The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ therefore the solution is $c_1 = 1,\ c_2 = 2,\ c_3 = -1$. This allows us to express $\mathbf{p} = 1\mathbf{p}_1 + 2\mathbf{p}_2 + (-1)\mathbf{p}_3$.

16. **(a)** $(0, \sqrt{2})$; **(b)** $(1,0)$; **(c)** $(-1, \sqrt{2})$; **(d)** $(a-b, \sqrt{2}b)$

4.5 Dimension

2. The augmented matrix of the linear system $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s,\ x_2 = -\frac{1}{4}s - t,\ x_3 = s,\ x_4 = t$. In vector form

$$(x_1, x_2, x_3, x_4) = \left(-\tfrac{1}{4}s, -\tfrac{1}{4}s - t, s, t\right) = s\left(-\tfrac{1}{4}, -\tfrac{1}{4}, 1,0\right) + t(0, -1,0,1)$$

therefore the solution space is spanned by the vectors $\mathbf{v}_1 = \left(-\frac{1}{4}, -\frac{1}{4}, 1,0\right)$ and $\mathbf{v}_2 = (0, -1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

4. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 3s - t,\ x_2 = s,\ x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (3s - t, s, t) = s(3,1,0) + t(-1,0,1)$$

therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (3,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

6. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x = 4t,\ y = -5t,\ z = t$. In vector form

$$(x, y, z) = (4t, -5t, t) = t(4, -5, 1)$$

therefore the solution space is spanned by the vector $\mathbf{v}_1 = (4, -5, 1)$. By Theorem 4.3.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

8. **(a)** The given subspace can be expressed as $\text{span}(S)$ where $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a, b, a+b, a-b) = a(1,0,1,1) + b(0,1,1,-1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,0,1,1), (0,1,1,-1)\}$. By Theorem 4.3.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a, a, a, a) = a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,1,1,1)\}$. By Theorem 4.3.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

10. The given subspace can be expressed as $\text{span}(S)$ where $S = \{x, x^2, x^3\}$ is a set of linearly independent vectors in P_3. Therefore S forms a basis for the subspace. The dimension of the subspace is 3.

12. **(a)** Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

16. One of the infinitely many ways to enlarge the given set to a basis for R^4 is by adding the vectors $(1,0,0,0)$ and $(0,1,0,0)$ to the set. Since the resulting set contains $\dim(R^4)=4$ vectors, by Theorem 4.5.4 we only need to establish the linear independence of the set to be able to conclude that it forms a basis for R^4. The homogeneous equation $k_1(1,-2,3,-5)+k_2(0,-1,2,-3)+k_3(1,0,0,0)+k_4(0,1,0,0)=(0,0,0,0)$ can be rewritten as a linear system whose coefficient matrix $\begin{bmatrix}1 & 0 & 1 & 0\\ -2 & -1 & 0 & 1\\ 3 & 2 & 0 & 0\\ -5 & -3 & 0 & 0\end{bmatrix}$ has determinant 1. Using parts (b) and (g) of Theorem 2.3.8, we conclude that there is only the trivial solution, therefore the enlarged set of four vectors is linearly independent (and, consequently , forms a basis for R^4).

20. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S=\{1,x,x^2\}$.

(a) Denote $\mathbf{v}_1=-1+x-2x^2$, $\mathbf{v}_2=3+3x+6x^2$, $\mathbf{v}_3=9$.
Then $(\mathbf{v}_1)_S=(-1,1,-2)$, $(\mathbf{v}_2)_S=(3,3,6)$, $(\mathbf{v}_3)_S=(9,0,0)$.
Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix}-1 & 3 & 9 & 0\\ 1 & 3 & 0 & 0\\ -2 & 6 & 0 & 0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2)=3$, from Theorem 4.5.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1=1+x$, $\mathbf{v}_2=x^2$, $\mathbf{v}_3=-2+2x^2$, $\mathbf{v}_4=-3x$.
Then $(\mathbf{v}_1)_S=(1,1,0)$, $(\mathbf{v}_2)_S=(0,0,1)$, $(\mathbf{v}_3)_S=(-2,0,2)$, $(\mathbf{v}_4)_S=(0,-3,0)$.
Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S+k_4(\mathbf{v}_4)_S=\mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix}1 & 0 & -2 & 0 & 0\\ 1 & 0 & 0 & -3 & 0\\ 0 & 1 & 2 & 0 & 0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1 & 0 & 0 & -3 & 0\\ 0 & 1 & 0 & 3 & 0\\ 0 & 0 & 1 & \frac{3}{2} & 0\end{bmatrix}$.
Based on the leading entries in the first three columns, the vector equation $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix}1 & 0 & -2 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 2 & 0\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$, $(\mathbf{v}_2)_S$, and $(\mathbf{v}_3)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2)=3$, from Theorem 4.5.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(c) Clearly, $1+x-3x^2=\frac{1}{2}(2+2x-6x^2)=\frac{1}{3}(3+3x-9x^2)$ therefore from Theorem 4.5.3(b), the subspace is spanned by $1+x-3x^2$. By Theorem 4.3.2(b), a set containing a single nonzero vector is linearly independent. We conclude that $1+x-3x^2$ forms a basis for this subspace of P_2.

4.6 Change of Basis

2. **(a)** Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(2,-1,3)=c_1(1,0,0)+c_2(2,2,0)+c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1=3$, $c_2=-2$, $c_3=1$ therefore the coordinate vector is $(\mathbf{v})_S=(3,-2,1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(5,-12,3)=c_1(1,2,3)+c_2(-4,5,6)+c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & - & 4c_2 & + & 7c_3 & = & 5 \\ 2c_1 & + & 5c_2 & - & 8c_3 & = & -12 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1=-2$, $c_2=0$, $c_3=1$ therefore the coordinate vector is $(\mathbf{v})_S=(-2,0,1)$.

