, which cannot be added to a 6×1 matrix $B^T E^T$)

(g) Defined; 3×6 matrix

(h) Undefined (DC is a 2×2 matrix, which cannot be added to a 1×1 matrix EA)

4. **(a)** $\begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(d) Undefined **(e)** $\begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{bmatrix}$ **(f)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$ **(h)** $\begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$ **(i)** $\begin{bmatrix} 65 & 26 & 69 \\ 185 & 69 & 182 \end{bmatrix}$

(j) Undefined **(k)** 46 **(l)** Undefined (trace is only defined for square matrices)

6. **(a)** $\begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$ **(b)** Undefined **(c)** $\begin{bmatrix} 2 & -10 & 11 \\ 13 & 2 & 5 \\ 4 & -3 & 13 \end{bmatrix}$

(d) $\begin{bmatrix} 10 & -6 \\ -14 & 2 \\ -1 & -8 \end{bmatrix}$ **(e)** $\begin{bmatrix} 40 & 72 \\ 26 & 42 \end{bmatrix}$ **(f)** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. **(a)** first column of $AB = A$ [first column of B] $= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix}$

(b) third column of $BB = B$ [third column of B] $= \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 18 \\ 74 \end{bmatrix}$

(c) second row of BB = [second row of B] $B = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 22 & 18 \end{bmatrix}$

(d) first column of $AA = A$ [first column of A] $= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 48 \\ 24 \end{bmatrix}$

(e) third column of $AB = A$ [third column of B] $= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix}$

(f) first row of BA = [first row of B] $A = \begin{bmatrix} 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 70 \end{bmatrix}$

10. (a) first column of $AB = 6\begin{bmatrix}3\\6\\0\end{bmatrix} + 0\begin{bmatrix}-2\\5\\4\end{bmatrix} + 7\begin{bmatrix}7\\4\\9\end{bmatrix} = \begin{bmatrix}67\\64\\63\end{bmatrix}$

second column of $AB = -2\begin{bmatrix}3\\6\\0\end{bmatrix} + 1\begin{bmatrix}-2\\5\\4\end{bmatrix} + 7\begin{bmatrix}7\\4\\9\end{bmatrix} = \begin{bmatrix}41\\21\\67\end{bmatrix}$

third column of $AB = 4\begin{bmatrix}3\\6\\0\end{bmatrix} + 3\begin{bmatrix}-2\\5\\4\end{bmatrix} + 5\begin{bmatrix}7\\4\\9\end{bmatrix} = \begin{bmatrix}41\\59\\57\end{bmatrix}$

(b) first column of $BA = 3\begin{bmatrix}6\\0\\7\end{bmatrix} + 6\begin{bmatrix}-2\\1\\7\end{bmatrix} + 0\begin{bmatrix}4\\3\\5\end{bmatrix} = \begin{bmatrix}6\\6\\63\end{bmatrix}$

second column of $BA = -2\begin{bmatrix}6\\0\\7\end{bmatrix} + 5\begin{bmatrix}-2\\1\\7\end{bmatrix} + 4\begin{bmatrix}4\\3\\5\end{bmatrix} = \begin{bmatrix}-6\\17\\41\end{bmatrix}$

third column of $BA = 7\begin{bmatrix}6\\0\\7\end{bmatrix} + 4\begin{bmatrix}-2\\1\\7\end{bmatrix} + 9\begin{bmatrix}4\\3\\5\end{bmatrix} = \begin{bmatrix}70\\31\\122\end{bmatrix}$

12. (a) $A = \begin{bmatrix}1 & -2 & 3\\2 & 1 & 0\\0 & -3 & 4\\1 & 0 & 1\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}-3\\0\\1\\5\end{bmatrix}$; the matrix equation: $\begin{bmatrix}1 & -2 & 3\\2 & 1 & 0\\0 & -3 & 4\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-3\\0\\1\\5\end{bmatrix}$

(b) $A = \begin{bmatrix}3 & 3 & 3\\-1 & -5 & -2\\0 & -4 & 1\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}-3\\3\\0\end{bmatrix}$; the matrix equation: $\begin{bmatrix}3 & 3 & 3\\-1 & -5 & -2\\0 & -4 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-3\\3\\0\end{bmatrix}$

14. (a)
$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 2\\ 4x_1 + 3x_2 + 7x_3 &= -1\\ -2x_1 + x_2 + 5x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 3w - 2x \qquad\quad + z &= 0\\ 5w \qquad + 2y - 2z &= 0\\ 3w + x + 4y + 7z &= 0\\ -2w + 5x + y + 6z &= 0 \end{aligned}$$

16. $[2 \quad 2 \quad k]\begin{bmatrix}1 & 2 & 0\\2 & 0 & 3\\0 & 3 & 1\end{bmatrix}\begin{bmatrix}2\\2\\k\end{bmatrix} = [2 \quad 2 \quad k]\begin{bmatrix}6\\3k+4\\k+6\end{bmatrix} = k^2 + 12k + 20 = (k+10)(k+2)$

The values of k that satisfy the equation are $k = -10$ and $k = -2$.

