CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

2. **(a)** $\mathbf{u}+\mathbf{v}=(0+1+1,4-3+1)=(2,2)$; $k\mathbf{u}=(2\cdot 0,2\cdot 4)=(0,8)$

(b) $(0,0)+(u_1,u_2)=(0+u_1+1,0+u_2+1)=(u_1+1,u_2+1)\neq(u_1,u_2)$ therefore $(0,0)$ is not the zero vector $\mathbf{0}$ required by Axiom 4

(c) For all real numbers u_1 and u_2, we have

$(-1,-1)+(u_1,u_2)=(-1+u_1+1,-1+u_2+1)=(u_1,u_2)$ and

$(u_1,u_2)+(-1,-1)=(u_1-1+1,u_2-1+1)=(u_1,u_2)$ therefore Axiom 4 holds for $\mathbf{0}=(-1,-1)$

(d) For any pair of real numbers $\mathbf{u}=(u_1,u_2)$, letting $-\mathbf{u}=(-2-u_1,-2-u_2)$ yields

$\mathbf{u}+(-\mathbf{u})=(u_1+(-2-u_1)+1,u_2+(-2-u_2)+1)=(-1,-1)=\mathbf{0}$;

Since $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ holds as well, Axiom 5 holds.

(e) Axiom 7 fails to hold:

$k(\mathbf{u}+\mathbf{v})=k(u_1+v_1+1,u_2+v_2+1)=(ku_1+kv_1+k,ku_2+kv_2+k)$

$k\mathbf{u}+k\mathbf{v}=(ku_1,ku_2)+(kv_1,kv_2)=(ku_1+kv_1+1,ku_2+kv_2+1)$

therefore in general $k(\mathbf{u}+\mathbf{v})\neq k\mathbf{u}+k\mathbf{v}$

Axiom 8 fails to hold:

$(k+m)\mathbf{u}=\big((k+m)u_1,(k+m)u_2\big)=(ku_1+mu_1,ku_2+mu_2)$

$k\mathbf{u}+m\mathbf{u}=(ku_1,ku_2)+(mu_1,mu_2)=(ku_1+mu_1+1,ku_2+mu_2+1)$

therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$

4. Let V denote the set of all pairs of real numbers of the form $(x,0)$.

Axiom 1: $(x,0)+(y,0)=(x+y,0)$ is in V for all real x and y;

Axiom 2: $(x,0)+(y,0)=(x+y,0)=(y+x,0)=(y,0)+(x,0)$ for all real x and y;

Axiom 3: $(x,0)+\big((y,0)+(z,0)\big)=(x,0)+(y+z,0)=(x+y+z,0)=(x+y,0)+(z,0)$
$=\big((x,0)+(y,0)\big)+(z,0)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(x,0)=(x,0)$ and $(x,0)+(0,0)=(x,0)$ for all real x;

Axiom 5: for each $\mathbf{u} = (x, 0)$, let $-\mathbf{u} = (-x, 0)$;
then $(x, 0) + (-x, 0) = (0,0)$ and $(-x, 0) + (x, 0) = (0,0)$;

Axiom 6: $k(x, 0) = (kx, 0)$ is in V for all real k and x;

Axiom 7: $k\big((x, 0) + (y, 0)\big) = k(x + y, 0) = (kx + ky, 0) = k(x, 0) + k(y, 0)$ for all real k, x, and y;

Axiom 8: $(k + m)(x, 0) = \big((k + m)x, 0\big) = (kx + mx, 0) = k(x, 0) + m(x, 0)$ for all real k, m, and x;

Axiom 9: $k\big(m(x, 0)\big) = k(mx, 0) = (kmx, 0) = (km)(x, 0)$ for all real k, m, and x;

Axiom 10: $1(x, 0) = (x, 0)$ for all real x.

This is a vector space – all axioms hold.

6. Let V denote the set of all n-tuples of real numbers of the form $(x, x, \ldots, x)$.

Axiom 1: $(x, x, \ldots, x) + (y, y, \ldots, y) = (x + y, x + y, \ldots, x + y)$ is in V for all real x and y;

Axiom 2: $(x, x, \ldots, x) + (y, y, \ldots, y) = (x + y, x + y, \ldots, x + y) = (y + x, y + x, \ldots, y + x)$
$= (y, y, \ldots, y) + (x, x, \ldots, x)$ for all real x and y;

Axiom 3: $(x, x, \ldots, x) + \big((y, y, \ldots, y) + (z, z, \ldots, z)\big) = (x, x, \ldots, x) + (y + z, y + z, \ldots, y + z)$
$= (x + y + z, x + y + z, \ldots, x + y + z) = (x + y, x + y, \ldots, x + y) + (z, z, \ldots, z)$
$= \big((x, x, \ldots, x) + (y, y, \ldots, y)\big) + (z, z, \ldots, z)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0} = (0,0, \ldots, 0)$, we have $(0,0, \ldots, 0) + (x, x, \ldots, x) = (x, x, \ldots, x)$ and
$(x, x, \ldots, x) + (0,0, \ldots, 0) = (x, x, \ldots, x)$ for all real x;