4. We express A as a linear combination of A_1, A_2, A_3, and A_4

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}=c_1\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}+c_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The right hand side can be rewritten as $\begin{bmatrix} -c_1+c_2 & c_1+c_2 \\ c_3 & c_4 \end{bmatrix}$; equating the corresponding entries in this matrix and the matrix A yields the linear system

$$\begin{array}{rcrcrcrcr} -c_1 & + & c_2 & & & & & = & 2 \\ c_1 & + & c_2 & & & & & = & 0 \\ & & & & c_3 & & & = & -1 \\ & & & & & & c_4 & = & 3 \end{array}$$

whose solution can be easily obtained (add the first equation to the second, then back-substitute) as $c_1 = -1,\ c_2 = 1,\ c_3 = -1,\ c_4 = 3$. Therefore, $(A)_S = (-1,1,-1,3)$.

6. (a) In this part, B' is the old basis and B is the new basis:

$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right] = [I \mid \text{transition from old to new}]$

No row operations were necessary to obtain the transition matrix $P_{B'\to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, B is the old basis and B' is the new basis: $[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$

The reduced row echelon form of this matrix is $[I \mid \text{transition from old to new}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$.

The transition matrix is $P_{B\to B'} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Clearly, $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & - & 3c_2 & = & 3 \\ c_1 & + & 4c_2 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{3}{11} \\ 0 & 1 & -\frac{13}{11} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{3}{11}$, $c_2 = -\frac{13}{11}$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$. This matches the result obtained in part (c).

8. (a) In this part, B is the old basis and B' is the new basis:

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{rrr|rrr} -6 & -2 & -2 & -3 & -3 & 1 \\ -6 & -6 & -3 & 0 & 2 & 6 \\ 0 & 4 & 7 & -3 & -1 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from old to new}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right].$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -3c_1 & - & 3c_2 & + & c_3 & = & -5 \\ & & 2c_2 & + & 6c_3 & = & 8 \\ -3c_1 & - & c_2 & - & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = 1$, $c_2 = 1$, $c_3 = 1$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ we obtain

$$\begin{bmatrix}-5\\8\\-5\end{bmatrix} = c_1\begin{bmatrix}-6\\-6\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\-6\\4\end{bmatrix} + c_3\begin{bmatrix}-2\\-3\\7\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} -6c_1 - 2c_2 - 2c_3 &= -5 \\ -6c_1 - 6c_2 - 3c_3 &= 8 \\ 4c_2 + 7c_3 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{19}{12} \\ 0 & 1 & 0 & -\frac{43}{12} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{19}{12}$, $c_2 = -\frac{43}{12}$, $c_3 = \frac{4}{3}$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$, which matches the result we obtained in part (b).

10. (a) We find the two columns of the transitions matrix $P_{B'\to B} = [[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B]$

$$\mathbf{q}_1 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2 \qquad\qquad \mathbf{q}_2 = b_1\mathbf{p}_1 + b_2\mathbf{p}_2$$

$$2 = a_1(6+3x) + a_2(10+2x) \qquad\qquad 3 + 2x = b_1(6+3x) + b_2(10+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each of the two equations

$$\begin{aligned} 6a_1 + 10a_2 &= 2 \\ 3a_1 + 2a_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} 6b_1 + 10b_2 &= 3 \\ 3b_1 + 2b_2 &= 2 \end{aligned}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & -\frac{2}{9} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

We obtain the transition matrix $P_{B'\to B} = [[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$.

(b) We find the two columns of the transitions matrix $P_{B\to B'} = [[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}]$

$$\mathbf{p}_1 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 \qquad \mathbf{p}_2 = b_1\mathbf{q}_1 + b_2\mathbf{q}_2$$

$$6 + 3x = a_1(2) + a_2(3 + 2x) \qquad 10 + 2x = b_1(2) + b_2(3 + 2x)$$

equate the coefficients corresponding to like powers of x on both sides of each of the two equations

$$\begin{array}{rcrcr} 2a_1 & + & 3a_2 & = & 6 \\ & & 2a_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} 2b_1 & + & 3b_2 & = & 10 \\ & & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

We obtain the transition matrix $P_{B\to B'} = [[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$.

(c) Since $-4 + x = (6 + 3x) - (10 + 2x)$, the coordinate vector is $[\mathbf{p}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{p}]_{B'} = P_{B\to B'}[\mathbf{p}]_B = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$.

(d) We are looking for the coordinate vector $[\mathbf{p}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ with c_1 and c_2 satisfying the equality

$$-4 + x = c_1(2) + c_2(3 + 2x)$$

for all real values x. Equating the coefficients associated with like powers of x on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 3c_2 & = & -4 \\ & & 2c_2 & = & 1 \end{array}$$

which can easily be solved by back-substitution: $c_2 = \frac{1}{2}$, $c_1 = \frac{-4-3\left(\frac{1}{2}\right)}{2} = -\frac{11}{4}$. We conclude that

$[\mathbf{p}]_{B'} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$, which matches the result obtained in part (c).

12. (a) By Theorem 4.6.2, $P_{B\to S}=\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}$.

(b) In this part, S is the old basis and B is the new basis: $[\text{new basis} \mid \text{old basis}]=\left[\begin{array}{cc|cc}2 & -3 & 1 & 0\\ 1 & 4 & 0 & 1\end{array}\right]$.

The reduced row echelon form of this matrix is $[I \mid \text{transition from old to new}]=\left[\begin{array}{cc|cc}1 & 0 & \frac{4}{11} & \frac{3}{11}\\ 0 & 1 & -\frac{1}{11} & \frac{2}{11}\end{array}\right]$.