18. The given matrix equation is equivalent to the linear system

$$\begin{array}{rcrcrcrcl} a & - & b & & & & & = & 8 \\ a & + & b & & & & & = & 1 \\ & & & & c & + & 3d & = & 7 \\ & & & - & c & + & 2d & = & 6 \end{array}$$

After subtracting first equation from the second, adding the third to the fourth, and back-substituting, we obtain the solution: $a = \frac{9}{2}$, $b = -\frac{7}{2}$, $c = -\frac{4}{5}$, and $d = \frac{13}{5}$.

24. **(a)** $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

28. There are infinitely many such matrices, e.g. for any real value r, the matrix $A = \begin{bmatrix} ry & (1-r)x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ will work. (There are other possible matrices as well, e.g., $\begin{bmatrix} y & 0 & 0 \\ z & 0 & -x \\ 0 & z & -y \end{bmatrix}$.)

1.4 Inverses; Algebraic Properties of Matrices

2. **(a)** $a(BC) = (aB)C = B(aC) = \begin{bmatrix} -72 & -248 & -132 \\ 28 & 68 & 88 \\ 44 & -108 & 152 \end{bmatrix}$

(b) $A(B-C) = AB - AC = \begin{bmatrix} 20 & -32 & -23 \\ 1 & -84 & -23 \\ -13 & -52 & 2 \end{bmatrix}$

(c) $(B+C)A = BA + CA = \begin{bmatrix} 20 & -30 & -9 \\ -10 & 37 & 67 \\ -16 & 0 & 71 \end{bmatrix}$

(d) $a(bC) = (ab)C = \begin{bmatrix} 0 & 56 & -84 \\ -28 & -196 & -112 \\ -84 & -140 & -252 \end{bmatrix}$

4. The determinant of A, $\det(A) = (3)(2) - (1)(5) = 1$, is nonzero. Therefore A is invertible and its inverse is $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

6. The determinant of C, $\det(C) = (6)(-1) - (4)(-2) = 2$, is nonzero. Therefore C is invertible and its inverse is $C^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{bmatrix}$.

8. The determinant of the matrix is $(\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta) = 1 \neq 0$. Therefore the matrix is invertible and its inverse is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

10. $(A^T)^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = (A^{-1})^T$

12. $AB = \begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix}$, $(AB)^{-1} = \frac{1}{20}\begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix} = \begin{bmatrix} -\frac{7}{20} & \frac{1}{4} \\ -\frac{9}{10} & \frac{1}{2} \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$, $B^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$, $B^{-1}A^{-1} = \begin{bmatrix} -\frac{7}{20} & \frac{1}{4} \\ -\frac{9}{10} & \frac{1}{2} \end{bmatrix}$

14. From part (a) of Theorem 1.4.7 we have $A = (A^{-1})^{-1}$. Therefore $A = \frac{1}{13}\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{1}{13} \\ -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$.

16. From part (a) of Theorem 1.4.7 it follows that the inverse of $(5A^T)^{-1}$ is $5A^T$.

Thus $5A^T = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$. Consequently, $A = \begin{bmatrix} -\frac{2}{5} & 1 \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$.

18. (a) $A^3 = AAA = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$ **(b)** $(A^3)^{-1} = \begin{bmatrix} \frac{1}{8} & 0 \\ -\frac{7}{2} & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$

(d) $A - 2I = \begin{bmatrix} 0 & 0 \\ 4 & -1 \end{bmatrix}$ **(e)** $2A^2 - A + I = \begin{bmatrix} 7 & 0 \\ 20 & 2 \end{bmatrix}$ **(f)** $A^3 - 2A + 4I = \begin{bmatrix} 8 & 0 \\ 20 & 3 \end{bmatrix}$

