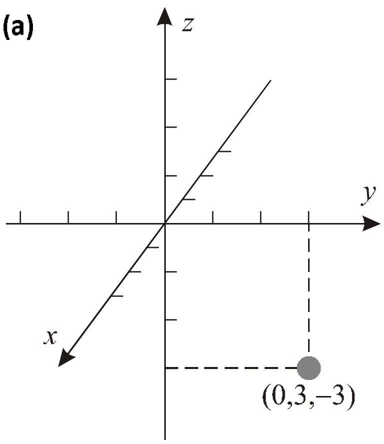
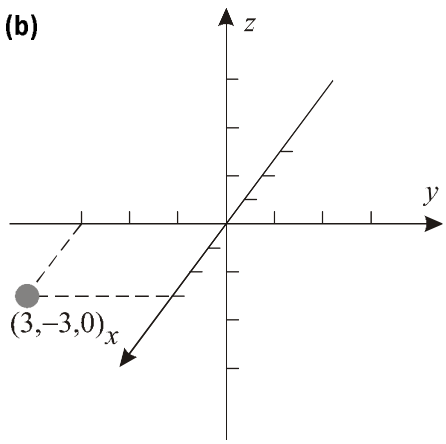
CHAPTER 3: EUCLIDEAN VECTOR SPACES

3.1Vectors in 2-Space, 3-Space, and n-Space

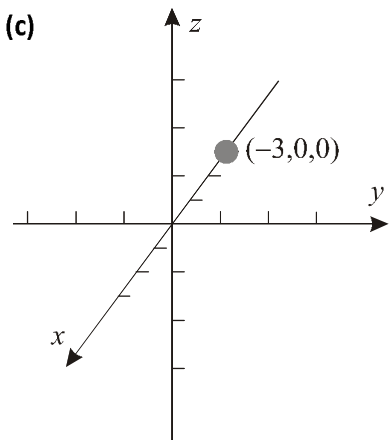
2. **(a)**



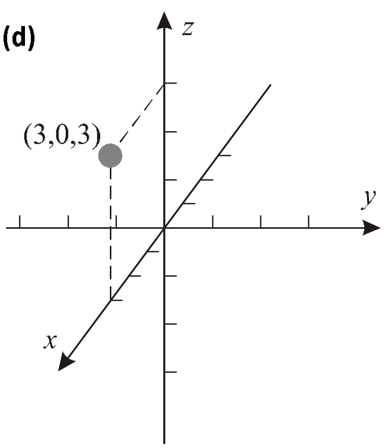
(b)



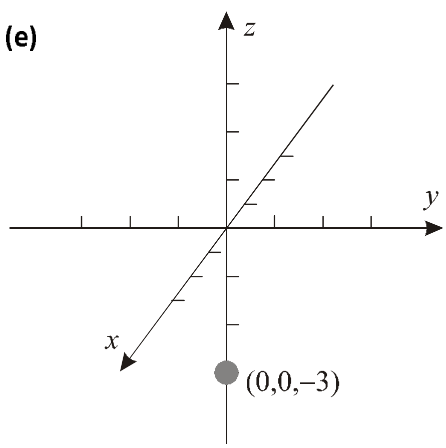
(c)



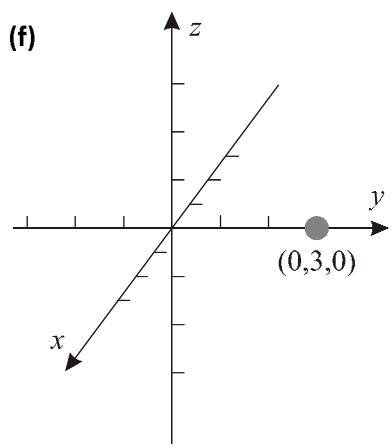
(d)



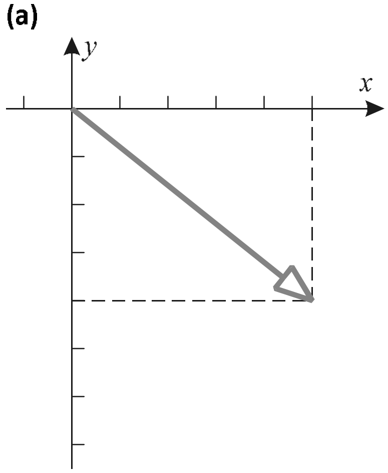
(e)



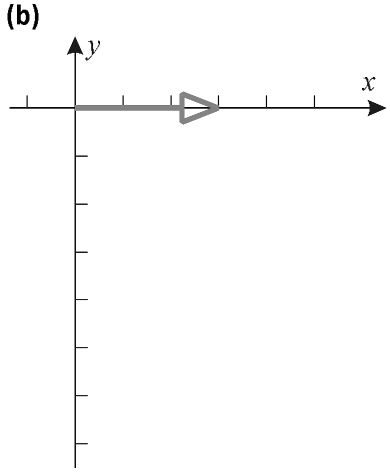
(f)



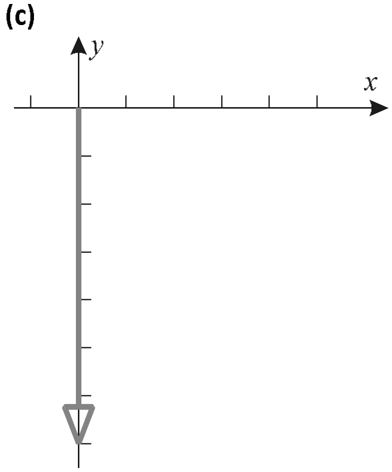
4. **(a)**

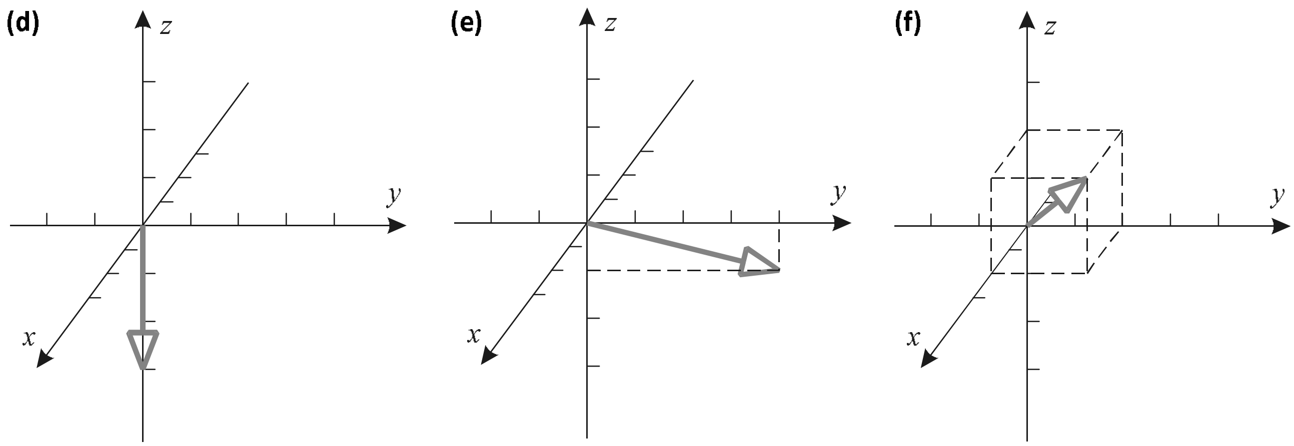


(b)



(c)





6. **(a)** $\overrightarrow{P_1P_2} = (-3-(-5),\ 0-1) = (2,1)$

(b) $\overrightarrow{P_1P_2} = (3-0,\ 4-0) = (3,4)$

(c) $\overrightarrow{P_1P_2} = (0-(-1), -1-0,\ 0-2) = (1,-1,-2)$

(d) $\overrightarrow{P_1P_2} = (0-2,\ 0-2,\ 0-2) = (-2,-2,-2)$

Illustration for part (a):

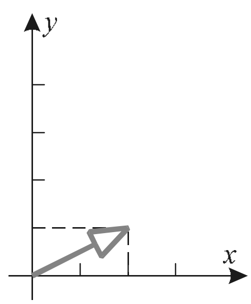


Illustration for part (b):