The transition matrix is $P_{S\to B}=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ -\frac{1}{11} & \frac{2}{11}\end{bmatrix}$.

(c) Since $\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ -\frac{1}{11} & \frac{2}{11}\end{bmatrix}\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ -\frac{1}{11} & \frac{2}{11}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Since $(5,-3)=(2,1)-(-3,4)$ the coordinate vector is $[\mathbf{w}]_B=\begin{bmatrix}1\\ -1\end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_S=P_{B\to S}[\mathbf{w}]_B=\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}5\\ -3\end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S=\begin{bmatrix}3\\ -5\end{bmatrix}$. From Formula (13), $[\mathbf{w}]_B=P_{S\to B}[\mathbf{w}]_S=\begin{bmatrix}\frac{4}{11} & \frac{3}{11}\\ -\frac{1}{11} & \frac{2}{11}\end{bmatrix}\begin{bmatrix}3\\ -5\end{bmatrix}=\begin{bmatrix}-\frac{3}{11}\\ -\frac{13}{11}\end{bmatrix}$.

14. (a) Here, B_2 is the old basis and B_1 is the new basis: $[\text{new basis} \mid \text{old basis}]=\left[\begin{array}{cc|cc}2 & 4 & 1 & -1\\ 2 & -1 & 3 & -1\end{array}\right]$.

The reduced row echelon form of this matrix is $[I \mid \text{transition from old to new}]=\left[\begin{array}{cc|cc}1 & 0 & \frac{13}{10} & -\frac{1}{2}\\ 0 & 1 & -\frac{2}{5} & 0\end{array}\right]$.

The transition matrix is $P_{B_2\to B_1}=\begin{bmatrix}\frac{13}{10} & -\frac{1}{2}\\ -\frac{2}{5} & 0\end{bmatrix}$.

(b) Here, B_1 is the old basis and B_2 is the new basis: $[\text{new basis} \mid \text{old basis}]=\left[\begin{array}{cc|cc}1 & -1 & 2 & 4\\ 3 & -1 & 2 & -1\end{array}\right]$.

The reduced row echelon form of this matrix is $[I \mid \text{transition from old to new}] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{array}\right]$.

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$.

(c) Since $\begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it follows that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 4c_2 & = & 5 \\ 2c_1 & - & c_2 & = & -3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{7}{10}$, $c_2 = \frac{8}{5}$, therefore the coordinate vector is $[\mathbf{w}]_{B_1} = \begin{bmatrix} -\frac{7}{10} \\ \frac{8}{5} \end{bmatrix}$.

$[\mathbf{w}]_{B_2} = P_{B_1 \to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} -\frac{7}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \end{bmatrix}$.

(e) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & - & c_2 & = & 3 \\ 3c_1 & - & c_2 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$. The solution of the linear system is $c_1 = -4$, $c_2 = -7$, therefore the coordinate vector is $[\mathbf{w}]_{B_2} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$.

$$[\mathbf{w}]_{B_1} = P_{B_2 \to B_1}[\mathbf{w}]_{B_2} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}\begin{bmatrix} -4 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}.$$

16. (a) B_1 is the old basis and B_2 is the new basis:

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{ccc|ccc} -6 & -2 & -2 & -3 & -3 & 1 \\ -6 & -6 & -3 & 0 & 2 & 6 \\ 0 & 4 & 7 & -3 & -1 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from old to new}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right].$$

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -3c_1 & - & 3c_2 & + & c_3 & = & -5 \\ & & 2c_2 & + & 6c_3 & = & 8 \\ -3c_1 & - & c_2 & - & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = 1$, $c_2 = 1$, $c_3 = 1$ therefore the coordinate vector is $[\mathbf{w}]_{B_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B_2} = P_{B_1\to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3}\end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ we obtain

$$\begin{bmatrix}-5\\8\\-5\end{bmatrix} = c_1\begin{bmatrix}-6\\-6\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\-6\\4\end{bmatrix} + c_3\begin{bmatrix}-2\\-3\\7\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -6c_1 & - & 2c_2 & - & 2c_3 & = & -5 \\ -6c_1 & - & 6c_2 & - & 3c_3 & = & 8 \\ & & 4c_2 & + & 7c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{19}{12} \\ 0 & 1 & 0 & -\frac{43}{12} \\ 0 & 0 & 1 & \frac{4}{3}\end{bmatrix}$. The solution of the linear system is $c_1 = \frac{19}{12}$, $c_2 = -\frac{43}{12}$, $c_3 = \frac{4}{3}$ therefore the coordinate vector is $[\mathbf{w}]_{B_2} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3}\end{bmatrix}$.

18. Reflecting $\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ about the line $y = x$ results in $\mathbf{v}_1 = \begin{bmatrix}0\\1\end{bmatrix}$. Likewise for $\mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ we obtain $\mathbf{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$.

(a) From Theorem 4.6.5, $P_{B\to S} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

(b) Denoting $P = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, it follows from Theorem 4.6.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

20. Since for every vector $\mathbf{v}$ in R^2 we have $[\mathbf{v}]_{B_2} = \begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}[\mathbf{v}]_{B_1}$ and $[\mathbf{v}]_{B_3} = \begin{bmatrix}7 & 2\\4 & -1\end{bmatrix}[\mathbf{v}]_{B_2}$ then it follows that $[\mathbf{v}]_{B_3} = \begin{bmatrix}7 & 2\\4 & -1\end{bmatrix}\begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}[\mathbf{v}]_{B_1} = \begin{bmatrix}31 & 11\\7 & 2\end{bmatrix}[\mathbf{v}]_{B_1}$ so that $P_{B_1\to B_3} = \begin{bmatrix}31 & 11\\7 & 2\end{bmatrix}$.