20. (a) $\begin{bmatrix} -13 & 0 & -14 \\ 0 & -8 & 0 \\ 70 & 0 & -27 \end{bmatrix}$ **(b)** $\begin{bmatrix} -\frac{27}{1331} & 0 & \frac{14}{1331} \\ 0 & -\frac{1}{8} & 0 \\ -\frac{70}{1331} & 0 & -\frac{13}{1331} \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 0 & -3 \\ 0 & 9 & 0 \\ 15 & 0 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ **(e)** $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 11 & 0 \\ 45 & 0 & -3 \end{bmatrix}$ **(f)** $\begin{bmatrix} -15 & 0 & -12 \\ 0 & 0 & 0 \\ 60 & 0 & -27 \end{bmatrix}$

22. $p_1(A) = A^2 - 9I = \begin{bmatrix} -5 & 0 \\ 12 & -8 \end{bmatrix}$,

$p_2(A) = A + 3I = \begin{bmatrix} 5 & 0 \\ 4 & 4 \end{bmatrix}$, $p_3(A) = A - 3I = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}$, $p_2(A)p_3(A) = \begin{bmatrix} -5 & 0 \\ 12 & -8 \end{bmatrix}$

24.

$$\begin{aligned} p_2(A)p_3(A) &= (A+3I)(A-3I) \\ &= A(A-3I) + (3I)(A-3I) && \longleftarrow \text{Theorem 1.4.1(e)} \\ &= (A^2 - A(3I)) + ((3I)A - (3I)(3I)) && \longleftarrow \text{Theorem 1.4.1(i)} \\ &= (A^2 - 3(AI)) + (3(IA) - 9II) && \longleftarrow \text{Theorem 1.4.1(m)} \\ &= (A^2 - 3A) + (3A - 9I) && \longleftarrow \text{Property } AI = IA = A \text{ on p. 41} \\ &= A^2 - 9I && \longleftarrow \text{Theorem 1.4.1(b)} \end{aligned}$$

28. The assumption $A^2 - 3A + I = O$ yields $I = 3A - A^2 = A(3I - A)$ which shows that $A^{-1} = 3I - A$.

30. Multiplying both sides on the left by $(ABC^T)^{-1} = (C^T)^{-1}B^{-1}A^{-1}$ and on the right by $(BA^TC)^{-1} = C^{-1}(A^T)^{-1}B^{-1}$ yields $D = (C^T)^{-1}B^{-1}A^{-1}AB^TC^{-1}(A^T)^{-1}B^{-1} = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1}$.

32. Yes, it is true. From part (e) of Theorem 1.4.8, it follows that $(A^2)^T = (AA)^T = A^TA^T = (A^T)^2$. This statement can be extended to n factors (see p. 47) so that

$$(A^n)^T = (\underbrace{AA\cdots A}_{n \text{ factors}})^T = \underbrace{A^TA^T\cdots A^T}_{n \text{ factors}} = (A^T)^n$$

34. $\underbrace{(AC^{-1})^{-1}}_{(C^{-1})^{-1}A^{-1}} \underbrace{(AC^{-1})(AC^{-1})^{-1}}_{\substack{AC^{-1} \text{ multiplied by} \\ \text{its inverse yields } I}} AD^{-1} = C\underbrace{A^{-1}A}_{I}D^{-1} = CD^{-1}$.

36. Letting $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, the matrix equation $AX = I$ becomes

$$\begin{bmatrix} x_{11}+x_{21}+x_{31} & x_{12}+x_{22}+x_{32} & x_{13}+x_{23}+x_{33} \\ x_{11} & x_{12} & x_{13} \\ x_{21}+x_{31} & x_{22}+x_{32} & x_{23}+x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Although this corresponds to a system of nine equations, it is sufficient to examine just the three equations corresponding to the first column

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 1 \\ x_{11} &= 0 \\ x_{21} + x_{31} &= 0 \end{aligned}$$

to see that subtracting the second and third equations from the first leads to a contradiction $0 = 1$. We conclude that A is not invertible.