Axiom 5: for each $\mathbf{u} = (x, x, \ldots, x)$, let $-\mathbf{u} = (-x, -x, \ldots, -x)$;
then $(x, x, \ldots, x) + (-x, -x, \ldots, -x) = (0,0, \ldots, 0)$ and
$(-x, -x, \ldots, -x) + (x, x, \ldots, x) = (0,0, \ldots, 0)$;

Axiom 6: $k(x, x, \ldots, x) = (kx, kx, \ldots, kx)$ is in V for all real k and x;

Axiom 7: $k\big((x, x, \ldots, x) + (y, y, \ldots, y)\big) = k(x + y, x + y, \ldots, x + y) = (kx + ky, kx + ky, \ldots, kx + ky)$
$= k(x, x, \ldots, x) + k(y, y, \ldots, y)$ for all real k, x, and y;

Axiom 8: $(k+m)(x,x,\ldots,x) = \big((k+m)x,(k+m)x,\ldots,(k+m)x\big)$
$= (kx+mx,kx+mx,\ldots,kx+mx) = k(x,x,\ldots,x)+m(x,x,\ldots,x)$ for all real k, m, and x;

Axiom 9: $k\big(m(x,x,\ldots,x)\big) = k(mx,mx,\ldots,mx) = (kmx,kmx,\ldots,kmx) = (km)(x,x,\ldots,x)$
for all real k, m, and x;

Axiom 10: $1(x,x,\ldots,x) = (x,x,\ldots,x)$ for all real x.

This is a vector space – all axioms hold.

8. Axiom 1 fails since a sum of two 2×2 invertible matrices may or may not be invertible, e.g. both $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ are invertible, but $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not invertible.

Axiom 6 fails whenever $k=0$.

10. Let V be the set of all real-valued functions f defined for all real numbers and such that $f(1)=0$.

Axiom 1: If f and g are in V then $f+g$ is a function defined for all real numbers and $(f+g)(1)=f(1)+g(1)=0$ therefore V is closed under the operation of addition defined by Formula (2) on p.175.

Axiom 6: If k is a scalar and f is in V then kf is a function defined for all real numbers and $(kf)(1)=k\big(f(1)\big)=0$ therefore V is closed under the operation of scalar multiplication defined by Formula (3) on p.175.

Verification of the eight remaining axioms proceeds analogously to Example 6 on p.175.

This is a vector space – all axioms hold.

12. Let V be the set of polynomials of the form $a+bx$.

Axiom 1: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x$ is in V for all real a_0, a_1, b_0, and b_1;

Axiom 2: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x=(a_1+a_0)+(b_1+b_0)x$
$=(a_1+b_1x)+(a_0+b_0x)$ for all real a_0, a_1, b_0, and b_1;

Axiom 3: $(a_0+b_0x)+\big((a_1+b_1x)+(a_2+b_2x)\big)=(a_0+a_1+a_2)+(b_0+b_1+b_2)x$
$\big((a_0+b_0x)+(a_1+b_1x)\big)+(a_2+b_2x)$ for all real a_0, a_1, a_2, b_0, b_1, and b_2;

Axiom 4: taking $\mathbf{0} = 0 + 0x,$ we have $(0 + 0x) + (a + bx) = a + bx$ and $(a + bx) + (0 + 0x) = a + bx$ for all real a and b;

Axiom 5: for each $\mathbf{u} = a + bx,$ let $-\mathbf{u} = -a - bx;$ then $(a + bx) + (-a - bx) = 0 + 0x = (-a - bx) + (a + bx)$ for all real a and b;

Axiom 6: $k(a + bx) = ka + (kb)x$ is in V for all real $a,$ $b,$ and k;

Axiom 7: $k\big((a_0 + b_0x) + (a_1 + b_1x)\big) = k\big((a_0 + a_1) + (b_0 + b_1)x\big) = k(a_0 + b_0x) + k(a_1 + b_1x)$ for all real $a_0, a_1, b_0, b_1,$ and k;

Axiom 8: $(k + m)(a + bx) = (k + m)a + (k + m)bx = k(a + bx) + m(a + bx)$ for all real $a, b, k,$ and m;

Axiom 9: $k\big(m(a + bx)\big) = k(ma + mbx) = kma + kmbx = (km)(a + bx)$ for all real $a, b, k,$ and m;

Axiom 10: $1(a + bx) = a + bx$ for all real a and $b.$

This is a vector space – all axioms hold.

4.2 Subspaces

2. **(a)** Let W be the set of all $n \times n$ diagonal matrices.

This set contains at least one matrix, e.g. the zero $n \times n$ matrix.

Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W:

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & 0 & \cdots & 0 \\ 0 & a_{22} + b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} + b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W:

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of $M_{nn}.$

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W. However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume $A = [a_{ij}]$ and $B = [b_{ij}]$ are both in W, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\operatorname{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$. Since $\operatorname{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$ $= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.