From Theorem 4.6.1, $P_{B_3\to B_1}$ is the inverse of this matrix, which can be easily computed as $\begin{bmatrix}-\frac{2}{15} & \frac{11}{15} \\ \frac{7}{15} & -\frac{31}{15}\end{bmatrix}$.

24. Let the given basis be denoted as $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{v}_3 = (1,0,0)$ and denote the unknown basis as $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We have $P_{B\to B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} = [[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid [\mathbf{u}_3]_{B'}]$. Equating the respective columns yields

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = (1,1,1)$$

$$[\mathbf{u}_2]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \Rightarrow \mathbf{u}_2 = 0\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = (5,3,0)$$

$$[\mathbf{u}_3]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_3 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = (3,2,0)$$

Thus the given matrix is the transition matrix from the basis $\{(1,1,1), (5,3,0), (3,2,0)\}$.

4.7 Row Space, Column Space, and Null Space

2. **(a)** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

4. **(a)** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

6. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = 16t,\ x_2 = 19t,\ x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = \frac{1}{2}t,\ x_2 = s,\ x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

(c) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -s + \frac{2}{7}t,\ x_2 = -s - \frac{4}{7}t,\ x_3 = s,\ x_4 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

(d) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -r - 2s - t,\ x_2 = -r - s - 2t,\ x_3 = r,\ x_4 = s,\ x_5 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

(e) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -2s - \frac{4}{3}t,\ x_2 = \frac{1}{6}t,\ x_3 = \frac{5}{12}t,\ x_4 = s,$ $x_5 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

8. In each part, we use the reduced row echelon form of the respective matrix A which was already obtained in our solution of Exercise 6.

(a) $[1 \quad 0 \quad -16]$, $[0 \quad 1 \quad -19]$ **(b)** $\left[1 \quad 0 \quad -\frac{1}{2}\right]$

(c) $\left[1 \quad 0 \quad 1 \quad -\frac{2}{7}\right]$, $\left[0 \quad 1 \quad 1 \quad \frac{4}{7}\right]$ **(d)** $[1 \quad 0 \quad 1 \quad 2 \quad 1]$, $[0 \quad 1 \quad 1 \quad 1 \quad 2]$

(e) $\left[1 \quad 0 \quad 0 \quad 2 \quad \frac{4}{3}\right]$, $\left[0 \quad 1 \quad 0 \quad 0 \quad -\frac{1}{6}\right]$, $\left[0 \quad 0 \quad 1 \quad 0 \quad -\frac{5}{12}\right]$

10. In each part, we are employing the procedure developed in Example 9 on p.232.

(a) The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -4 & -6 \\ 3 & -4 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $[1 \quad -1 \quad 3]$ and $[5 \quad -4 \quad -4]$, form a basis for the row space of A.

(b) The reduced row echelon form of $A^T = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since only the first column of the reduced row echelon form contains a leading 1, by Theorems 4.7.5 and 4.7.6(b) the first column of A^T forms a basis for the column space of A^T. Consequently, the first row of A, $[2 \quad 0 \quad -1]$, forms a basis for the row space of A.

(c) The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $[1 \quad 4 \quad 5 \quad 2]$ and $[2 \quad 1 \quad 3 \quad 0]$, form a basis for the row space of A.

(d) The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \frac{13}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $[1 \quad 4 \quad 5 \quad 6 \quad 9]$ and $[3 \quad -2 \quad 1 \quad 4 \quad -1]$, form a basis for the row space of A.

(e) The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 0 & 2 & 3 & -2 \\ -3 & 3 & -3 & -6 & 9 \\ 2 & 6 & -2 & 0 & 2 \\ 2 & 0 & 4 & 6 & -4 \\ 1 & -3 & 4 & 5 & -5 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Since the first three columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first three columns of A^T form a basis for the column space of A^T. Consequently, the first three rows of A, $[1 \quad -3 \quad 2 \quad 2 \quad 1]$, $[0 \quad 3 \quad 6 \quad 0 \quad -3]$, and $[2 \quad -3 \quad -2 \quad 4 \quad 4]$, form a basis for the row space of A.

12. (a) Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:
$A = \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{cccc} \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} -2 \\ 1 \\ 0 \\ 0 \end{matrix}\right] \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 \end{array}$$

contains leading 1's in the first two columns, then by Theorems 4.7.5 and 4.7.6(b), the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the column space of A, and for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_4 = -2\mathbf{w}_1 + \mathbf{w}_2$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$.

(b) Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:

$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -4 & 1 & -1 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{cccc} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 \end{array} \end{array}$$

contains leading 1's in columns 1 and 3, then by Theorems 4.7.5 and 4.7.6(b), the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ form a basis for the column space of A, and for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_2 = 2\mathbf{w}_1$ and $\mathbf{w}_4 = \mathbf{w}_1 + \mathbf{w}_3$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_2 = 2\mathbf{v}_1$ and $\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_3$.

(c) Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$:

$A = \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array} \end{array}$$

contains leading 1's in columns 1, 2, and 4, then by Theorems 4.7.5 and 4.7.6(b), the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ form a basis for the column space of A, and for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{w}_5 = -\mathbf{w}_1 + 3\mathbf{w}_2 + 2\mathbf{w}_4$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4$.

14. Let $B = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \\ 2 & 4 \end{bmatrix}$. We are looking for a matrix A such that $AB = O$. Taking a transpose on both sides results in $B^T A^T = O^T$. We proceed to solve the homogeneous linear system $B^T\mathbf{u} = \mathbf{0}$. The reduced row echelon form of its augmented matrix $\begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 2 & 0 & -2 & 4 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{bmatrix}$ therefore the general solution in the vector form is $s\begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. We can take $A^T = \begin{bmatrix} 1 & -2 \\ 4 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ thus

$A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$.