40. $x_1 = \frac{(-3)(4)-(5)(1)}{(-1)(-3)-(5)(-1)} = -\frac{17}{8}$, $x_2 = \frac{(-1)(1)-(-1)(4)}{(-1)(-3)-(5)(-1)} = \frac{3}{8}$

42. $x_1 = \frac{(4)(4)-(-2)(4)}{(2)(4)-(-2)(1)} = \frac{24}{10} = \frac{12}{5}$, $x_2 = \frac{(2)(4)-(1)(4)}{(2)(4)-(-2)(1)} = \frac{4}{10} = \frac{2}{5}$

1.5 Elementary Matrices and a Method for Finding A^{-1}

2. **(a)** Elementary matrix (corresponds to multiplying the second row by $\sqrt{3}$)

(b) Elementary matrix (corresponds to interchanging the first row and the third row)

(c) Elementary matrix (corresponds to adding 9 times the third row to the second row)

(d) Not an elementary matrix

4. **(a)** Add 3 times the first row to the second row: $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(b) Multiply the third row by $\frac{1}{3}$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(c) Interchange the first and fourth rows: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(d) Add $\frac{1}{7}$ times the third row to the first row: $\begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. (a) Multiply the first row by -6 : $EA = \begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) Add -4 times the first row to the second row: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) Multiply the second row by 5 : $EA = \begin{bmatrix} 1 & 4 \\ 10 & 25 \\ 3 & 6 \end{bmatrix}$

8. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D was obtained from B by multiplying the second row by -3)

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B was obtained from D by multiplying the second row by $-\frac{1}{3}$)

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (F was obtained from B by adding 2 times the third row to the second row)

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (B was obtained from F by adding -2 times the third row to the second row)

10.

$\left[\begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$ ← The first row was multiplied by $-\frac{1}{3}$.

$\left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 13 & \frac{4}{3} & 1 \end{array}\right]$ ← -4 times the first row was added to the second row.

$\left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array}\right]$ ← The second row was multiplied by $\frac{1}{13}$.

$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array}\right]$ ← 2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$.

12. $\left[\begin{array}{rr|rr} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{rr|rr} 0 & 0 & 1 & 2 \\ -3 & 2 & 0 & 1 \end{array}\right]$ ← 2 times the second row was added to the first row.

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

14. $\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ ← -2 times the first row was added to the second row.

$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ ← The second row was multiplied by $-\frac{1}{3}$.

$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{4}{3} & \frac{2}{3} & 1 \end{array}\right]$ ← -2 times the second row was added to the third row.

$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{array}\right]$ ← The third row was multiplied by $\frac{3}{7}$.

$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ 0 & 0 & 1 & -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{array}\right]$ ← $\frac{2}{3}$ times the third row was added to the second row.

$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{7} & \frac{2}{7} & -\frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ 0 & 0 & 1 & -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{array}\right]$ ← -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{3}{7} & \frac{2}{7} & -\frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix}$.

16. $\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 5 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{array}\right]$ ← Each row was multiplied by 5.

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & -5 & 5 & 0 \\ 0 & -5 & \frac{5}{2} & -5 & 0 & 5 \end{array}\right]$ ← -1 times the first row was added to the second row and -1 times the first row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & \frac{5}{2} & -5 & 0 & 5 \\ 0 & 0 & \frac{5}{2} & -5 & 5 & 0 \end{array}\right]$ ← The second and third rows were interchanged.

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$ ← The second row was multiplied by $-\frac{1}{5}$ and the third row was multiplied by $\frac{2}{5}$.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$ ← $\frac{1}{2}$ times the third row was added to the second row and 2 times the third row was added to the first row.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$ ← -1 times the second row was added to the first row.

The inverse is $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$.

18. $\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -4 & 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ ← Each of the first two rows was multiplied by $\frac{1}{\sqrt{2}}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 13 & 0 & 2\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← 4 times the first row was added to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 & \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← The second row was multiplied by $\frac{1}{13}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ 0 & 1 & 0 & \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← -3 times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

20.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right]$$ ← -1 times the first row was added to each of the remaining rows.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right]$$ ← -1 times the second row was added to the third row and to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right]$$ ← -1 times the third row was added to the fourth row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{array}\right]$$ ← The second row was multiplied by $\frac{1}{3}$, the third row was multiplied by $\frac{1}{5}$, and the fourth row was multiplied by $\frac{1}{7}$.

The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$.

22. This matrix is singular because its third row contains only zeros. (Refer to Example 6 in Section 1.4.)

24.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

← The first and second rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$

← -2 times the first row was added to the fourth row and to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$

← The second and third rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$

← The second row was multiplied by -1.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 8 & -5 & 0 & -2 & 1 & 1 \end{array}\right]$$

← -1 times the second row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 & -2 & 1 & 1 \end{array}\right]$$

← -4 times the third row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

← The third row was multiplied by $\frac{1}{2}$, and the fourth row was multiplied by $-\frac{1}{5}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

← -1 times the fourth row was added to the first row and
3 times the third row was added to the second row.