A scalar multiple of the same matrix A with a scalar k has $tr(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).

This set contains at least one matrix, e.g., I_n is in W.

Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A + B)^T = A^T + B^T = A + B$ therefore W is closed under addition.

From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(e) Let W be the set of all $n \times n$ matrices such that $A^T = -A$.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume A and B are both in W, i.e. $A^T = -A$ and $B^T = -B$. By Theorem 1.4.8(b), their sum satisfies $(A + B)^T = A^T + B^T = -A - B = -(A + B)$ therefore W is closed under addition.

From Theorem 1.4.8(d), we have $(kA)^T = kA^T = k(-A) = -kA$ which makes W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(f) Let W be the set of $n \times n$ matrices for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. It follows from Theorem 1.5.3 that the set W consists of all $n \times n$ matrices that are invertible. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

(g) Let B be some fixed $n \times n$ matrix, and let W be the set of all $n \times n$ matrices A such that $AB = BA$. This set contains at least one matrix, e.g., I_n is in W.

Let us assume A and C are both in W, i.e. $AB = BA$ and $CB = BC$. By Theorem 1.4.1(d,e), their sum satisfies $(A + C)B = AB + CB = BA + BC = B(A + C)$ therefore W is closed under addition.

From Theorem 1.4.1(m), we have $(kA)B = k(AB) = k(BA) = B(kA)$ which makes W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of M_{nn}.

4. **(a)** Let W be the set of all functions f in $F(-\infty, \infty)$ for which $f(0) = 0$.

This set contains at least one function, e.g., the constant function $f(x) = 0$.

Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty, \infty)$ and satisfies $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.

A scalar multiple of a function f in W, kf, is also a function in $F(-\infty, \infty)$ for which $(kf)(0) = k\big(f(0)\big) = 0$ making W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of $F(-\infty, \infty)$.

(b) Let W be the set of all functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.

We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be two functions in W. Their sum, $f + g$, is not in W since $(f + g)(0) = f(0) + g(0) = 1 + 1 = 2$.

We conclude that W is not a subspace of $F(-\infty, \infty)$.

(c) Let W be the set of all functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

This set contains at least one function, e.g., the constant function $f(x) = 0$.

Assume we have two functions f and g in W, i.e., $f(-x) = f(x)$ and $g(-x) = g(x)$. Their sum $f + g$ is also a function in $F(-\infty, \infty)$ and satisfies $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$ therefore W is closed under addition.

A scalar multiple of a function f in W, kf, is also a function in $F(-\infty, \infty)$ for which $(kf)(-x) = k\big(f(-x)\big) = k\big(f(x)\big) = (kf)(x)$ making W closed under scalar multiplication.

According to Theorem 4.2.1, W is a subspace of $F(-\infty, \infty)$.

(d) A sum of two polynomials of degree 2 may be a polynomial of lower degree, e.g.,

$$(1 + x^2) + (x - x^2) = 1 + x$$

therefore the set is not closed under addition, and consequently is not a subspace of $F(-\infty, \infty)$.

6. The line L contains at least one point – e.g., the origin.

If the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are both on L, then there must exist real numbers t_1 and t_2 such that $x_1 = at_1$, $y_1 = bt_1$, $z_1 = ct_1$, $x_2 = at_2$, $y_2 = bt_2$, and $z_2 = ct_2$.

L is closed under addition since $(x_1, y_1, z_1) + (x_2, y_2, z_2) = ((a)(t_1 + t_2), (b)(t_1 + t_2), (c)(t_1 + t_2))$.

It is also closed under scalar multiplication because $k(x_1, y_1, z_1) = ((a)(kt_1), (b)(kt_1), (c)(kt_1))$.

It follows from Theorem 4.2.1 that L is a subspace of R^3.

8. **(a)** For $(-9, -7, -15)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (-9,-7,-15)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a = -2$, $b = 1$, $c = -2$, therefore $(-9, -7, -15) = -2\mathbf{u} + 1\mathbf{v} - 2\mathbf{w}$.

(b) For $(6,11,6)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (6,11,6)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a = 4$, $b = -5$, $c = 1$, therefore $(6,11,6) = 4\mathbf{u} - 5\mathbf{v} + 1\mathbf{w}$.

(c) For $(0,0,0)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (0,0,0)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = 0,\ c = 0,$ therefore $(0,0,0) = 0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w}$.

(d) For $(7,8,9)$ to be a linear combination of the vectors $\mathbf{u},\ \mathbf{v}$, and $\mathbf{w},$ there must exist scalars $a,\ b,$ and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (7,8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = -2,\ c = 3,$ therefore $(7,8,9) = 0\mathbf{u} - 2\mathbf{v} + 3\mathbf{w}$.

10. (a) For $-9 - 7x - 15x^2$ to be a linear combination of the vectors $\mathbf{p}_1,\ \mathbf{p}_2$, and $\mathbf{p}_3,$ there must exist scalars $a,\ b,$ and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = -9 - 7x - 15x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = -9 - 7x - 15x^2$$

Since this equality must hold for every real value $x,$ the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a = -2$, $b = 1$, $c = -2$, therefore $-9 - 7x - 15x^2 = -2\mathbf{p}_1 + 1\mathbf{p}_2 - 2\mathbf{p}_3$.