16. **(a)** e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ null space is the origin

(b) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the z-axis

(c) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the yz-plane

18. The corresponding homogeneous system $x_1 + x_2 + x_3 = 0$ has a general solution $x_1 = -s - t$, $x_2 = s$, $x_3 = t$. The original nonhomogeneous system has a general solution $x_1 = 1 - s - t$, $x_2 = s$, $x_3 = t$, which can be expressed in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - s - t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\substack{\text{particular} \\ \text{solution}}} + \underbrace{s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\substack{\text{general} \\ \text{solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{system}}}$$

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

2. In Exercises 6 and 8 of Section 4.7, we found a basis for the null space and a basis for the row space of each of these five matrices.

The number of vectors in a basis for the row space is the rank of the matrix.

Likewise, the number of vectors in a basis for the null space is the nullity of the matrix.

(a) $\operatorname{rank}(A) = 2$; $\operatorname{nullity}(A) = 1$; $\operatorname{rank}(A) + \operatorname{nullity}(A) = 2 + 1 = 3 = n$ ← number of columns of A

(b) $\operatorname{rank}(A) = 1$; $\operatorname{nullity}(A) = 2$; $\operatorname{rank}(A) + \operatorname{nullity}(A) = 1 + 2 = 3 = n$ ← number of columns of A

(c) $\operatorname{rank}(A) = 2$; $\operatorname{nullity}(A) = 2$; $\operatorname{rank}(A) + \operatorname{nullity}(A) = 2 + 2 = 4 = n$ ← number of columns of A

(d) $\operatorname{rank}(A) = 2$; $\operatorname{nullity}(A) = 3$; $\operatorname{rank}(A) + \operatorname{nullity}(A) = 2 + 3 = 5 = n$ ← number of columns of A

(e) $\operatorname{rank}(A) = 3$; $\operatorname{nullity}(A) = 2$; $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 2 = 5 = n$ ← number of columns of A

4.

		(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
Size of A: $\operatorname{rank}(A)$	$m \times n$ $= r$	3×3 3	3×3 2	3×3 1	5×9 2	9×5 2	4×4 0	6×2 2
dimension of the row space of A	$= r$	3	2	1	2	2	0	2
dimension of the column space of A	$= r$	3	2	1	2	2	0	2
dimension of the null space of A	$= n - r$	0	1	2	7	3	4	0
dimension of the null space of A^T	$= m - r$	0	1	2	3	7	4	4

6. The largest possible value for the rank of an $m \times n$ matrix A is the smaller of the two dimensions of A:

- n if $m \geq n$ (when every column of the reduced row echelon form of A contains a leading 1),
- m if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

The smallest possible value for the nullity of an $m \times n$ matrix A is

- 0 if $m \geq n$ (when every column of the reduced row echelon form of A contains a leading 1),
- $n - m$ if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

8.

		(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
Size of A: $\operatorname{rank}(A)$ $\operatorname{rank}[A\vert\mathbf{b}]$ is not relevant here	$m \times n$ $= r$	3×3 3	3×3 2	3×3 1	5×9 2	9×5 2	4×4 0	6×2 2
nullity of A	$= n - r$	0	1	2	7	3	4	0
number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$	$= n - r$	0	1	2	7	3	4	0

12. (a) The determinant of A is

$$\begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ t-1 & 0 & 1-t \end{vmatrix} \longleftarrow$$

-1 times the first row was added to the second row and to the third row.

$$= \begin{vmatrix} 1 & 1+t & t \\ 0 & 0 & 1-t \\ t-1 & 1-t & 1-t \end{vmatrix}$$ ← Last column was added to the second column.

$$= -(1-\mathrm{t}) \begin{vmatrix} 1 & 1+t \\ t-1 & 1-t \end{vmatrix}$$ ← Cofactor expansion along the second row.

$$= -(1-t)((1-t)-(1+t)(t-1))$$

$$= -(1-t)^2(2+t)$$

From parts (g) and (n) of Theorem 4.8.4, $\mathrm{rank}(A) = 3$ when $\det(A) \neq 0$, i.e. for all t values other than 1 or -2.

If $t = 1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 1.

If $t = -2$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

(b) The determinant of A is

$$\begin{vmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{vmatrix} = \begin{vmatrix} t & 3 & -1 \\ 3-2t & 0 & 0 \\ -1 & -3 & t \end{vmatrix}$$ ← -2 times the first row was added to the second row.

$$= -(3-2t) \begin{vmatrix} 3 & -1 \\ -3 & t \end{vmatrix}$$ ← Cofactor expansion along the second row.

$$= -(3-2t)(3t-3)$$

$$= 3(2t-3)(t-1)$$

From parts (g) and (n) of Theorem 4.8.4, $\mathrm{rank}(A) = 3$ when $\det(A) \neq 0$, i.e. for all t values other than 1 or $\frac{3}{2}$.

If $t = 1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

If $t = \frac{3}{2}$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

16. (a) e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ - the column space is the xy-plane in R^3

(b) The general solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = (0,0,t)$. The null space is the z-axis.

