$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow$$ *inverted operations (g):* The third row was added to the first row and $\frac{5}{9}$ times the third row was added to the second row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \longleftarrow$$ *inverted operation (f):* The third row was multiplied by $-\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{5}{9} \\ 0 & -3 & -2 \end{bmatrix} \longleftarrow$$ *inverted operation (e):* -3 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 9 & 5 \\ 0 & -3 & -2 \end{bmatrix} \longleftarrow$$ *inverted operation (d):* The second row was multiplied by 9.

$$\begin{bmatrix} 1 & 2 & 1 \\ -5 & -1 & 0 \\ 6 & 9 & 4 \end{bmatrix} \longleftarrow$$ *inverted operations (c):* 6 times the first row was added to the third row and -5 times the first row was added to the second row.

$$\begin{bmatrix} -1 & -2 & -1 \\ -5 & -1 & 0 \\ 6 & 9 & 4 \end{bmatrix} \longleftarrow$$ *inverted operation (b):* The first row was multiplied by -1.

$$B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \longleftarrow$$ *inverted operation (a):* The first and third rows were interchanged.

1.6 More on Linear Systems and Invertible Matrices

2. We begin by inverting the coefficient matrix $\begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] \longleftarrow$$ The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 4 & -3 & 1 & 0 \end{array}\right] \longleftarrow$$ The first and second rows were interchanged.

$$\left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{array}\right] \longleftarrow$$ -2 times the first row was added to the second row.

$$\left[\begin{array}{cc|cc} 1 & -\frac{5}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right] \longleftarrow$$ The first row was multiplied by $\frac{1}{2}$ and the second row was multiplied by $\frac{1}{7}$.

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{14} & -\frac{3}{14} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right]$$ ← $\frac{5}{2}$ times the second row was added to the first row.

Since the inverse of the coefficient matrix is $\begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, i.e., $x_1 = x_2 = -3$.

4. We begin by inverting the coefficient matrix $\begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← -1 times the second row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 2 & -\frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$ ← -3 times the first row was added to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & -\frac{3}{2} & \frac{5}{2} & 0 \end{array}\right]$$ ← The second and third rows were interchanged.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -\frac{3}{2} & \frac{5}{2} & -3 \end{array}\right]$$ ← -3 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right]$$ ← The third row was multiplied by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right]$$ ← -1 times the third row was added to the second row.

Since the inverse of the coefficient matrix is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$, i.e., $x_1 = 1, x_2 = -11$, and $x_3 = 16$.

6. We begin by inverting the coefficient matrix $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 1 & 4 & 4 \\ 1 & 3 & 7 & 9 \\ -1 & -2 & -4 & -6 \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← The first and second rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 & 0 & 1 \end{array}\right]$$ ← -1 times the first row was added to the third row and the first row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 & 0 & 1 \end{array}\right]$$ ← The second row was multiplied by -1 .

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$ ← -2 times the second row was added to the third row and the second row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$ ← The third row was multiplied by -1 .

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 2 & 1 \end{array}\right]$$ ← -2 times the third row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right]$$ ← The fourth row was multiplied by -1 .

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 0 & 12 & -3 & 8 & 4 \\ 0 & 1 & 2 & 0 & 8 & -3 & 6 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right]$$ ← -1 times the last row was added to the third row, -3 times the last row was added to the second row and -4 times the last row was added to the first row.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 8 & -3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right]$$ ← -2 times the third row was added to the second row and -4 times the third row was added to the first row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right]$$ ← -1 times the second row was added to the first row

Since the inverse of the coefficient matrix is $\begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution: $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix}$,

i.e., $w = -6$, $x = 1$, $y = 10$, and $z = -7$.

8. We begin by inverting the coefficient matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right]$$ ← -2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right]$$ ← The second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$ ← The third row was multiplied by $-\frac{1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{13}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

← The third row was added to the second row and -3 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

← -2 times the second row was added to the first row.

Since the inverse of the coefficient matrix is $\begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3 \\ \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 \\ \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3 \end{bmatrix}$, i.e.,

$x_1 = -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3$, $x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3$, and $x_3 = \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3$.

10.

$$\left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right]$$

← We augmented the coefficient matrix with two columns of constants on the right hand sides of the systems (i) and (ii) – refer to Example 2 on p. 62.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right]$$

← The first row was multiplied by -1.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 13 & -1 & 1 & 1 \\ 0 & 28 & -2 & 0 & -23 \end{array}\right]$$

← -1 times the first row was added to the second row and -6 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 28 & -2 & 0 & -23 \end{array}\right]$$

← The second row was multiplied by $\frac{1}{13}$.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 0 & \frac{2}{13} & -\frac{28}{13} & -\frac{327}{13} \end{array}\right]$$

← -28 times the second row was added to the third row.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right]$$ ← The third row was multiplied by $\frac{13}{2}$.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & 0 & -14 & -\frac{321}{2} \\ 0 & 1 & 0 & -1 & -\frac{25}{2} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right]$$ ← $\frac{1}{13}$ times the third row was added to the second row and the third row was added to the first row.