(b) For $6 + 11x + 6x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 6 + 11x + 6x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 6 + 11x + 6x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a = 4$, $b = -5$, $c = 1$, therefore $6 + 11x + 6x^2 = 4\mathbf{p}_1 - 5\mathbf{p}_2 + 1\mathbf{p}_3$.

(c) For 0 to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 0$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 0 + 0x + 0x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = 0,\ c = 0,$ therefore $0 = 0\mathbf{p}_1 + 0\mathbf{p}_2 + 0\mathbf{p}_3$.

(d) For $7 + 8x + 9x^2$ to be a linear combination of the vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3,$ there must exist scalars $a,\ b,$ and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 7 + 8x + 9x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 7 + 8x + 9x^2$$

Since this equality must hold for every real value $x,$ the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = -2,\ c = 3,$ therefore $7 + 8x + 9x^2 = 0\mathbf{p}_1 - 2\mathbf{p}_2 + 3\mathbf{p}_3$.

12. (a) In order for the vector $(2,3,-7,3)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\},$ there must exist scalars $a,\ b,$ and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (2,3,-7,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 2 \\ 1a & - & 1b & + & 0c & = & 3 \\ 0a & + & 5b & + & 2c & = & -7 \\ 3a & + & 2b & + & 1c & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This system is consistent (its only solution is $a = 2,\ b = -1,\ c = -1$), therefore $(2,3,-7,3)$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) The vector $(0,0,0,0)$ is obviously in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ since

$$0(2,1,0,3)+0(3,-1,5,2)+0(-1,0,2,1)=(0,0,0,0)$$

(c) In order for the vector $(1,1,1,1)$ to be in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(1,1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 1 \\ 1a & - & 1b & + & 0c & = & 1 \\ 0a & + & 5b & + & 2c & = & 1 \\ 3a & + & 2b & + & 1c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This system is inconsistent therefore $(1,1,1,1)$ is not in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

(d) In order for the vector $(-4,6,-13,4)$ to be in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(-4,6,-13,4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & -4 \\ 1a & - & 1b & + & 0c & = & 6 \\ 0a & + & 5b & + & 2c & = & -13 \\ 3a & + & 2b & + & 1c & = & 4 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This system is consistent (its only solution is $a=3$, $b=-3$, $c=1$), therefore $(-4,6,-13,4)$ is in $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

14. (a) It follows from the trigonometric identity $\cos 2x=\cos^2 x-\sin^2 x$ that $\cos 2x$ is in $\text{span}\{\mathbf{f},\mathbf{g}\}$.

(b) In order for $3+x^2$ to be in $\text{span}\{\mathbf{f},\mathbf{g}\}$, there must exist scalars a and b such that

$$a\cos^2 x + b\sin^2 x = 3 + x^2$$

holds for all real x values. When $x = 0$ the equation becomes $a = 3$, however if $x = \pi$ then it yields $a = 3 + \pi^2$ - a contradiction. We conclude that $3 + x^2$ is not in $\operatorname{span}\{\mathbf{f}, \mathbf{g}\}$.

(c) It follows from the trigonometric identity $\cos^2 x + \sin^2 x = 1$ that 1 is in $\operatorname{span}\{\mathbf{f}, \mathbf{g}\}$.

(d) In order for $\sin x$ to be in $\operatorname{span}\{\mathbf{f}, \mathbf{g}\}$, there must exist scalars a and b such that

$$a\cos^2 x + b\sin^2 x = \sin x$$

holds for all real x values. When $x = \frac{\pi}{2}$ the equation becomes $b = 1$, however if $x = -\frac{\pi}{2}$ then it yields $b = -1$ - a contradiction. We conclude that $\sin x$ is not in $\operatorname{span}\{\mathbf{f}, \mathbf{g}\}$.

(e) Since $0\cos^2 x + 0\sin^2 x = 0$ holds for all real x values, we conclude that 0 is in $\operatorname{span}\{\mathbf{f}, \mathbf{g}\}$.

20. We begin by showing that the vector $\mathbf{w}_1$ is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, i.e., that there exist scalars a, b, and c such that

$$a(1,6,4) + b(2,4,-1) + c(-1,2,5) = (1,-2,-5)$$

Equating corresponding components on both sides leads to the linear system

$$\begin{array}{rcrcrcr} 1a & + & 2b & - & 1c & = & 1 \\ 6a & + & 4b & + & 2c & = & -2 \\ 4a & - & 1b & + & 5c & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system is $a = -1 - t,\ b = 1 + t,\ c = t.$ E.g., letting $t = 0$ yields a solution $a = -1,\ b = 1,\ c = 0.$

Applying the same procedure repeatedly to each of the remaining four vectors, we can show that

$$\begin{aligned} \mathbf{w}_1 &= -1\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3 \\ \mathbf{w}_2 &= 2\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 \\ \mathbf{v}_1 &= 1\mathbf{w}_1 + 1\mathbf{w}_2 \\ \mathbf{v}_2 &= 2\mathbf{w}_1 + 1\mathbf{w}_2 \\ \mathbf{v}_3 &= -1\mathbf{w}_1 + 0\mathbf{w}_2 \end{aligned}$$

It follows from Theorem 4.2.5 that the sets $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ span the same subspace of R^3.