(c) The row space of A is the xy-plane in R^3

18. (a) 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A^T can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A the zero matrix, then the general solution of $A^T\mathbf{x} = \mathbf{0}$ has three parameters;

4.9 Matrix Transformations from R^n to R^m

2. (a) domain: R^5; codomain: R^4 **(b)** domain: R^4; codomain: R^5

(c) domain: R^4; codomain: R^4 **(d)** domain: R^1; codomain: R^3

4. R^3; R^2; $(-2,2)$

6. (a), (b), and (c) are matrix transformations, but (d) and (e) are not

8. (a) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -3 & 0 & 1 \\ 3 & 5 & 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$

(d) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

10. (a) $T(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $T(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $T(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $T(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

12. (a) $T(\mathbf{x}) = [T]\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $T(\mathbf{x}) = [T]\mathbf{x} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

(c) $T(\mathbf{x}) = [T]\mathbf{x} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 + 4x_3 \\ 3x_1 + 5x_2 + 7x_3 \\ 6x_1 - x_3 \end{bmatrix}$

(d) $T(\mathbf{x}) = [T]\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ 2x_1 + 4x_2 \\ 7x_1 + 8x_2 \end{bmatrix}$

14. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

16. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$

18. (a) $\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{2} + 2 \\ \frac{3}{2} - 2\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 4.60 \\ -1.96 \end{bmatrix}$

(b) $\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - 2\sqrt{3} \\ -\frac{3\sqrt{3}}{2} - 2 \end{bmatrix} \approx \begin{bmatrix} -1.96 \\ -4.60 \end{bmatrix}$

(c) $\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \approx \begin{bmatrix} 4.95 \\ -0.71 \end{bmatrix}$

(d) $\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$

20. (a) $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(-60^\circ) & -\sin(-60^\circ)\\ 0 & \sin(-60^\circ) & \cos(-60^\circ)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

(b) $\begin{bmatrix}\cos(-60^\circ) & 0 & \sin(-60^\circ)\\ 0 & 1 & 0\\ -\sin(-60^\circ) & 0 & \cos(-60^\circ)\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{bmatrix}$

(c) $\begin{bmatrix}\cos(-60^\circ) & -\sin(-60^\circ) & 0\\ \sin(-60^\circ) & \cos(-60^\circ) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$

22. (a) $[T_1]=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$; $[T_2]=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$; $[T_3]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

24. A unit vector in the direction of $(1,1,1)$ is $\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{3}}(1,1,1)$ so in Formula (15), we take $a=b=c=\frac{1}{\sqrt{3}}$. Formula (15) yields the standard matrix

$$\begin{bmatrix}\frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2}\\ \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2}\\ \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}}\\ \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}}\\ \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3}\end{bmatrix}$$

26. The columns of A are orthonormal since $\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\cdot\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}=\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)=\frac{1}{2}-\frac{1}{2}=0$,

$\left\|\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right\|=\sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1$, and $\left\|\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}\right\|=\sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2+\left(-\frac{1}{\sqrt{2}}\right)^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1.$

Also, $\det(A)=\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)-\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}+\frac{1}{2}=1$. Therefore, the stated conditions are satisfied.

To find θ, the angle of rotation, we use Formula (13). Setting $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$ we obtain $\cos\theta=-\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$. The only angle in the interval $[0,2\pi)$ that satisfies these equations is $\theta=\frac{3\pi}{4}$. (Generally, for any integer k, $\theta=\frac{3\pi}{4}+2k\pi$ is a solution.)

28. (a) Let us denote the columns of A as $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{9} \\ \frac{8}{9} \\ -\frac{4}{9} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -\frac{4}{9} \\ \frac{4}{9} \\ \frac{7}{9} \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} \frac{8}{9} \\ \frac{1}{9} \\ \frac{4}{9} \end{bmatrix}$. We have

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(\tfrac{1}{9}\right)\left(-\tfrac{4}{9}\right) + \left(\tfrac{8}{9}\right)\left(\tfrac{4}{9}\right) + \left(-\tfrac{4}{9}\right)\left(\tfrac{7}{9}\right) = \tfrac{-4+32-28}{81} = 0 \qquad \|\mathbf{v}_1\| = \sqrt{\left(\tfrac{1}{9}\right)^2 + \left(\tfrac{8}{9}\right)^2 + \left(-\tfrac{4}{9}\right)^2} = \sqrt{\tfrac{1+64+16}{81}} = 1$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \left(\tfrac{1}{9}\right)\left(\tfrac{8}{9}\right) + \left(\tfrac{8}{9}\right)\left(\tfrac{1}{9}\right) + \left(-\tfrac{4}{9}\right)\left(\tfrac{4}{9}\right) = \tfrac{8+8-16}{81} = 0 \qquad \|\mathbf{v}_2\| = \sqrt{\left(-\tfrac{4}{9}\right)^2 + \left(\tfrac{4}{9}\right)^2 + \left(\tfrac{7}{9}\right)^2} = \sqrt{\tfrac{16+16+49}{81}} = 1$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = \left(-\tfrac{4}{9}\right)\left(\tfrac{8}{9}\right) + \left(\tfrac{4}{9}\right)\left(\tfrac{1}{9}\right) + \left(\tfrac{7}{9}\right)\left(\tfrac{4}{9}\right) = \tfrac{-32+4+28}{81} = 0 \qquad \|\mathbf{v}_3\| = \sqrt{\left(\tfrac{8}{9}\right)^2 + \left(\tfrac{1}{9}\right)^2 + \left(\tfrac{4}{9}\right)^2} = \sqrt{\tfrac{64+1+16}{81}} = 1$$

which shows that the columns of A are orthonormal vectors. Furthermore,

$$\begin{vmatrix} \frac{1}{9} & -\frac{4}{9} & \frac{8}{9} \\ \frac{8}{9} & \frac{4}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{7}{9} & \frac{4}{9} \end{vmatrix} = \left(\tfrac{1}{9}\right)\left(\tfrac{1}{9}\right)\left(\tfrac{1}{9}\right)\begin{vmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{vmatrix}$$

A common factor of $\frac{1}{9}$ from each row was taken through the determinant sign.

$$= \left(\tfrac{1}{729}\right)\begin{vmatrix} 1 & -4 & 8 \\ 0 & 36 & -63 \\ 0 & -9 & 36 \end{vmatrix}$$

-8 times the first row was added to the second row and 4 times the first row was added to the third row.

$$= \left(\tfrac{1}{729}\right)(1)\begin{vmatrix} 36 & -63 \\ -9 & 36 \end{vmatrix}$$

Cofactor expansion along the first column

$$= \left(\tfrac{1}{729}\right)(1)[(36)(36) - (-63)(-23)]$$

$$= \left(\tfrac{1}{729}\right)(1)[1296 - 567]$$

$$= \left(\tfrac{1}{729}\right)(1)(729) = 1$$

We conclude that multiplication by the given matrix A is a rotation.