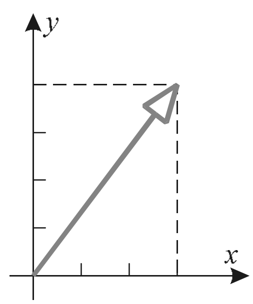


Illustration for part (c):

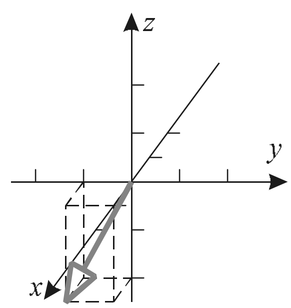
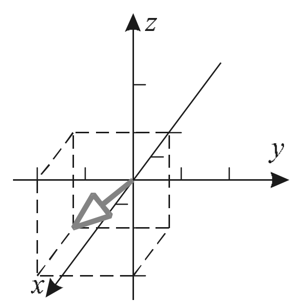


Illustration for part (d):



8. **(a)** $\overrightarrow{P_1P_2} = (-4-(-6), -1-2) = (2,-3)$

(b) $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0) = (-1,6,1)$

10. **(a)** Denote the initial point by $A(a_1, a_2)$. Since the vector $\overrightarrow{AB} = (2-a_1, 0-a_2) = (2-a_1, -a_2)$ is to be equivalent to the vector $\mathbf{u} = (1,2)$, the coordinates of A must satisfy the equations

$$2 - a_1 = 1 \text{ and } -a_2 = 2$$

therefore $a_1 = 1$ and $a_2 = -2$. The initial point is $A(1,-2)$.

(b) Denote the terminal point by $B(b_1, b_2, b_3)$. Since the vector $\overrightarrow{AB} = (b_1 - 0, b_2 - 2, b_3 - 0) = (b_1, b_2 - 2, b_3)$ is to be equivalent to the vector $\mathbf{u} = (1, 1, 3)$, the coordinates of B must satisfy the equations

$$b_1 = 1, \quad b_2 - 2 = 1, \quad \text{and} \quad b_3 = 3$$

therefore $b_1 = 1, \quad b_2 = 3, \quad$ and $\; b_3 = 3$. The terminal point is $B(1, 3, 3)$.

12. **(a)** For any positive real number k, the vector $\mathbf{u} = k\mathbf{v}$ has the same direction as $\mathbf{v}$. For example, letting $k = 1$, we have $\mathbf{u} = (6, 7, -3)$. If the initial point is $P(-1{,}3, -5)$ then the terminal point has coordinates $(-1 + 6, 3 + 7, -5 - 3)$, i.e., $(5{,}10, -8)$.

(b) For any negative real number k, the vector $\mathbf{u} = k\mathbf{v}$ is oppositely directed to $\mathbf{v}$. For example, letting $k = -1$, we have $\mathbf{u} = (-6, -7, 3)$. If the initial point is $P(-1{,}3, -5)$ then the terminal point has coordinates $(-1 - 6, 3 - 7, -5 + 3)$, i.e., $(-7, -4, -2)$.

14. **(a)** $\mathbf{v} - \mathbf{w} = \big(4 - 6, 0 - (-1), -8 - (-4)\big) = (-2, 1, -4)$

(b) $6\mathbf{u} + 2\mathbf{v} = (-18, 6, 12) + (8, 0, -16) = (-10, 6, -4)$

(c) $-\mathbf{v} + \mathbf{u} = (-4, 0, 8) + (-3, 1, 2) = (-7, 1, 10)$

(d) $5(\mathbf{v} - 4\mathbf{u}) = 5[(4, 0, -8) - (-12, 4, 8)] = 5(16, -4, -16) = (80, -20, -80)$

(e) $-3(\mathbf{v} - 8\mathbf{w}) = -3[(4, 0, -8) - (48, -8, -32)] = -3(-44, 8, 24) = (132, -24, -72)$

(f) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u}) = [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)]$

$$= (-48,\; 9, 32) - (29, 1, -62) = (-77, 8, 94)$$

(an alternate method to evaluate this expression is to simplify the original expression to $\mathbf{u} - 7\mathbf{w} - 8\mathbf{v}$ prior to substituting the vectors in)

16. Solve the vector equation using the properties listed in Theorems 3.1.1 and 3.1.2:

$5\mathbf{x} + (-2)\mathbf{v} = 2(\mathbf{w} + (-5)\mathbf{x})$	[Part (c) of Theorem 3.1.2 and part (g) of Theorem 3.1.1]
$(-2)\mathbf{v} + 5\mathbf{x} = 2\mathbf{w} + 2((-5)\mathbf{x})$	[Parts (a) and (e) of Theorem 3.1.1]
$(-2)\mathbf{v} + 5\mathbf{x} = 2\mathbf{w} + (-10)\mathbf{x}$	[Part (g) of Theorem 3.1.1]

$\left((-2)\mathbf{v}+5\mathbf{x}\right)+10\mathbf{x}=(2\mathbf{w}+(-10)\mathbf{x})+10\mathbf{x}$	[Add $10\mathbf{x}$ to both sides]
$(-2)\mathbf{v}+(5\mathbf{x}+10\mathbf{x})=2\mathbf{w}+((-10)\mathbf{x}+10\mathbf{x})$	[Part (b) of Theorem 3.1.1]
$(-2)\mathbf{v}+(5+10)\mathbf{x}=2\mathbf{w}+(-10+10)\mathbf{x}$	[Part (f) of Theorem 3.1.1]
$2\mathbf{v}+\left((-2)\mathbf{v}+15\mathbf{x}\right)=2\mathbf{v}+2\mathbf{w}$	[Add $2\mathbf{v}$ to both sides and use part (a) of Theorem 3.1.2]
$(2-2)\mathbf{v}+15\mathbf{x}=2\mathbf{v}+2\mathbf{w}$	[Parts (b) and (f) of Theorem 3.1.1]
$15\mathbf{x}=2\mathbf{v}+2\mathbf{w}$	[Part (a) of Theorem 3.1.2]
$\frac{1}{15}(15\mathbf{x})=\frac{1}{15}(2\mathbf{v}+2\mathbf{w})$	[Multiply both sides by $\frac{1}{15}$]
$\mathbf{x}=\frac{1}{15}(2\mathbf{v}+2\mathbf{w})$	[Parts (g) and (h) of Theorem 3.1.1]

Therefore $\mathbf{x}=\frac{1}{15}[(8,14,-6,4)+(10,-4,16,2)]=(\frac{6}{5},\frac{2}{3},\frac{2}{3},\frac{2}{5})$.

18. (a) $\mathbf{v}+\mathbf{w}=(0+7,4+1,-1-4,1-2,2+3)=(7,5,-5,-1,5)$

(b) $3(2\mathbf{u}-\mathbf{v})=3[(2,4,-6,10,0)-(0,4,-1,1,2)]=3(2,0,-5,9,-2)=(6,0,-15,27,-6)$

(c) $(3\mathbf{u}-\mathbf{v})-(2\mathbf{u}+4\mathbf{w})$

$$=[(3,6,-9,15,0)-(0,4,-1,1,2)]-[(2,4,-6,10,0)+(28,4,-16,-8,12)]$$
$$=(3,2,-8,14,-2)-(30,8,-22,2,12)=(-27,-6,14,12,-14)$$

20. Solve the vector equation using the properties listed in Theorems 3.1.1 and 3.1.2:

$3\mathbf{u}+\mathbf{v}+(-2)\mathbf{w}=3\mathbf{x}+2\mathbf{w}$	[Part (c) of Theorem 3.1.2 and part (g) of Theorem 3.1.1]
$(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}=3\mathbf{x}+0\mathbf{w}$	[Add $-2\mathbf{w}$ to both sides, use parts (b) and (d) of Th. 3.1.1]
$(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}=3\mathbf{x}$	[Use part (a) of Theorem 3.1.2]
$\frac{1}{3}[(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}]=\frac{1}{3}(3\mathbf{x})$	[Multiply both sides by $\frac{1}{3}$]
$\frac{1}{3}[(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}]=\mathbf{x}$	[Parts (g) and (h) of Theorem 3.1.1]

Therefore $\mathbf{x}=\frac{1}{3}[(3,10,-10,16,2)+(-28,-4,16,8,-12)]=(-\frac{25}{3},2,2,8,-\frac{10}{3})$.

22. Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel (collinear) if one of them is a scalar multiple of the other one, i.e. either $\mathbf{u}=a\mathbf{v}$ for some scalar a or $\mathbf{v}=b\mathbf{u}$ for some scalar b or both (the two conditions are not equivalent if one of the vectors is a zero vector, but the other one is not.)

(a) Let $\mathbf{v} = (8t, -2)$.

$\mathbf{u} = a\mathbf{v} \quad \Leftrightarrow \quad 4 = 8at \text{ and } -1 = -2a \quad \Leftrightarrow \quad a = \frac{1}{2} \text{ and } t = 1$

$\mathbf{v} = b\mathbf{u} \quad \Leftrightarrow \quad 8t = 4b \text{ and } -2 = -b \quad \Leftrightarrow \quad b = 2 \text{ and } t = 1$

Therefore the vector $(8t, -2)$ is parallel to $(4, -1)$ if and only if $t = 1$.

(b) Let $\mathbf{v} = (8t, 2t)$.

$\mathbf{u} = a\mathbf{v} \quad \Leftrightarrow \quad 4 = 8at \text{ and } -1 = 2at \quad \Leftrightarrow \quad 1 = 2at \text{ and } -1 = 2at$ - contradiction

$\mathbf{v} = b\mathbf{u} \quad \Leftrightarrow \quad 8t = 4b \text{ and } 2t = -b \quad \Leftrightarrow \quad b = 0 \text{ and } t = 0$

Therefore the vector $(8t, 2t)$ is parallel to $(4, -1)$ if and only if $t = 0$.

(c) Let $\mathbf{v} = (1, t^2)$.

$\mathbf{u} = a\mathbf{v} \quad \Leftrightarrow \quad 4 = a \text{ and } -1 = at^2$ - contradiction

$\mathbf{v} = b\mathbf{u} \quad \Leftrightarrow \quad 1 = 4b \text{ and } t^2 = -b$ - contradiction

Therefore the vector $(1, t^2)$ is not parallel to $(4, -1)$ for any real value t.

24. The vector equation $a(2,1,0,1,-1) + b(-2,3,1,0,2) = (-8,8,3,-1,7)$ is equivalent to the linear system

$$\begin{array}{rcrcr} 2a & - & 2b & = & -8 \\ 1a & + & 3b & = & 8 \\ 0a & + & 1b & = & 3 \\ 1a & + & 0b & = & -1 \\ -1a & + & 2b & = & 7 \end{array}$$

whose augmented matrix $\begin{bmatrix} 2 & -2 & 8 \\ 1 & 3 & 8 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the unique solution is $a = -1$ and $b = 3$.

26. The vector equation $c_1(1,2,0)+c_2(2,1,1)+c_3(0,3,1)=(0,0,0)$ is equivalent to the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 2c_2 & + & 0c_3 & = & 0 \\ 2c_1 & + & 1c_2 & + & 3c_3 & = & 0 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Therefore, the unique solution is $c_1=c_2=c_3=0$.

28. The vector equation $c_1(-1,0,2)+c_2(2,2,-2)+c_3(1,-2,1)=(-6,12,4)$ is equivalent to the linear system

$$\begin{array}{rcrcrcr} -1c_1 & + & 2c_2 & + & 1c_3 & = & -6 \\ 0c_1 & + & 2c_2 & - & 2c_3 & = & 12 \\ 2c_1 & - & 2c_2 & + & 1c_3 & = & 4 \end{array}$$

whose augmented matrix $\begin{bmatrix} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 2 & -2 & 1 & 4 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

Therefore, the unique solution is $c_1=6,\ c_2=2,$ and $c_3=-4$.

30. Equating the second components on both sides yields a contradictory equation $0=-2$.

32. When the vector $\mathbf{u}=\overrightarrow{OP_1}+\frac{1}{2}\left(\overrightarrow{OP_2}-\overrightarrow{OP_1}\right)$ is positioned so its initial point is at the origin, its terminal point is the midpoint of the line segment connecting the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ since

$$\mathbf{u}=(x_1,y_1)+\frac{1}{2}(x_2-x_1,y_2-y_1)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

34. The midpoint of the line segment connecting the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

Therefore we have

$$\left(\frac{1+x_2}{2},\frac{3+y_2}{2},\frac{7+z_2}{2}\right)=(4,0,-6).$$

This vector equation is equivalent to a system of three linear equations in three unknowns that is easy to solve:

$$\frac{1+x_2}{2} = 4 \quad \Leftrightarrow \quad x_2 = 7$$
$$\frac{3+y_2}{2} = 0 \quad \Leftrightarrow \quad y_2 = -3$$
$$\frac{7+z_2}{2} = -6 \quad \Leftrightarrow \quad z_2 = -19$$

We conclude that the point Q is $(7,-3,-19)$.

3.2 Norm, Dot Product, and Distance in Rⁿ

2. **(a)** $\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{13}(-5,12) = (-\frac{5}{13}, \frac{12}{13});\quad -\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{13}(-5,12) = (\frac{5}{13}, -\frac{12}{13})$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}\,;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{6}}(1,-1,2) = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}});\quad -\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{6}}(1,-1,2) = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$

(c) $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2 + 3^2 + (-1)^2} = \sqrt{23}\,;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{23}}(-2,3,3,-1) = (-\frac{2}{\sqrt{23}}, \frac{3}{\sqrt{23}}, \frac{3}{\sqrt{23}}, -\frac{1}{\sqrt{23}});$
$-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{23}}(-2,3,3,-1) = (\frac{2}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, \frac{1}{\sqrt{23}})$

4. **(a)** $\mathbf{u}+\mathbf{v}+\mathbf{w} = (6,1,3);\quad \|\mathbf{u}+\mathbf{v}+\mathbf{w}\| = \sqrt{6^2+1^2+3^2} = \sqrt{46}$

(b) $\mathbf{u}-\mathbf{v} = (1,1,-1);\quad \|\mathbf{u}-\mathbf{v}\| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$

(c) $3\mathbf{v} = (3,-9,12);\quad \|3\mathbf{v}\| - 3\|\mathbf{v}\| = \sqrt{3^2+(-9)^2+12^2} - 3\sqrt{1^2+(-3)^2+4^2} = \sqrt{234} - 3\sqrt{26} = 0$

(d) $\|\mathbf{u}\| - \|\mathbf{v}\| = \sqrt{2^2+(-2)^2+3^2} - \sqrt{1^2+(-3)^2+4^2} = \sqrt{17} - \sqrt{26}$

6. **(a)** $\|\mathbf{u}\| - 2\|\mathbf{v}\| - 3\|\mathbf{w}\|$

$$= \sqrt{(-2)^2+(-1)^2+4^2+5^2} - 2\sqrt{3^2+1^2+(-5)^2+7^2} - 3\sqrt{(-6)^2+2^2+1^2+1^2}$$
$$= \sqrt{46} - 2\sqrt{84} - 3\sqrt{42} = \sqrt{46} - 4\sqrt{21} - 3\sqrt{42}$$

(b) $-2\mathbf{v} = (-6,-2,10,-14)$, $-3\mathbf{w} = (18,-6,-3,-3)$

$\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$
$= \sqrt{(-2)^2+(-1)^2+4^2+5^2} + \sqrt{(-6)^2+(-2)^2+10^2+(-14)^2} + \sqrt{18^2+(-6)^2+(-3)^2+(-3)^2}$
$= \sqrt{46}+\sqrt{336}+\sqrt{378} = \sqrt{46}+4\sqrt{21}+3\sqrt{42}$

(c) $\mathbf{u}-\mathbf{v} = (-5,-2,9,-2)$, $\|\mathbf{u}-\mathbf{v}\| = \sqrt{(-5)^2+(-2)^2+9^2+(-2)^2} = \sqrt{114}$

$\|\mathbf{u}-\mathbf{v}\|\mathbf{w} = (-6\sqrt{114}, 2\sqrt{114}, \sqrt{114}, \sqrt{114})$; $\big\|\|\mathbf{u}-\mathbf{v}\|\mathbf{w}\big\| = \sqrt{4788} = 6\sqrt{133}$

8. $\|k\mathbf{v}\| = \sqrt{k^2+k^2+(2k)^2+(-3k)^2+k^2} = \sqrt{16k^2} = 4\sqrt{k^2}$; this quantity equals 4 if $k=1$ or $k=-1$