4.3 Linear Independence

2. **(a)** The vector equation $a(4,-1,2)+b(-4,10,2)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcl} 4a & - & 4b & = & 0 \\ -1a & + & 10b & = & 0 \\ 2a & + & 2b & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=0$. We conclude that the given set of vectors is linearly independent.

(b) The vector equation $a(-3,0,4)+b(5,-1,2)+c(1,1,3)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -3a & + & 5b & + & 1c & = & 0 \\ 0a & - & 1b & + & 1c & = & 0 \\ 4a & + & 2b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given set of vectors is linearly independent.

(c) The vector equation $a(8,-1,3)+b(4,0,1)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcl} 8a & + & 4b & = & 0 \\ -1a & + & 0b & = & 0 \\ 3a & + & 1b & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=0$. We conclude that the given set of vectors is linearly independent.

(d) The vector equation $a(-2,0,1)+b(3,2,5)+c(6,-1,1)+d(7,0,-2)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} -2a & + & 3b & + & 6c & + & 7d & = & 0 \\ 0a & + & 2b & - & 1c & + & 0d & = & 0 \\ 1a & + & 5b & + & 1c & - & 2d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{bmatrix}$ therefore a general solution of the system is

$$a=\frac{79}{29}t,\ b=-\frac{3}{29}t,\ c=-\frac{6}{29}t,\ d=t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

4. **(a)** The terms in the equation

$$a(2-x+4x^2)+b(3+6x+2x^2)+c(2+10x-4x^2)=0$$

can be grouped according to the powers of x

$$(2a+3b+2c)+(-a+6b+10c)x+(4a+2b-4c)x^2=0+0x+0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & 2c & = & 0 \\ -a & + & 6b & + & 10c & = & 0 \\ 4a & + & 2b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given set of vectors in P_2 is linearly independent.

(b) The terms in the equation

$$a(3+x+x^2)+b(2-x+5x^2)+c(4-3x^2)=0$$

can be grouped according to the powers of x

$$(3a+2b+4c)+(a-b)x+(a+5b-3c)x^2=0+0x+0x^2$$

For this to hold for all real values of $x,$ the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcl} 3a & + & 2b & + & 4c & = & 0 \\ a & - & b & & & = & 0 \\ a & + & 5b & - & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0.$ We conclude that the given set of vectors in P_2 is linearly independent.

(c) The terms in the equation

$$a(6-x^2)+b(1+x+4x^2)=0$$

can be grouped according to the powers of x

$$(6a+b)+bx+(-a+4b)x^2=0+0x+0x^2$$

For this to hold for all real values of $x,$ the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcl} 6a & + & b & = & 0 \\ & & b & = & 0 \\ -a & + & 4b & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=0.$ We conclude that the given set of vectors in P_2 is linearly independent.

(d) The terms in the equation

$$a(1+3x+3x^2)+b(x+4x^2)+c(5+6x+3x^2)+d(7+2x-x^2)=0$$

can be grouped according to the powers of x

$$(a+5c+7d)+(3a+b+6c+2d)x+(3a+4b+3c-d)x^2=0+0x+0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{ccccccccc} a & & & + & 5c & + & 7d & = & 0 \\ 3a & + & b & + & 6c & + & 2d & = & 0 \\ 3a & + & 4b & + & 3c & - & d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{17}{4} & 0 \\ 0 & 1 & 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 & \frac{9}{4} & 0 \end{bmatrix}$ therefore a general solution of the system is

$$a=\frac{17}{4}t,\ b=-\frac{5}{4}t,\ c=-\frac{9}{4}t,\ d=t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

6. **(a)** The set $\{\mathbf{v}_1,\mathbf{v}_3\}$ can be shown to be linearly independent since $a(-1,2,3)+b(-3,6,0)=(0,0,0)$ has only the trivial solution $a=b=0$. Therefore the three vectors do not lie on the same line (even though the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *are* collinear).

(b) Any subset of two vectors chosen from these three vectors can be shown to be linearly independent (e.g., $a(2,-1,4)+b(4,2,3)=(0,0,0)$ has only the trivial solution $a=b=0$). Therefore the three vectors do not lie on the same line.

(An alternate way to show this would be to demonstrate that the three vectors form a linearly independent set, therefore they do not even lie on the same plane, so that they cannot possibly lie on the same line.)

(c) Each subset of two vectors chosen from these three vectors can be shown to be linearly dependent since $-1\mathbf{v}_1+2\mathbf{v}_2=\mathbf{0}$, $1\mathbf{v}_1+2\mathbf{v}_3=\mathbf{0}$, and $1\mathbf{v}_2+1\mathbf{v}_3=\mathbf{0}$. Therefore all three vectors lie on the same line.