The inverse is $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{2} & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}$.

26. (a) $\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

← The first and fourth rows were interchanged;
the second and third rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 0 & 0 & 0 \end{array}\right]$$

← The first row was multiplied by $1/k_4$,
the second row was multiplied by $1/k_3$,
the third row was multiplied by $1/k_2$, and
the fourth row was multiplied by $1/k_1$.

The inverse is $\begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}$.

(b) $\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right]$

← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$

Each row was multiplied by $1/k$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$

$-\frac{1}{k}$ times the first row was added to the second row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$

$-\frac{1}{k}$ times the second row was added to the third row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{array}\right]$$

$-\frac{1}{k}$ times the third row was added to the fourth row.

The inverse is $\begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{bmatrix}$.

28. It follows from parts (a) and (d) of Theorem 1.5.3 that a square matrix is invertible if and only if its reduced row echelon form is identity.

$$\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & c & 1 \\ c & 1 & 0 \\ 0 & 1 & c \end{bmatrix}$$

The first and second rows were interchanged.

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ c & 1 & 0 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 1-c^2 & -c \end{bmatrix} \longleftarrow -c \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & c^3-2c \end{bmatrix} \longleftarrow c^2-1 \text{ times the second row was added to the third row.}$$

If $c^3 - 2c = c(c^2 - 2) = 0$, i.e. if $c = 0$, $c = \sqrt{2}$ or $c = -\sqrt{2}$ the last matrix contains a row of zeros, therefore it cannot be reduced to I by elementary row operations.

Otherwise (if $c^3 - 2c \neq 0$), multiplying the last row by $\frac{1}{c^3-2c}$ would result in a row echelon form with 1's on the main diagonal. Subsequent elementary row operations would then lead to the identity matrix.

We conclude that for any value of c other than 0, $\sqrt{2}$ and $-\sqrt{2}$ the matrix is invertible.

30. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrices:

$$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \longleftarrow 5 \text{ times the first row was added to the second row.} \quad E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{2}. \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Since $E_2 E_1 A = I$, then $A = (E_2 E_1)^{-1} I = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

32. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longleftarrow -1 \text{ times the first row was added to the second row.} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ← The second and third rows were interchanged $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← -1 times the third row was added to the second row. $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← -1 times the second row was added to the first row. $E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $E_4E_3E_2E_1A = I$, then

$$A = (E_4E_3E_2E_1)^{-1}I = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

34. We use the elementary matrices already obtained in the solution of Exercise 30. Since $E_2E_1A = I$, then

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

36. We use the elementary matrices already obtained in the solution of Exercise 32. Since $E_4E_3E_2E_1A = I$, then

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38. Let us begin by finding a sequence of elementary row operations that will reduce B to I:

$B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 & -1 \\ -5 & -1 & 0 \\ 6 & 9 & 4 \end{bmatrix}$ ← (a) The first and third rows were interchanged.

$\begin{bmatrix} 1 & 2 & 1 \\ -5 & -1 & 0 \\ 6 & 9 & 4 \end{bmatrix}$ ← (b) The first row was multiplied by -1.

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 9 & 5 \\ 0 & -3 & -2 \end{bmatrix}$ ← (c) 5 times the first row was added to the second row and -6 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & -3 & -2 \end{bmatrix}$ ← (d) The second row was multiplied by $\frac{1}{9}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \quad \xleftarrow{\text{(e)}} \quad \text{3 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & 1 \end{bmatrix} \quad \xleftarrow{\text{(f)}} \quad \text{The third row was multiplied by } -3.$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \xleftarrow{\text{(g)}} \quad -\tfrac{5}{9} \text{ times the third row was added to the second row and } -1 \text{ times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \xleftarrow{\text{(h)}} \quad -2 \text{ times the second row was added to the first row.}$$

Now we reduce A to I, then continue applying the inverse operations of the ones listed above, in reverse order, to obtain B.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \quad \longleftarrow \quad \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \quad \longleftarrow \quad \text{The first row was multiplied by } -1.$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & -1 \end{bmatrix} \quad \longleftarrow \quad -2 \text{ times the first row was added to the second row and } -3 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \quad \longleftarrow \quad \text{The second row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \longleftarrow \quad -3 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \longleftarrow \quad \text{The third row was multiplied by } -1.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \longleftarrow \quad \text{The second row was added to the first row.}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \longleftarrow \quad \textit{inverted operation (h):}\ 2 \text{ times the second row was added to the first row}$$