(b) Any nonzero $\mathbf{x}$ vector can be used - e.g., with $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we obtain a vector that defines an axis for the rotation: $\mathbf{u} = A\mathbf{x} + A^T\mathbf{x} + \left(1 - \operatorname{tr}(A)\right)\mathbf{x} = \begin{bmatrix} \frac{1}{9} \\ \frac{8}{9} \\ -\frac{4}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{9} \\ -\frac{4}{9} \\ \frac{8}{9} \end{bmatrix} + (1-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ \frac{4}{9} \\ \frac{4}{9} \end{bmatrix}$. A unit vector in the direction of $\mathbf{u}$ is $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\frac{2}{3}}\begin{bmatrix} \frac{2}{9} \\ \frac{4}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$.

(c) $\cos\theta = \frac{\mathrm{tr}(A)-1}{2} = \frac{1-1}{2} = 0$. On the interval $[0,2\pi)$ this equation has two solutions: $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, however, substituting them into Formula (15) we can verify that $\theta = \frac{\pi}{2}$ yields the original matrix A, whereas $\theta = \frac{3\pi}{2}$ does not. (Generally, for any integer k, $\theta = \frac{\pi}{2} + 2k\pi$ is a solution.)

30. (a) Expansion of R^2 in the x-direction with factor 2 and in the y-direction with factor 3.

(b) Rotation through the angle $\frac{\pi}{6}$.

4.10 Properties of Matrix Transformations

2. $[T_B \circ T_A] = [T_B][T_A] = BA = \begin{bmatrix} 40 & 0 & 20 \\ 12 & -9 & 18 \\ 38 & -18 & 43 \end{bmatrix}$; $\quad [T_A \circ T_B] = [T_A][T_B] = AB = \begin{bmatrix} 19 & 18 & 22 \\ 10 & -3 & 16 \\ 31 & -33 & 58 \end{bmatrix}$

4. (a) In vector form, $T_1(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$.

Likewise, $T_2(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 \\ -x_3 \\ 4x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$.

(b) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$

(c) $T_1(T_2(x_1, x_2, x_3)) = (4x_1 + 8x_2, -2x_1 - 4x_2 - x_3, -x_1 - 2x_2 + 3x_3)$

$T_2(T_1(x_1, x_2, x_3)) = (2x_2, x_1 + 3x_2, 17x_1 + 3x_2)$

6. (a) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the rotation of $60°$, T_2 is the orthogonal projection on the x-axis, and T_3 is the reflection about the line $y = x$. From Tables 5, 1, and 3 in Section 4.9, $[T_1] = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the dilation with factor $k = 2$, T_2 is the rotation of $45°$, and T_3 is the reflection about the y-axis. From Tables 7, 5, and 1 in Section 4.9,

$[T_1] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, and $[T_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the rotation of $15°$, T_2 is the rotation of $105°$, and T_3 is the rotation of $60°$. The net effect of the three rotations is a single rotation of $15° + 105° + 60° = 180°$. From Table 5 in Section 4.9, $[T] = \begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

8. **(a)** We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the rotation of $30°$ about the x-axis, T_2 is the rotation of $30°$ about the z-axis, and T_3 is the contraction with factor $k = \frac{1}{4}$. From Tables 6 and 8 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} =$ $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & -\frac{\sqrt{3}}{16} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the reflection about the xy-plane, T_2 is the reflection about the xz-plane, and T_3 is the orthogonal projection on the yz-plane. From Tables 2 and 4 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the rotation of $270°$ about the x-axis, T_2 is the rotation of $90°$ about the y-axis, and T_3 is the rotation of $180°$ about the z-axis. From Table 6 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 270° & -\sin 270° \\ 0 & \sin 270° & \cos 270° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} =$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$.

10. (a) From Tables 8 and 6 in Section 4.9, $[T_1] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} k\cos\theta & -k\sin\theta & 0 \\ k\sin\theta & k\cos\theta & 0 \\ 0 & 0 & k \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} k\cos\theta & -k\sin\theta & 0 \\ k\sin\theta & k\cos\theta & 0 \\ 0 & 0 & k \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 = T_2 \circ T_1$.

(b) From Table 6 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1 \end{bmatrix}$;

$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} \cos\theta_2 & -\cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 \\ \sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

12. (a) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 8x_1 + 4x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$; since $\begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is not one-to-one

(b) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ 5x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$; since $\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = 17 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is one-to-one

(c) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 3x_2 + 2x_3 \\ 2x_1 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}$; since $\begin{vmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is not one-to-one

(d) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 5x_2 + 3x_3 \\ x_1 + 8x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$; since $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix} = -1 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is one-to-one

14. (a) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; since

$\begin{vmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 \neq 0,$ it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is one-to-one;

the reduced row echelon form of the matrix $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$

therefore the standard matrix of T^{-1} is $\begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$;

$T^{-1}(w_1, w_2, w_3) = (x_1 - 2x_2 + 4x_3, -x_1 + 2x_2 - 3x_3, -x_1 + 3x_2 - 5x_3)$

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 + x_3 \\ -2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix}$; since

$\begin{vmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{vmatrix} = 0,$ it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is not one-to-one