8. **(a)** The vector equation $a(1,2,3,4)+b(0,1,0,-1)+c(1,3,3,3)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{ccccccc} 1a & + & 0b & + & 1c & = & 0 \\ 2a & + & 1b & + & 3c & = & 0 \\ 3a & + & 0b & + & 3c & = & 0 \\ 4a & - & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is

$$a=-t,\ b=-t,\ c=t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) In the general solution we obtained in part (a), let the parameter t have a nonzero value, e.g., $t=1$. Then $a=-1$, $b=-1$, and $c=1$ so that $-\mathbf{v}_1-\mathbf{v}_2+\mathbf{v}_3=\mathbf{0}$. This can be solved for each of the three vectors: $\mathbf{v}_1=-\mathbf{v}_2+\mathbf{v}_3$, $\mathbf{v}_2=-\mathbf{v}_1+\mathbf{v}_3$, and $\mathbf{v}_3=\mathbf{v}_1+\mathbf{v}_2$.

20. **(a)** From the identity $\sin^2 x+\cos^2 x=1$ we have $(-1)(6)+(2)(3\sin^2 x)+(3)(2\cos^2 x)=0$ for all real x. Therefore, the set is linearly dependent.

(b) The equality $ax+b\cos x=0$ is to hold for all real x. Taking $x=0$ yields $b=0$, whereas taking $x=\frac{\pi}{2}$ implies $a=0$. The set is linearly independent.

(c) The equality $(a)(1)+b\sin x+c\sin 2x=0$ is to hold for all real x. Taking $x=0$ yields $a=0$. When $x=\frac{\pi}{2}$, we obtain $b=0$. Finally, substituting $x=\frac{\pi}{4}$ results in $c=0$. The set is linearly independent.

(d) From the identity $\cos^2 x-\sin^2 x=\cos 2x$ we have $(1)(\cos 2x)+(1)(\sin^2 x)+(-1)(\cos^2 x)=0$ for all real x. Therefore, the set is linearly dependent.

(e) Since $(3-x)^2=9-6x+x^2$ we can write $(3-x)^2-(x^2-6x)-9=0$ or $(1)(3-x)^2+(-1)(x^2-6x)+\left(-\frac{9}{5}\right)(5)=0$. The set is linearly dependent.

(f) From Theorem 4.3.2(a), this set is linearly dependent.

22. The Wronskian is $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$. Since $W(x)$ is not identically 0 on $(-\infty, \infty)$, $\sin x$ and $\cos x$ are linearly independent.

24.

$$\begin{aligned}
W(x) &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} && \longleftarrow \text{The Wronskian} \\
&= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 + x & 2x + x^2 \\ 1 & 2 + x & 2 + 4x + x^2 \end{vmatrix} && \longleftarrow \text{A common factor of } e^x \text{ from each row was taken through the determinant sign.} \\
&= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2 + 4x \end{vmatrix} && \longleftarrow -1 \text{ times the first row was added to the second row and to the third row.} \\
&= (e^{3x})(1) \begin{vmatrix} 1 & 2x \\ 2 & 2 + 4x \end{vmatrix} && \longleftarrow \text{Cofactor expansion along the first column} \\
&= (\mathrm{e}^{3x})(1)(2 + 4x - 4x) = 2e^{3x}
\end{aligned}$$

Since $W(x)$ is not identically 0 on $(-\infty, \infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

4.4 Coordinates and Basis

2. **(a)** Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1) + c_2(3,0) = (0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector **b** $= (b_1, b_2)$ in R^2 as

$$c_1(2,1) + c_2(3,0) = (b_1, b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & b_1 \\ c_1 & & & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independent and span R^2 so that they form a basis for R^2.

(b) Vectors $(4,1)$ and $(-7,-8)$ are linearly independent if the vector equation

$$c_1(4,1) + c_2(-7.-8) = (0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b} = (b_1, b_2)$ in R^2 as

$$c_1(4,1) + c_2(-7,-8) = (b_1, b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 4c_1 & - & 7c_2 & = & 0 \\ c_1 & - & 8c_2 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcl} 4c_1 & - & 7c_2 & = & b_1 \\ c_1 & - & 8c_2 & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 4 & -7 \\ 1 & -8 \end{vmatrix} = -25 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(4,1)$ and $(-7,-8)$ are linearly independent and span R^2 so that they form a basis for R^2.

(c) From Theorem 4.3.2(a), the set is linearly dependent, therefore it does not form a basis for R^2.

(d) Since $4(3,9) + 3(-4,-12) = (0,0)$, the set is linearly dependent, therefore it does not form a basis for R^2.