10. (a) $\mathbf{u}\cdot\mathbf{v} = (1)(-1)+(1)(0)+(-2)(5)+(3)(1) = -8$
$\mathbf{u}\cdot\mathbf{u} = (1)(1)+(1)(1)+(-2)(-2)+(3)(3) = 15$
$\mathbf{v}\cdot\mathbf{v} = (-1)(-1)+(0)(0)+(5)(5)+(1)(1) = 27$

(b) $\mathbf{u}\cdot\mathbf{v} = (2)(1)+(-1)(2)+(1)(2)+(0)(2)+(-2)(1) = 0$
$\mathbf{u}\cdot\mathbf{u} = (2)(2)+(-1)(-1)+(1)(1)+(0)(0)+(-2)(-2) = 10$
$\mathbf{v}\cdot\mathbf{v} = (1)(1)+(2)(2)+(2)(2)+(2)(2)+(1)(1) = 14$

12. (a) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(1-5)^2+(2-1)^2+(-3-2)^2+\big(0-(-2)\big)^2} = \sqrt{46}$

(b) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| =$
$\sqrt{\big(2-(-2)\big)^2+\big(-1-(-1)\big)^2+(-4-0)^2+(1-3)^3+(0-7)^2+(6-2)^2+\big(-3-(-5)\big)^2+(1-1)^2}$

$= \sqrt{105}$

(c) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(0-2)^2+(1-1)^2+(1-0)^2+\big(1-(-1)\big)^2+(2-3)^2} = \sqrt{10}$

14. (a) $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1)(5)+(2)(1)+(-3)(2)+(0)(-2)}{\sqrt{1^2+2^2+(-3)^2+0^2}\sqrt{5^2+1^2+2^2+(-2)^2}} = \frac{1}{\sqrt{14}\sqrt{34}}$; the angle is acute since $\mathbf{u}\cdot\mathbf{v} > 0$

(b) $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(2)(-2)+(-1)(-1)+(-4)(0)+(1)(3)+(0)(7)+(6)(2)+(-3)(-5)+(1)(1)}{\sqrt{2^2+(-1)^2+(-4)^2+1^2+0^2+6^2+(-3)^2+1^2}\sqrt{(-2)^2+(-1)^2+0^2+3^2+7^2+2^2+(-5)^2+1^2}} = \frac{28}{2\sqrt{17}\sqrt{93}}$;

the angle is acute since $\mathbf{u}\cdot\mathbf{v} > 0$

(c) $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(0)(2)+(1)(1)+(1)(0)+(1)(-1)+(2)(3)}{\sqrt{0^2+1^2+1^2+1^2+2^2}\sqrt{2^2+1^2+0^2+(-1)^2+3^2}} = \frac{6}{\sqrt{7}\sqrt{15}}$; the angle is acute since $\mathbf{u}\cdot\mathbf{v} > 0$

16. $\mathbf{a}\cdot\mathbf{b} = 0$ since the angle between the two vectors is $90°$

18. **(a)** $\|\mathbf{u}\|\cdot\|\mathbf{v}\|$ does not make sense: $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are scalars, whereas the dot product is only defined for vectors

(b) $(\mathbf{u}\cdot\mathbf{v}) - \mathbf{w}$ does not make sense: $\mathbf{u}\cdot\mathbf{v}$ is a scalar so the vector $\mathbf{w}$ cannot be subtracted from it

(c) $(\mathbf{u}\cdot\mathbf{v}) - k$ makes sense (the result is a scalar)

(d) $k\cdot\mathbf{u}$ does not make sense: k is a scalar, whereas the dot product is only defined for vectors

20. **(a)** $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-12)^2+(-5)^2}}(-12,-5) = -\frac{1}{13}(-12,-5) = (\frac{12}{13},\frac{5}{13})$

(b) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{3^2+(-3)^2+(-3)^2}}(3,-3,-3) = -\frac{1}{3\sqrt{3}}(3,-3,-3) = (-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$

(c) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-6)^2+8^2}}(-6,8) = -\frac{1}{10}(-6,8) = (\frac{3}{5},-\frac{4}{5})$

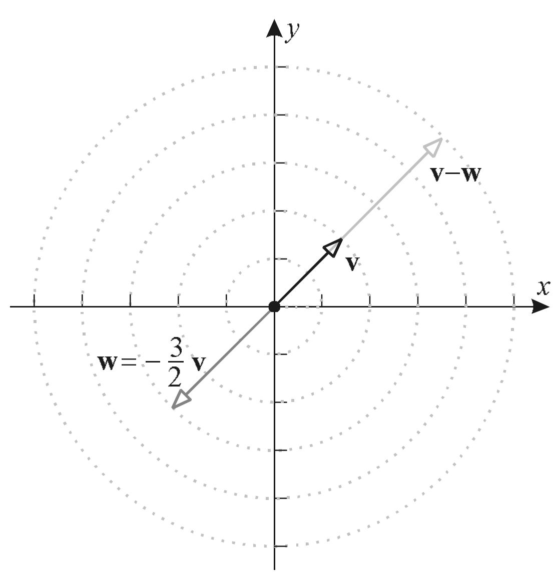
(d) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-3)^2+1^2+(\sqrt{6})^2+3^2}}(-3,1,\sqrt{6},3) = -\frac{1}{5}(-3,1,\sqrt{6},3) = (\frac{3}{5},-\frac{1}{5},-\frac{\sqrt{6}}{5},-\frac{3}{5})$

22. Let us assume both vectors $\mathbf{v}$ and $\mathbf{w}$ have the same number of components (otherwise $\mathbf{v}-\mathbf{w}$ would be undefined).

From Theorem 3.2.5(a), we obtain two inequalities:
$\|\mathbf{v}-\mathbf{w}\| \le \|\mathbf{v}\| + \|-\mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\| = 5$.

The norm $\|\mathbf{v}-\mathbf{w}\|$ can actually attain this upper bound if $\mathbf{w} = -\frac{3}{2}\mathbf{v}$ (so that the two vectors have opposite directions):

$$\|\mathbf{v}-\mathbf{w}\| = \left\|\mathbf{v} - \left(-\frac{3}{2}\mathbf{v}\right)\right\| = \left\|\frac{5}{2}\mathbf{v}\right\| \underbrace{=}_{\text{Theorem 3.2.1c}} \left|\frac{5}{2}\right|\|\mathbf{v}\| = 5$$



Applying Theorem 3.2.5(a) to $-\mathbf{w} = (\mathbf{v} - \mathbf{w}) + (-\mathbf{v})$ yields

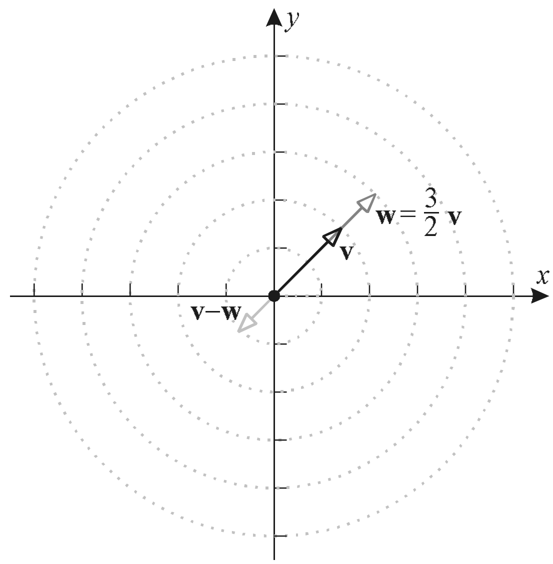
$$\|-\mathbf{w}\| \le \|\mathbf{v} - \mathbf{w}\| + \|-\mathbf{v}\|$$

thus

$$\|\mathbf{v} - \mathbf{w}\| \ge \|\mathbf{w}\| - \|\mathbf{v}\| = 1.$$

The norm $\|\mathbf{v} - \mathbf{w}\|$ attains this lower bound if $\mathbf{w} = \frac{3}{2}\mathbf{v}$ (so that the two vectors have the same direction):

$$\|\mathbf{v} - \mathbf{w}\| = \left\|\mathbf{v} - \frac{3}{2}\mathbf{v}\right\| = \left\|-\frac{1}{2}\mathbf{v}\right\| \underbrace{=}_{\substack{\text{Theorem}\\ \text{3.2.1c}}} \left|-\frac{1}{2}\right| \|\mathbf{v}\| = 1$$