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -18 & -\frac{421}{2} \\ 0 & 1 & 0 & -1 & -\frac{25}{2} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right]$$ ← 4 times the second row was added to the first row.

We conclude that the solutions of the two systems are:

(i) $x_1 = -18,\ x_2 = -1,\ x_3 = -14$ **(ii)** $x_1 = -\frac{421}{2},\ x_2 = -\frac{25}{2},\ x_3 = -\frac{327}{2}$

12. $$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ -1 & -2 & 0 & 0 & 1 & -1 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right]$$ ← We augmented the coefficient matrix with three columns of constants on the right hand sides of the systems (i), (ii) and (iii) – refer to Example 2 on p. 62.

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & -1 & -6 & -3 & 1 & 2 \end{array}\right]$$ ← The first row was added to the second row and -2 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{array}\right]$$ ← The second row was added to the third row.

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$ ← The third row was multiplied by -1.

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 0 & -9 & 10 & -1 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$ ← -5 times the third row was added to the first row and to the second row.

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 18 & -23 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$ ← -3 times the second row was added to the first row.

We conclude that the solutions of the three systems are:

(i) $x_1 = 18,\ x_2 = -9,\ x_3 = 2$

(ii) $x_1 = -23,\ x_2 = 11,\ x_3 = -2$

(iii) $x_1 = 5,\ x_2 = -2,\ x_3 = 0$

14.

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right]$$ ← The augmented matrix for the system.

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 3 & -2 & b_2 \end{array}\right]$$ ← The first row was multiplied by $\frac{1}{6}$.

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right]$$ ← -3 times the first row was added to the second row.

The system is consistent if and only if $-\frac{1}{2}b_1 + b_2 = 0$, i.e. $b_1 = 2b_2$.

16.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right]$$ ← The augmented matrix for the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & 4b_1 + b_2 \\ 0 & -1 & 0 & 4b_1 + b_3 \end{array}\right]$$ ← 4 times the first row was added to the second row and to the third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -1 & 0 & 4b_1 + b_3 \\ 0 & -3 & -2 & 4b_1 + b_2 \end{array}\right]$$ ← The second and third rows were interchanged.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & -3 & -2 & 4b_1 + b_2 \end{array}\right]$$ ← The second row was multiplied by -1.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & -2 & -8b_1 + b_2 - 3b_3 \end{array}\right]$$ ← 3 times the second row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & 1 & 4b_1 - \frac{1}{2}b_2 + \frac{3}{2}b_3 \end{array}\right]$$ ← The third row was multiplied by $-\frac{1}{2}$.

The system is consistent for all values of b_1, b_2, and b_3.

18. (a) The equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $A\mathbf{x} = I\mathbf{x}$, which yields $A\mathbf{x} - I\mathbf{x} = \mathbf{0}$ and $(A - I)\mathbf{x} = \mathbf{0}$.

This is a matrix form of a homogeneous linear system - to solve it, we reduce its augmented matrix to a row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$ ← The augmented matrix for the homogeneous system $(A-I)\mathbf{x}=\mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right]$$ ← -2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right]$$ ← The second row was multiplied by -1.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$ ← 2 times the second row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ ← The third row was multiplied by $\frac{1}{6}$.

Using back-substitution, we obtain the unique solution: $x_1 = x_2 = x_3 = 0$.

(b) As was done in part (a), the equation $A\mathbf{x} = 4\mathbf{x}$ can be rewritten as $(A-4I)\mathbf{x} = \mathbf{0}$. We solve the latter system by Gauss-Jordan elimination

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$ ← The augmented matrix for the homogeneous system $(A-4I)\mathbf{x}=\mathbf{0}$.

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$ ← The first and second rows were interchanged.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$ ← 2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$ ← The second row was multiplied by -1.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ ← -4 times the second row was added to the third row and the second row was added to the first row.

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$$x_1 = t\,,\ x_2 = 0\,,\ \text{and}\ x_3 = t.$$

20. $X = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$. Let us find $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1}$:

$$\left[\begin{array}{rrr|rrr} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the matrix.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$ ← The first and third rows were interchanged.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -7 & 1 & 0 & 2 \end{array}\right]$$ ← 2 times the first row was added to the third row.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & -7 & 1 & 0 & 2 \end{array}\right]$$ ← The second row was multiplied by -1 .