(c) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 - x_3 \\ 2x_1 + 7x_2 + x_3 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$; since

$\begin{vmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 2 \neq 0,$ it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is one-to-one;

the reduced row echelon form of the matrix $\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

therefore the standard matrix of T^{-1} is $\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$;

$T^{-1}(w_1, w_2, w_3) = (-\frac{3}{2}x_1 - \frac{3}{2}x_2 + \frac{11}{2}x_3,\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{3}{2}x_3,\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3)$

(d) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -2x_1 + x_2 + 4x_3 \\ 7x_1 + 4x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$; since

$\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 7 & 4 & -5 \end{vmatrix} = 0,$ it follows from parts (g) and (s) of Theorem 4.10.4 that the operator is not one-to-one

16. (a) We check whether the two properties of Theorem 4.10.2 hold for all real x, x', y, y', and k:

$T\big((x,y) + (x',y')\big) = T(x + x', y + y') = (2(x + x'), y + y') = (2x, y) + (2x', y') = T(x,y) + T(x',y')$

$T\big(k(x,y)\big) = T(kx, ky) = (2kx, ky) = k(2x, y) = kT(x,y)$

Both properties hold, therefore T is a matrix operator.

(b) When $x = y = k = 2$, property (ii) of Theorem 4.10.2 does not hold: $T\big(2(2,2)\big) = T(4,4) = (16,4)$ does not equal $2T(2,2) = 2(4,2) = (8,4)$. T is not a matrix operator.

(c) We check whether the two properties of Theorem 4.10.2 hold for all real x, x', y, y', and k:

$T\big((x,y) + (x',y')\big) = T(x + x', y + y') = (-(y + y'), x + x') = (-y, x) + (-y', x') = T(x,y) + T(x',y')$

$T\big(k(x,y)\big) = T(kx, ky) = (-ky, kx) = k(-y, x) = kT(x,y)$

Both properties hold, therefore T is a matrix operator.

(d) We check whether the two properties of Theorem 4.10.2 hold for all real x, x', y, y', and k:

$T\big((x,y) + (x',y')\big) = T(x + x', y + y') = (x + x', 0) = (x, 0) + (x', 0) = T(x,y) + T(x',y')$

$T\big(k(x,y)\big) = T(kx, ky) = (kx, 0) = k(x, 0) = kT(x,y)$

Both properties hold, therefore T is a matrix operator.

18. (a) We check whether the two properties of Theorem 4.10.2 hold for all real x, x', y, y', z, z', and k:

$T\big((x,y,z) + (x',y',z')\big) = T(x + x', y + y', z + z') = (x + x', x + x' + y + y' + z + z')$

$= (x, x + y + z) + (x', x' + y' + z') = T(x,y,z) + T(x',y',z')$

$T\big(k(x,y,z)\big) = T(kx, ky, kz) = (kx, kx + ky + kz) = k(x, x + y + z) = kT(x,y,z)$

Both properties hold, therefore T is a matrix transformation.

(b) $T\big((x,y,z) + (x',y',z')\big) = T(x + x', y + y', z + z') = (1,1)$ does not equal $T(x,y,z) + T(x',y',z') = (1,1) + (1,1) = (2,2)$. Since property (i) of Theorem 4.10.2 does not hold, T is not a matrix transformation.

20. (a) Reflection about the y-axis: $A = [T(1,0) \,|\, T(0,1)] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$;

Reflection about the x-axis: $A = [T(1,0) \,|\, T(0,1)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

Reflection about the line $y = x$: $A = [T(1,0) \,|\, T(0,1)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) Reflection about the xy-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$;

Reflection about the xz-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

Reflection about the yz-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Orthogonal projection on the x-axis: $A = [T(1,0) \mid T(0,1)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;

Orthogonal projection on the y-axis: $A = [T(1,0) \mid T(0,1)] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) Orthogonal projection on the xy-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

Orthogonal projection on the xz-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

Orthogonal projection on the yz-plane: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) Rotation through an angle θ: $A = [T(1,0) \mid T(0,1)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(f) Dilation or contraction by a factor of k: $A = [T(1,0,0) \mid T(0,1,0) \mid T(0,0,1)] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

22. (a) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is the reflection about the xz-plane and T_2 is the contraction with factor $k = \frac{1}{5}$. From Tables 2 and 8 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is the orthogonal projection onto the xz-plane and T_2 is the orthogonal projection onto the xy-plane. From Table 4 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the reflection about the xy-plane, T_2 is the reflection about the xz-plane, and T_3 is the reflection about the yz-plane. From Table 2 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

24. (a) Suppose $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Subtracting both equations yields $A\begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ therefore the transformation is one-to-one if and only if the nullity of A is 0 since that is equivalent to stating that $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ implies $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so we conclude that the nullity of A is 0, thus T_A is one-to-one.

(b) Proceeding as in part (a), we determine the reduced row echelon form of A to be $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$.

Therefore A has nullity 1 and T_A is not one-to-one (e.g., $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).

(c) Proceeding as in part (a), we determine the reduced row echelon form of A to be $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore A has nullity 1 and T_A is not one-to-one (e.g., $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$).

26. Both properties of Theorem 4.10.2 hold for $T(x,y) = (0,0)$:

$T\big((x,y) + (x',y')\big) = T(x + x', y + y') = (0,0) = (0,0) + (0,0) = T(x,y) + T(x',y')$

$T\big(k(x,y)\big) = T(kx, ky) = (0,0) = k(0,0) = kT(x,y)$

On the other hand, neither property holds in general for $T(x,y) = (1,1)$, e.g.,

$T\big((x,y) + (x',y')\big) = T(x + x', y + y') = (1,1)$ does not equal

$T(x,y) + T(x',y') = (1,1) + (1,1) = (2,2)$