24. **(a)** $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{(1)(21)+(-7)(3)}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1} 0 = \frac{\pi}{2}$

(b) $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{(0)(3)+(2)(-3)}{\sqrt{0^2+2^2}\sqrt{3^2+(-3)^2}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

(c) $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{(-1)(0)+(1)(-1)+(0)(1)}{\sqrt{(-1)^2+1^2+0^2}\sqrt{0^2+(-1)^2+1^2}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

(d) $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{(1)(1)+(-1)(0)+(0)(0)}{\sqrt{1^2+(-1)^2+0^2}\sqrt{1^2+0^2+0^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

26. **(a)** $|\mathbf{u}\cdot\mathbf{v}| = |(4)(1)+(1)(2)+(1)(3)| = 9$; $\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{4^2+1^2+1^2}\sqrt{1^2+2^2+3^2} = \sqrt{18}\sqrt{14}$

Since $|\mathbf{u}\cdot\mathbf{v}| = 9 = \sqrt{81} \le \sqrt{252} = \sqrt{18}\sqrt{14} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

(b) $|\mathbf{u}\cdot\mathbf{v}| = |(1)(0)+(2)(1)+(1)(1)+(2)(5)+(3)(-2)| = 7$

$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{1^2+2^2+1^2+2^2+3^2}\sqrt{0^2+1^2+1^2+5^2+(-2)^2} = \sqrt{19}\sqrt{31}$

Since $|\mathbf{u}\cdot\mathbf{v}| = 7 = \sqrt{49} \le \sqrt{589} = \sqrt{19}\sqrt{31} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

(c) $|\mathbf{u}\cdot\mathbf{v}| = |(1)(0)+(3)(2)+(5)(4)+(2)(1)+(0)(3)+(1)(5)| = 33$

$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{1^2+3^2+5^2+2^2+0^2+1^2}\sqrt{0^2+2^2+4^2+1^2+3^2+5^2} = \sqrt{40}\sqrt{55}$

Since $|\mathbf{u}\cdot\mathbf{v}| = 33 = \sqrt{1089} \le \sqrt{2200} = \sqrt{40}\sqrt{55} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

28. **(b)** $\mathbf{u} = (3\cos 30°, 3\sin 30°) = (\frac{3\sqrt{3}}{2}, \frac{3}{2})$; $\mathbf{v} = (2\cos 135°, 2\sin 135°) = (-\sqrt{2}, \sqrt{2})$

$4\mathbf{u} - 5\mathbf{v} = (6\sqrt{3}, 6) - (-5\sqrt{2}, 5\sqrt{2}) = (6\sqrt{3} + 5\sqrt{2}, 6 - 5\sqrt{2})$

3.3 Orthogonality

2. **(a)** $\mathbf{u} \cdot \mathbf{v} = (2)(5) + (3)(-7) = -11 \neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

(b) $\mathbf{u} \cdot \mathbf{v} = (-6)(4) + (-2)(0) = -24 \neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

(c) $\mathbf{u} \cdot \mathbf{v} = (1)(3) + (-5)(3) + (4)(3) = 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

(d) $\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (2)(7) + (3)(-4) = 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

4. **(a)** $\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-3) + (3)(2) = 0$ therefore the vectors form an orthogonal set

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = (1)(-2) + (-2)(1) = -4 \neq 0$ therefore the vectors do not form an orthogonal set

(c) $\mathbf{v}_1 \cdot \mathbf{v}_2 = (1)(1) + (0)(1) + (1)(1) = 2 \neq 0$ therefore the vectors do not form an orthogonal set (even though $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$)

(d) $\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(2) + (-2)(1) + (1)(-2) = 0$,
$\mathbf{v}_1 \cdot \mathbf{v}_3 = (2)(1) + (-2)(2) + (1)(2) = 0$, and
$\mathbf{v}_2 \cdot \mathbf{v}_3 = (2)(1) + (1)(2) + (-2)(2) = 0$ therefore the vectors form an orthogonal set

6. **(a)** $\mathbf{v} \cdot \mathbf{w} = (a)(-b) + (b)(a) = 0$ therefore $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors

(b) $(3,2)$ and $(-3,-2)$

(c) $\frac{1}{\sqrt{4^2+3^2}}(4,3) = (\frac{4}{5}, \frac{3}{5})$ and $-\frac{1}{\sqrt{4^2+3^2}}(4,3) = (-\frac{4}{5}, -\frac{3}{5})$

8. $\overrightarrow{AB} = (4-3, 3-0, 0-2) = (1,3,-2)$, $\overrightarrow{AC} = (8-3, 1-0, -1-2) = (5,1,-3)$,
$\overrightarrow{BC} = (8-4, 1-3, -1-0) = (4,-2,-1)$
$\overrightarrow{AB} \cdot \overrightarrow{BC} = (1)(4) + (3)(-2) + (-2)(-1) = 0$
therefore the points A, B, and C form the vertices of a right triangle

10. $x - 1 + 9(y-1) + 8(z-4) = 0$

12. $x+2y+3z=0$

14. The plane $x-4y-3z-2=0$ has a normal vector $(1,-4,-3)$.

The plane $3x-12y-9z-7=0$ has a normal vector $(3,-12,-9)$.

The two normal vectors are parallel: $(3,-12,-9)=3(1,-4,-3)$ therefore the planes are parallel as well.

16. The normal vectors of the two planes are parallel: $(8,-2,-4)=-2(-4,1,2)$ therefore the planes are parallel as well.

18. The normal vectors of the two planes are orthogonal:

$$(1,-2,3)\cdot(-2,5,4)=(1)(-2)+(-2)(5)+(3)(4)=0$$

therefore the given planes are perpendicular.

20. (a) From Formula (12) on p.148,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\|=\frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|}=\frac{|(5)(2)+(6)(-1)|}{\sqrt{2^2+(-1)^2}}=\frac{4}{\sqrt{5}}$$

(b) From Formula (12) on p.148,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\|=\frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|}=\frac{|(3)(1)+(-2)(2)+(6)(-7)|}{\sqrt{1^2+2^2+(-7)^2}}=\frac{43}{\sqrt{54}}=\frac{43}{3\sqrt{6}}$$

22. $\mathbf{u}\cdot\mathbf{a}=(-1)(-2)+(-2)(3)=-4,\ \ \|\mathbf{a}\|^2=(-2)^2+3^2=13,$

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}=-\frac{4}{13}(-2,3)=(\frac{8}{13},-\frac{12}{13})$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u}-\text{proj}_{\mathbf{a}}\mathbf{u}=(-1,-2)-\left(\frac{8}{13},-\frac{12}{13}\right)=(-\frac{21}{13},-\frac{14}{13})$

24. $\mathbf{u}\cdot\mathbf{a}=(1)(4)+(0)(3)+(0)(8)=4,\ \ \|\mathbf{a}\|^2=4^2+3^2+8^2=89,$

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}=\frac{4}{89}(4,3,8)=(\frac{16}{89},\frac{12}{89},\frac{32}{89})$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u}-\text{proj}_{\mathbf{a}}\mathbf{u}=(1,0,0)-(\frac{16}{89},\frac{12}{89},\frac{32}{89})=(\frac{73}{89},-\frac{12}{89},-\frac{32}{89})$

26. $\mathbf{u}\cdot\mathbf{a}=(2)(1)+(0)(2)+(1)(3)=5,\ \ \|\mathbf{a}\|^2=1^2+2^2+3^2=14,$

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}=\frac{5}{14}(1,2,3)=(\frac{5}{14},\frac{5}{7},\frac{15}{14})$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u}-\text{proj}_{\mathbf{a}}\mathbf{u}=(2,0,1)-(\frac{5}{14},\frac{5}{7},\frac{15}{14})=(\frac{23}{14},-\frac{5}{7},-\frac{1}{14})$

28. $\mathbf{u}\cdot\mathbf{a}=(5)(2)+(0)(1)+(-3)(-1)+(7)(-1)=6,\quad \|\mathbf{a}\|^2=2^2+1^2+(-1)^2+(-1)^2=7,$