4. **(a)** Vectors $\mathbf{p}_1 = 1 - 3x + 2x^2$, $\mathbf{p}_2 = 1 + x + 4x^2$, and $\mathbf{p}_3 = 1 - 7x$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.
By grouping the terms on the left hand side as $c_1(1 - 3x + 2x^2) + c_2(1 + x + 4x^2) + c_3(1 - 7x) = (c_1 + c_2 + c_3) + (-3c_1 + c_2 - 7c_3)x + (2c_1 + 4c_2)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & 0 \\ -3c_1 & + & c_2 & - & 7c_3 & = & 0 \\ 2c_1 & + & 4c_2 & & & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

(b) Vectors $\mathbf{p}_1 = 4 + 6x + x^2$, $\mathbf{p}_2 = -1 + 4x + 2x^2$, and $\mathbf{p}_3 = 5 + 2x - x^2$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.

Grouping the terms on the left hand side as $c_1(4 + 6x + x^2) + c_2(-1 + 4x + 2x^2) + c_3(5 + 2x - x^2) = (4c_1 - c_2 + 5c_3) + (6c_1 + 4c_2 + 2c_3)x + (c_1 + 2c_2 - c_3)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcl} 4c_1 & - & c_2 & + & 5c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ c_1 & + & 2c_2 & - & c_3 & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

(c) Vectors $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x + x^2$, and $\mathbf{p}_3 = x^2$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

By grouping the terms on the left hand sides as $c_1(1 + x + x^2) + c_2(x + x^2) + c_3(x^2) = c_1 + (c_1 + c_2)x + (c_1 + c_2 + c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

(d) Vectors $\mathbf{p}_1 = -4 + x + 3x^2$, $\mathbf{p}_2 = 6 + 5x + 2x^2$, and $\mathbf{p}_3 = 8 + 4x + x^2$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1(-4 + x + 3x^2) + c_2(6 + 5x + 2x^2) + c_3(8 + 4x + x^2) = (-4c_1 + 6c_2 + 8c_3) + (c_1 + 5c_2 + 4c_3)x + (3c_1 + 2c_2 + c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} -4c_1 & + & 6c_2 & + & 8c_3 & = & 0 \\ c_1 & + & 5c_2 & + & 4c_3 & = & 0 \\ 3c_1 & + & 2c_2 & + & c_3 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcl} -4c_1 & + & 6c_2 & + & 8c_3 & = & a_0 \\ c_1 & + & 5c_2 & + & 4c_3 & = & a_1 \\ 3c_1 & + & 2c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} = -26 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

6. **(a)** The identity $\cos^2 x - \sin^2 x = \cos 2x$ can be rewritten as $1\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$ which shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, therefore it is not a basis for V.

(b) For the equation $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0$ (required when $x = 0$) and $c_2 = 0$ (required when $x = \frac{\pi}{2}$). Therefore the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ are linearly independent.

Any vector $\mathbf{v}$ in V can be expressed as $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$. However, from the identity $\cos^2 x - \sin^2 x = \cos 2x$ it follows that we can express $\mathbf{v}$ as a linear combination of $\cos^2 x$ and $\sin^2 x$ alone: $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3(\cos^2 x - \sin^2 x) = (k_1 + k_3)\cos^2 x + (k_2 - k_3)\sin^2 x$. This proves that the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ span V.

We conclude that $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ form a basis for V.

8. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 1 \\ -c_1 & + & c_2 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = (\frac{1}{2}, \frac{1}{2})$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(0,1) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ -c_1 & + & c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = (-\frac{1}{2}, \frac{1}{2})$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obviously have $(1,1) = 0(1,-1) + 1(1,1)$ therefore the coordinate vector is $(\mathbf{w})_S = (\,0,1\,)$.

10. **(a)** Since $\mathbf{p} = 4\mathbf{p}_1 + (-3)\mathbf{p}_2 + 1\mathbf{p}_3$ we conclude that the coordinate vector is $(\mathbf{p})_S = (4,-3,1)$.

(b) Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{ccccccc} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0$, $c_2 = 2$, $c_3 = -1$, therefore the coordinate vector is $(\mathbf{p})_S = (0,2,-1)$.

12. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 + k_2 + k_3 & k_2 \\ k_1 + k_4 & k_3 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 0 \\ & & k_2 & & & & & = & 0 \\ k_1 & & & & & + & k_4 & = & 0 \\ & & & & k_3 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & a \\ & & k_2 & & & & & = & b \\ k_1 & & & & & + & k_4 & = & c \\ & & & & k_3 & & & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 6 \\ & & k_2 & & & & & = & 2 \\ k_1 & & & & & + & k_4 & = & 5 \\ & & & & k_3 & & & = & 3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ therefore the solution is $k_1 = 1,\ k_2 = 2,\ k_3 = 3,\ k_4 = 4$. This allows us to express $A = 1A_1 + 2A_2 + 3A_3 + 4A_4$.

14. Vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = (c_1 + 2c_2 + 3c_3) + (2c_1 + 9c_2 + 3c_3)x + (c_1 + 4c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & 3c_3 & = & 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 0 \\ c_1 & & & + & 4c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccc} c_1 & + & 2c_2 & + & 3c_3 & = & a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & a_1 \\ c_1 & & & + & 4c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values $a_0,\ a_1,$ and a_2. Therefore the vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 2 + 17x - 3x^2$ as a linear combination of the vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 17 \\ c_1 & & & + & 4c_3 & = & -3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}$ therefore the solution is $c_1 = 1,\ c_2 = 2,\ c_3 = -1$. This allows us to express $\mathbf{p} = 1\mathbf{p}_1 + 2\mathbf{p}_2 + (-1)\mathbf{p}_3$.

16. **(a)** $(0,\sqrt{2})$; **(b)** $(1,0)$; **(c)** $(-1,\sqrt{2})$; **(d)** $(a-b,\sqrt{2}b)$

4.5 Dimension

2. The augmented matrix of the linear system $\begin{bmatrix}3&1&1&1&0\\5&-1&1&-1&0\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1&0&\frac{1}{4}&0&0\\0&1&\frac{1}{4}&1&0\end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s,\ x_2 = -\frac{1}{4}s - t,\ x_3 = s,\ x_4 = t$. In vector form

$$(x_1,x_2,x_3,x_4) = \left(-\tfrac{1}{4}s, -\tfrac{1}{4}s - t, s, t\right) = s\left(-\tfrac{1}{4}, -\tfrac{1}{4}, 1,0\right) + t(0,-1,0,1)$$

therefore the solution space is spanned by the vectors $\mathbf{v}_1 = \left(-\frac{1}{4}, -\frac{1}{4}, 1,0\right)$ and $\mathbf{v}_2 = (0,-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

4. The augmented matrix of the linear system $\begin{bmatrix}1&-3&1&0\\2&-6&2&0\\3&-9&3&0\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1&-3&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_1 = 3s - t,\ x_2 = s,\ x_3 = t$. In vector form

$$(x_1,x_2,x_3) = (3s - t, s, t) = s(3,1,0) + t(-1,0,1)$$

therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (3,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

6. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x = 4t,\ y = -5t,\ z = t$. In vector form

$$(x, y, z) = (4t, -5t, t) = t(4, -5, 1)$$

therefore the solution space is spanned by the vector $\mathbf{v}_1 = (4, -5, 1)$. By Theorem 4.3.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

8. **(a)** The given subspace can be expressed as $\text{span}(S)$ where $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a, b, a+b, a-b) = a(1,0,1,1) + b(0,1,1,-1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,0,1,1), (0,1,1,-1)\}$. By Theorem 4.3.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a, a, a, a) = a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,1,1,1)\}$. By Theorem 4.3.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

10. The given subspace can be expressed as $\text{span}(S)$ where $S = \{x, x^2, x^3\}$ is a set of linearly independent vectors in P_3. Therefore S forms a basis for the subspace. The dimension of the subspace is 3.

12. **(a)** Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

16. One of the infinitely many ways to enlarge the given set to a basis for R^4 is by adding the vectors $(1,0,0,0)$ and $(0,1,0,0)$ to the set. Since the resulting set contains $\dim(R^4) = 4$ vectors, by Theorem 4.5.4 we only need to establish the linear independence of the set to be able to conclude that it forms a basis for R^4. The homogeneous equation $k_1(1,-2,3,-5) + k_2(0,-1,2,-3) + k_3(1,0,0,0) + k_4(0,1,0,0) = (0,0,0,0)$ can be rewritten as a linear system whose coefficient matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ -5 & -3 & 0 & 0 \end{bmatrix}$ has determinant 1. Using parts (b) and (g) of Theorem 2.3.8, we conclude that there is only the trivial solution, therefore the enlarged set of four vectors is linearly independent (and, consequently, forms a basis for R^4).

20. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S = \{1, x, x^2\}$.

(a) Denote $\mathbf{v}_1 = -1 + x - 2x^2$, $\mathbf{v}_2 = 3 + 3x + 6x^2$, $\mathbf{v}_3 = 9$.
Then $(\mathbf{v}_1)_S = (-1,1,-2)$, $(\mathbf{v}_2)_S = (3,3,6)$, $(\mathbf{v}_3)_S = (9,0,0)$.
Setting $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S + k_3(\mathbf{v}_3)_S = \mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix} -1 & 3 & 9 & 0 \\ 1 & 3 & 0 & 0 \\ -2 & 6 & 0 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2) = 3$, from Theorem 4.5.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1 = 1 + x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = -2 + 2x^2$, $\mathbf{v}_4 = -3x$.
Then $(\mathbf{v}_1)_S = (1,1,0)$, $(\mathbf{v}_2)_S = (0,0,1)$, $(\mathbf{v}_3)_S = (-2,0,2)$, $(\mathbf{v}_4)_S = (0,-3,0)$.
Setting $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S + k_3(\mathbf{v}_3)_S + k_4(\mathbf{v}_4)_S = \mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 \end{bmatrix}$.
Based on the leading entries in the first three columns, the vector equation
$k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S + k_3(\mathbf{v}_3)_S = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$, $(\mathbf{v}_2)_S$, and $(\mathbf{v}_3)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2) = 3$, from Theorem 4.5.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.