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\mathrm{proj}_{\mathbf{a}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}=\frac{6}{7}(2,1,-1,-1)=\left(\frac{12}{7},\frac{6}{7},-\frac{6}{7},-\frac{6}{7}\right)$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$\mathbf{u}-\mathrm{proj}_{\mathbf{a}}\mathbf{u}=(5,0,-3,7)-\left(\frac{12}{7},\frac{6}{7},-\frac{6}{7},-\frac{6}{7}\right)=\left(\frac{23}{7},-\frac{6}{7},-\frac{15}{7},\frac{55}{7}\right)$

30. From Theorem 3.3.4(a) the distance between the point and the line is $D=\frac{|(1)(-1)+(-3)(4)+2|}{\sqrt{1^2+(-3)^2}}=\frac{11}{\sqrt{10}}$

32. From Theorem 3.3.4(a) the distance between the point and the line is $D=\frac{|(3)(1)+(1)(8)-5|}{\sqrt{3^2+1^2}}=\frac{6}{\sqrt{10}}$

(the equation of the line had to be rewritten in the form $ax+by+c=0$ as $3x+y-5=0$)

34. From Theorem 3.3.4(b) the distance between the point and the plane is

$D=\frac{|(2)(-1)+(5)(-1)+(-6)(2)-4|}{\sqrt{2^2+5^2+(-6)^2}}=\frac{23}{\sqrt{65}}$ (the equation of the plane had to be rewritten in the form $ax+by+cz+d=0$ as $2x+5y-6z-4=0$)

36. From Theorem 3.3.4(b) the distance between the point and the plane is

$D=\frac{|(1)(0)+(-1)(3)+(-1)(-2)-3|}{\sqrt{1^2+(-1)^2+(-1)^2}}=\frac{4}{\sqrt{3}}$

(the equation of the plane had to be rewritten in the form $ax+by+cz+d=0$ as $x-y-z-3=0$)

38. First, select an arbitrary point in the plane $3x-4y+z=1$ by setting $x=y=0$; we obtain $P_0(0,0,1)$.

From Theorem 3.3.4(b) the distance between P_0 and the plane $6x-8y+2z-3=0$ is

$D=\frac{|(6)(0)+(-8)(0)+(2)(1)-3|}{\sqrt{6^2+(-8)^2+2^2}}=\frac{1}{\sqrt{104}}=\frac{1}{2\sqrt{26}}$

40. First, select an arbitrary point in the plane $2x-y+z=1$ by setting $x=y=0$; we obtain $P_0(0,0,1)$.

From Theorem 3.3.4(b) the distance between P_0 and the plane $2x-y+z+1=0$ is

$D=\frac{|(2)(0)+(-1)(0)+(1)(1)+1|}{\sqrt{2^2+(-1)^2+1^2}}=\frac{2}{\sqrt{6}}$

42. Align the edges of the box with the coordinate axes so that the diagonal becomes the vector $\mathbf{v}=(10,15,25)$.

Then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{\sqrt{10^2+15^2+25^2}}(10,15,25)=\left(\frac{2}{\sqrt{38}},\frac{3}{\sqrt{38}},\frac{5}{\sqrt{38}}\right)$ so that the approximate values of the angles formed by the diagonal with the edges (and the axes) are

$$\cos^{-1}\left(\frac{2}{\sqrt{38}}\right)\approx 71^\circ,\ \cos^{-1}\left(\frac{3}{\sqrt{38}}\right)\approx 61^\circ,\text{ and }\cos^{-1}\left(\frac{5}{\sqrt{38}}\right)\approx 36^\circ.$$

3.4 The Geometry of Linear Systems

2. The vector equation in Formula (5) on p.153 can be expressed as $(x,y) = (2,-1) + t(-4,-2)$.

This yields the parametric equations $x = 2 - 4t,\ y = -1 - 2t$.

4. The vector equation in Formula (5) on p.153 can be expressed as $(x,y,z) = (-9,3,4) + t(-1,6,0)$.

This yields the parametric equations $x = -9 - t,\ y = 3 + 6t,\ z = 4$.

6. A point on the line: $(0,7,4)$; a vector parallel to the line: $(4,0,3)$.

8. A point on the line: $(0,-5,1)$; a vector parallel to the line: $(0,5,-1)$.

10. The vector equation in Formula (6) on p.154 can be expressed as

$(x,y,z) = (0,6,-2) + t_1(0,9,-1) + t_2(0,-3,0)$.

This yields the parametric equations $x = 0,\ y = 6 + 9t_1 - 3t_2,\ z = -2 - t_1$.

12. The vector equation in Formula (6) on p.154 can be expressed as

$(x,y,z) = (0,5,-4) + t_1(0,0,-5) + t_2(1,-3,-2)$.

This yields the parametric equations $x = t_2,\ y = 5 - 3t_2,\ z = -4 - 5t_1 - 2t_2$.

14. We find a nonzero vector orthogonal to $\mathbf{v}$, e.g., $(4,1)$. The vector equation of the line passing through $(0,0)$ and parallel to $(4,1)$ can be expressed as $(x,y) = t(4,1)$. Parametric equations are $x = 4t$ and $y = t$.

16. We find two nonparallel nonzero vectors orthogonal to $\mathbf{v}$, e.g., $(-1,3,0)$ and $(0,6,1)$. The vector equation of the plane that contains the origin and these two vectors can be expressed as $(x,y,z) = t_1(-1,3,0) + t_2(0,6,1)$. Parametric equations are $x = -t_1,\ y = 3t_1 + 6t_2$, and $z = t_2$.

18. The augmented matrix of the linear system $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 2 & 6 & -8 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -3s + 4t$, $x_2 = s$, $x_3 = t$ expressed in vector form as $\mathbf{x} = (-3s + 4t, s, t)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,3,-4)$ and $\mathbf{r}_2 = (2,6,-8)$ since

$\mathbf{r}_1 \cdot \mathbf{x} = (1)(-3s + 4t) + (3)(s) + (-4)(t) = 0$ and $\mathbf{r}_2 \cdot \mathbf{x} = (2)(-3s + 4t) + (6)(s) + (-8)(t) = 0$.

20. The augmented matrix of the linear system $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 17 & 0 \\ 0 & 1 & -7 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -17t$, $x_2 = 7t$, $x_3 = t$ expressed in vector form as $\mathbf{x} = (-17t, 7t, t)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,3,-4)$ and $\mathbf{r}_2 = (1,2,3)$ since

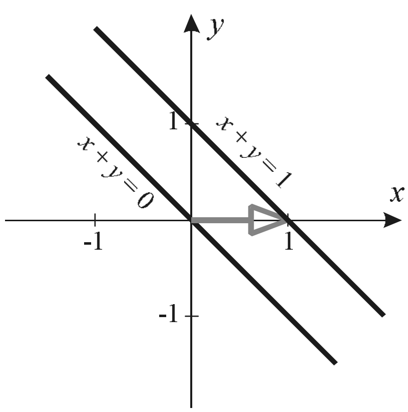
$$\mathbf{r}_1 \cdot \mathbf{x} = (1)(-17t) + (3)(7t) + (-4)(t) = 0 \text{ and } \mathbf{r}_2 \cdot \mathbf{x} = (1)(-17t) + (2)(7t) + (3)(t) = 0\,.$$

22. (a) Associated homogeneous system $x + y = 0$ has a general solution $x = -t,\ y = t$.

The original nonhomogeneous system has a general solution $x = 1 - t,\ y = t$, which can be expressed in vector form as

$$(x, y) = (1 - t, t) = \underbrace{(1,0)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-t, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(b)



24. (a) Theorem 3.4.3 yields the following homogeneous linear system that satisfies our requirements:

$$\begin{aligned} -3x + 2y - z &= 0 \\ -2y - 2z &= 0 \end{aligned}$$

(b) A straight line passing through the origin – this line is parallel to any vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(c) The augmented matrix of the system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. A general solution of the system is $x = -t,\ y = -t,\ z = t$. It can also be expressed in vector form as $\mathbf{u} = (x, y, z) = (-t, -t, t)$. To confirm that Theorem 3.4.3 holds, we verify that $\mathbf{u}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{u} \cdot \mathbf{a} = (-t)(-3) + (-t)(2) + (t)(-1) = 0,\ \mathbf{u} \cdot \mathbf{b} = (-t)(0) + (-t)(-2) + (t)(-2) = 0.$$

26. (a) The augmented matrix of the homogeneous system has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & \frac{11}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{11}{5}t,\ x_2 = \frac{2}{5}t,\ x_3 = t$.

(b) Multiplying $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$ therefore $x_1 = x_2 = x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,1,1)}_{\substack{\text{particular} \\ \text{solution} \\ \text{of the} \\ \text{nonhomogeneous} \\ \text{system}}} + \underbrace{\left(-\frac{11}{5}t, \frac{2}{5}t, t\right)}_{\substack{\text{general} \\ \text{solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & \frac{11}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{16}{5} - \frac{11}{5}s,\ x_2 = \frac{3}{5} + \frac{2}{5}s,\ x_3 = s$.

If we let $s = 1 + t$ then this agrees with the solution we obtained in part (c).

28. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{13}{3} & \frac{13}{3} \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system

$$x_1 = \frac{13}{3} + \frac{1}{3}s + \frac{13}{3}t, \quad x_2 = s, \quad x_3 = -7 - 9t, \quad x_4 = t$$

can be expressed in vector form as

$$(x_1, x_2, x_3, x_4) = \underbrace{\left(\frac{13}{3}, 0, -7, 0\right)}_{\substack{\text{particular} \\ \text{solution} \\ \text{of the} \\ \text{nonhomogeneous} \\ \text{system}}} + \underbrace{\left(\frac{1}{3}s + \frac{13}{3}t, s, -9t, t\right)}_{\substack{\text{general} \\ \text{solution} \\ \text{of the} \\ \text{associated} \\ \text{homogeneous} \\ \text{system}}}$$

3.5 Cross Product

2. **(a)** $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}2 & -1\\2 & -3\end{vmatrix}, -\begin{vmatrix}3 & -1\\0 & -3\end{vmatrix}, \begin{vmatrix}3 & 2\\0 & 2\end{vmatrix}\right) = (-4,9,6)$

$\mathbf{v}\times\mathbf{w} = \left(\begin{vmatrix}2 & -3\\6 & 7\end{vmatrix}, -\begin{vmatrix}0 & -3\\2 & 7\end{vmatrix}, \begin{vmatrix}0 & 2\\2 & 6\end{vmatrix}\right) = (32,-6,-4)$

$(\mathbf{u}\times\mathbf{v})\times(\mathbf{v}\times\mathbf{w}) = \left(\begin{vmatrix}9 & 6\\-6 & -4\end{vmatrix}, -\begin{vmatrix}-4 & 6\\32 & -4\end{vmatrix}, \begin{vmatrix}-4 & 9\\32 & -6\end{vmatrix}\right) = (0,176,-264)$

(b) $\mathbf{v}-2\mathbf{w} = (0,2,-3)-(4,12,14) = (-4,-10,-17)$

$\mathbf{u}\times(\mathbf{v}-2\mathbf{w}) = \left(\begin{vmatrix}2 & -1\\-10 & -17\end{vmatrix}, -\begin{vmatrix}3 & -1\\-4 & -17\end{vmatrix}, \begin{vmatrix}3 & 2\\-4 & -10\end{vmatrix}\right) = (-44,55,-22)$

(c) $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}2 & -1\\2 & -3\end{vmatrix}, -\begin{vmatrix}3 & -1\\0 & -3\end{vmatrix}, \begin{vmatrix}3 & 2\\0 & 2\end{vmatrix}\right) = (-4,9,6)$

$(\mathbf{u}\times\mathbf{v})-2\mathbf{w} = (-4,9,6)-(4,12,14) = (-8,-3,-8)$

4. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}1 & -2\\-1 & 2\end{vmatrix}, -\begin{vmatrix}1 & -2\\2 & 2\end{vmatrix}, \begin{vmatrix}1 & 1\\2 & -1\end{vmatrix}\right) = (0,-6,-3)$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

6. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}3 & 1\\4 & 2\end{vmatrix}, -\begin{vmatrix}3 & 1\\0 & 2\end{vmatrix}, \begin{vmatrix}3 & 3\\0 & 4\end{vmatrix}\right) = (2,-6,12)$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

8. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}-1 & 4\\-2 & 8\end{vmatrix}, -\begin{vmatrix}3 & 4\\6 & 8\end{vmatrix}, \begin{vmatrix}3 & -1\\6 & -2\end{vmatrix}\right) = (0,0,0)$

The area of the parallelogram determined by both $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{0^2+0^2+0^2} = 0$.

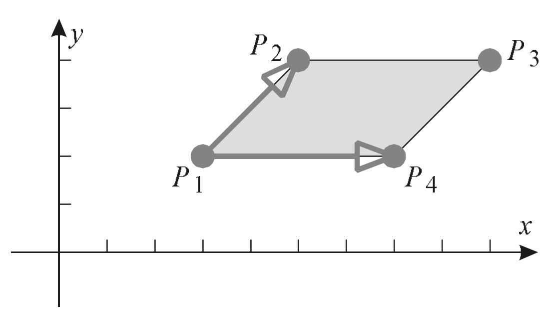
10. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}1 & 1\\2 & -5\end{vmatrix}, -\begin{vmatrix}1 & 1\\3 & -5\end{vmatrix}, \begin{vmatrix}1 & 1\\3 & 2\end{vmatrix}\right) = (-7,8,-1)$

The area of the parallelogram determined by both $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-7)^2+8^2+(-1)^2} = \sqrt{114}$.

12. $\overrightarrow{P_1P_2} = (2,2) = \overrightarrow{P_4P_3}, \quad \overrightarrow{P_1P_4} = (4,0) = \overrightarrow{P_2P_3}$

Viewing these as vectors in 3-space, we obtain

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4} = (2,2,0)\times(4,0,0)$$
$$= \left(\begin{vmatrix}2 & 0\\0 & 0\end{vmatrix}, -\begin{vmatrix}2 & 0\\4 & 0\end{vmatrix}, \begin{vmatrix}2 & 2\\4 & 0\end{vmatrix}\right) = (0,0,-8)$$



The area of the parallelogram is

$$\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4}\right\| = \sqrt{0^2+0^2+(-8)^2} = 8.$$

14. We have $\overrightarrow{AB} = (1,1)$ and $\overrightarrow{AC} = (2,-4)$. Viewing these as vectors in 3-space, we obtain

$$\overrightarrow{AB} \times \overrightarrow{AC} = (1,1,0) \times (2,-4,0) = \left(\begin{vmatrix} 1 & 0 \\ -4 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \right) = (0,0,-6).$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{0^2 + 0^2 + (-6)^2} = 3.$

16. $\overrightarrow{PQ} = (-1,4,2), \quad \overrightarrow{PR} = (5,2,6)$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(\begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix}, \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} \right) = (20,16,-22).$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{20^2 + 16^2 + (-22)^2} = \sqrt{285}.$

18. From Theorem 3.5.4(b), the volume of the parallelepiped is equal to $\left|\det\begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right| = 45.$

20. $\begin{vmatrix} 5 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0$ therefore by Theorem 3.5.5 these vectors lie in the same plane when they have the same initial point.

22. From Formula (7) on p.165, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix} = -10.$

24. From Formula (7) on p.165, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix} = -110.$

26. (a) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ can be obtained from $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ by interchanging the first row and the second row. This reverses the sign of the determinant, therefore $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = -3.$

(b) $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = -3$ as shown in part (a) above

(c) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$ since this determinant has two equal rows (this follows from Theorem 2.2.5)

28. $\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix}, -\begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right) = (36,-24,0)$; $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{36^2 + (-24)^2 + 0^2} = 12\sqrt{13}$;

$\|\mathbf{u}\| = \sqrt{2^2 + 3^2 + (-6)^2} = 7$; $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 6^2} = 7$

From Formula (6) on p.164, $\sin\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{12}{49}\sqrt{13}$.

Chapter 3 Supplementary Exercises

2. Rewrite $\mathbf{u} = (3, -5, 1)$, $\mathbf{v} = (-2, 0, 2)$, and $\mathbf{w} = (0, -1, 4)$.

(a) $3\mathbf{v} - 2\mathbf{u} = (-6,0,6) - (6,-10,2) = (-12,10,4)$

(b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (1,-6,7)$; $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{1^2 + (-6)^2 + 7^2} = \sqrt{86}$

(c) $-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w}) = (-9,15,-3) - \big((-2,0,2) + (0,-5,20)\big) = (-7,20,-25)$

$$d(-3\mathbf{u}, \mathbf{v} + 5\mathbf{w}) = \|-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w})\| = \sqrt{(-7)^2 + 20^2 + (-25)^2} = \sqrt{1074}$$

(d) $\mathbf{u} \cdot \mathbf{w} = (3)(0) + (-5)(-1) + (1)(4) = 9$; $\|\mathbf{w}\|^2 = 0^2 + (-1)^2 + 4^2 = 17$;

$$\text{proj}_{\mathbf{w}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{9}{17}(0,-1,4) = (0, -\frac{9}{17}, \frac{36}{17})$$

(e) From Formula (7) on p.165, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -5 & 1 \\ -2 & 0 & 2 \\ 0 & -1 & 4 \end{vmatrix} = -32$

(f) $-5\mathbf{v} + \mathbf{w} = (10,0,-10) + (0,-1,4) = (10,-1,-6)$

$(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = [(3)(-2) + (-5)(0) + (1)(2)]\mathbf{w} = -4\mathbf{w} = (0,4,-16)$

$(-5\mathbf{v} + \mathbf{w}) \times \big((\mathbf{u} \cdot \mathbf{v})\mathbf{w}\big) = \left(\begin{vmatrix} -1 & -6 \\ 4 & -16 \end{vmatrix}, -\begin{vmatrix} 10 & -6 \\ 0 & -16 \end{vmatrix}, \begin{vmatrix} 10 & -1 \\ 0 & 4 \end{vmatrix}\right) = (40,160,40)$

4. **(a)** $3\mathbf{v} - 2\mathbf{u} = (3,-3,18,-6,0) - (0,10,0,-2,-4) = (3,-13,18,-4,4)$

(b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (-3,3,10,-3,0)$; $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{(-3)^2 + 3^2 + 10^2 + (-3)^2 + 0^2} = \sqrt{127}$

(c) $-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w}) = (0,-15,0,3,6) - \big((1,-1,6,-2,0) + (-20,-5,20,0,10)\big) = (19,-9,-26,5,-4)$

$d(-3\mathbf{u}, \mathbf{v} + 5\mathbf{w}) = \|-3\mathbf{u} - (\mathbf{v} + 5\mathbf{w})\| = \sqrt{19^2 + (-9)^2 + (-26)^2 + 5^2 + (-4)^2} = \sqrt{1159}$

(d) $\mathbf{u} \cdot \mathbf{w} = (0)(-4) + (5)(-1) + (0)(4) + (-1)(0) + (-2)(2) = -9$;

$$\|\mathbf{w}\|^2 = (-4)^2 + (-1)^2 + 4^2 + 0^2 + 2^2 = 37$$

$$\text{proj}_{\mathbf{w}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-9}{37}(-4,-1,4,0,2) = (\frac{36}{37}, \frac{9}{37}, -\frac{36}{37}, 0, -\frac{18}{37})$$

6. $(-2,0,1)\cdot(1,1,2)=(-2)(1)+(0)(1)+(1)(2)=0$

$(-2,0,1)\cdot(1,-5,2)=(-2)(1)+(0)(-5)+(1)(2)=0$

$(1,1,2)\cdot(1,-5,2)=(1)(1)+(1)(-5)+(2)(2)=0$

therefore the given vectors form an orthogonal set;

An orthonormal set is formed by the vectors

$\frac{1}{\sqrt{(-2)^2+0^2+1^2}}(-2,0,1)=(-\frac{2}{\sqrt{5}},0,\frac{1}{\sqrt{5}})$

$\frac{1}{\sqrt{1^2+1^2+2^2}}(1,1,2)=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}})$

$\frac{1}{\sqrt{1^2+(-5)^2+2^2}}(1,-5,2)=(\frac{1}{\sqrt{30}},-\frac{5}{\sqrt{30}},\frac{2}{\sqrt{30}})$

8. $\mathbf{v}_1\cdot\mathbf{v}_2=\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(-\frac{2}{3}\right)=0$;

$\|\mathbf{v}_1\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}=1$; $\|\mathbf{v}_2\|=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^2}=1$

therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthonormal vectors.

Using the cross product, we can create a vector orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{v}_1\times\mathbf{v}_2=\left(\begin{vmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3}\end{vmatrix},-\begin{vmatrix}\frac{2}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3}\end{vmatrix},\begin{vmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{vmatrix}\right)=\left(-\frac{6}{9},\frac{6}{9},\frac{3}{9}\right)=\left(-\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$$

Since this cross product has magnitude 1, we can let $\mathbf{v}_3=\left(-\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$.

10. False: e.g., take $\mathbf{u}=(1,0)$, $\mathbf{v}=(1,0)$, and $\mathbf{w}=(-1,1)$.

12. Denoting $S(a,b,6,c)$ we have $\overrightarrow{RS}=(a+4,b-1,2,c)$. For this vector to be parallel to $\overrightarrow{PQ}=(3,4,1,-8)$ there must exist a scalar k such that $\overrightarrow{RS}=k\,\overrightarrow{PQ}$. The equality of the third components immediately leads to $k=2$. Equating the remaining pairs of components yields the equations:

$$a+4=(2)(3),\quad b-1=(2)(4),\quad c=(2)(-8)$$

therefore $a=2$, $b=9$, and $c=-16$. We conclude that the point S has coordinates $(2,9,6,-16)$.

14. $\overrightarrow{PQ}=(3,4,1,-8)$; $\overrightarrow{PR}=(-1,0,4,-6)$

$$\cos\theta=\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|}=\frac{(3)(-1)+(4)(0)+(1)(4)+(-8)(-6)}{\sqrt{3^2+4^2+1^2+(-8)^2}\sqrt{(-1)^2+0^2+4^2+(-6)^2}}=\frac{49}{\sqrt{90}\sqrt{53}}=\frac{49}{3\sqrt{530}}$$

16. The planes are parallel since their normal vectors, $(3,-1,6)$ and $(-6,2,-12)$, are parallel: $(-6,2,-12)=-2(3,-1,6)$. We select an arbitrary point in the plane $3x-y+6z=7$ by setting $x=z=0$ to obtain $P_0(0,-7,0)$.

From Theorem 3.3.4(b) the distance between P_0 and the plane $-6x+2y-12z-1=0$ is

$$D=\frac{|(-6)(0)+(2)(-7)+(-12)(0)-1|}{\sqrt{(-6)^2+2^2+(-12)^2}}=\frac{15}{\sqrt{184}}=\frac{15}{2\sqrt{46}}$$

18. Since the line is to be orthogonal to the plane $4x-z=5$, it must be parallel to a normal vector to the plane $(4,0,-1)$.

The vector equation in Formula (5) on p.153 can be expressed as $(x,y,z)=(-1,6,0)+t(4,0,-1)$.

This yields the parametric equations $x=-1+4t,\ y=6,\ z=-t$.

20. Since the plane is to be parallel to the plane $-8x+6y-z=4$, it must be orthogonal to a normal vector to the given plane $(-8,6,-1)$. To find a vector form and parametric form of the plane equation, we construct two nonzero nonparallel vectors orthogonal to $(-8,6,-1)$, e.g., $(1,0,-8)$ and $(0,1,6)$.

The vector equation of the plane that contains $P(-2,1,0)$ and these two vectors can be expressed as $(x,y,z)=(-2,1,0)+t_1(1,0,-8)+t_2(0,1,6)$.

Parametric equations are $x=-2+t_1,\ y=1+t_2$, and $z=-8t_1+6t_2$.

22. To find a vector form and parametric form of the plane equation, we construct two nonzero nonparallel vectors orthogonal to the plane's normal vector $(2,-6,3)$, e.g., $(-3,0,2)$ and $(3,1,0)$.

We also need a point on the plane $2x-6y+3z=5$, e.g., $(1,0,1)$ - note that any one of the infinitely many solutions can be used here.

The vector equation of the plane that contains the point $(1,0,1)$ and the vectors $(-3,0,2)$ and $(3,1,0)$ can be expressed as $(x,y,z)=(1,0,1)+t_1(-3,0,2)+t_2(3,1,0)$.

Parametric equations are $x=1-3t_1+3t_2,\ y=t_2$, and $z=1+2t_1$.

24. Since the plane is to be orthogonal to the line $x=3-5t,\ y=2t,\ z=7$, we can use the vector $(-5,2,0)$ as a normal vector for the plane. This yields the point-normal equation $-5(x+5)+2(y-1)=0$.