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -9 & 1 & 2 & 2 \end{array}\right]$$ ← -2 times the second row was added to the third row.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right]$$ ← The third row was multiplied by $-\frac{1}{9}$.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right]$$ ← -1 times the third row was added to the second row and 4 times the third row was added to the first row.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right]$$ ← -1 times the second row was added to the first row.

Using $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix} = \begin{bmatrix} -3 & -\frac{25}{9} & -\frac{25}{9} & -\frac{23}{9} \\ -4 & -\frac{40}{9} & -\frac{40}{9} & -\frac{44}{9} \\ -2 & -\frac{23}{9} & -\frac{32}{9} & -\frac{37}{9} \end{bmatrix}$$

1.7 Diagonal, Triangular, and Symmetric Matrices

2. A diagonal matrix is invertible if and only if its diagonal entries are all nonzero. Since this is a diagonal matrix with a 0 on its diagonal, this matrix is not invertible.

4. A diagonal matrix is invertible if and only if its diagonal entries are all nonzero. All four diagonal entries of this diagonal matrix are nonzero, therefore the matrix is invertible.

6. $\begin{bmatrix} 1 & 2 & -5 \\ -3 & -1 & 0 \end{bmatrix}\begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} (1)(-4) & (2)(3) & (-5)(2) \\ (-3)(-4) & (-1)(3) & (0)(2) \end{bmatrix} = \begin{bmatrix} -4 & 6 & -10 \\ 12 & -3 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} (2)(4)(-3) & (2)(-1)(5) & (2)(3)(2) \\ (-1)(1)(-3) & (-1)(2)(5) & (-1)(0)(2) \\ (4)(-5)(-3) & (4)(1)(5) & (4)(-2)(2) \end{bmatrix}$

$= \begin{bmatrix} -24 & -10 & 12 \\ 3 & -10 & 0 \\ 60 & 20 & -16 \end{bmatrix}$

10. $A^2 = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$, $A^{-2} = \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{25} \end{bmatrix}$, $A^{-k} = \begin{bmatrix} (-6)^{-k} & 0 & 0 \\ 0 & 3^{-k} & 0 \\ 0 & 0 & 5^{-k} \end{bmatrix}$

12. $A^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$, $A^{-k} = \begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix}$

14. Not symmetric ($a_{12} \neq a_{21}$)

16. Symmetric

18. Symmetric

20. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this upper triangular matrix has all three diagonal entries nonzero, it is invertible.

22. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this lower triangular matrix has a 0 on its diagonal, it is not invertible.

24. The matrix is symmetric if and only if the following equations must be satisfied

$$\begin{array}{rcrcrcr} a & - & 2b & + & 2c & = & 3 \\ 2a & + & b & + & c & = & 0 \\ a & & & + & c & = & -2 \end{array}$$

We solve this system by Gauss-Jordan elimination

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -2 \end{array}\right]$$ ← The augmented matrix for the system.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & 2 & 3 \end{array}\right]$$ ← The first and third rows were interchanged.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & -2 & 1 & 5 \end{array}\right]$$ ← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 13 \end{array}\right]$$ ← 2 times the second row was added to the third row.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -13 \end{array}\right]$$ ← The third row was multiplied by -1.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -13 \end{array}\right]$$ ← The third row was added to the second row and -1 times the third row was added to the first row.

In order for A to be symmetric, we must have $a = 11$, $b = -9$, and $c = -13$.

26. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given lower triangular matrix is invertible for any real number x such that $x \neq \frac{1}{2}$, $x \neq \frac{1}{3}$, and $x \neq -\frac{1}{4}$.

28. For example $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (there are seven other possible answers, e.g., $\begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$, etc.)

30. $$\left[\begin{array}{rrr|rrr} -1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the matrix A.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -5 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$ ← The first row was multiplied by -1 and the third row was multiplied by $-\frac{1}{4}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & -\frac{5}{4} \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$ ← -3 times the third row was added to the second row and 5 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$ ← 2 times the second row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 2 & -8 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the matrix B.

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$ ← The first row was multiplied by $\frac{1}{2}$, the second row was multiplied by $\frac{1}{2}$, and the third row was multiplied by $\frac{1}{3}$.

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$ ← $-\frac{1}{2}$ times the third row was added to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 2 & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$ ← 4 times the second row was added to the first row.

Since both $A^{-1} = \begin{bmatrix} -1 & 2 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 2 & -\frac{2}{3} \\ 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ are upper triangular, we have verified Theorem 1.7.1(d).

32. (a) Theorem 1.4.8(e) states that $(AB)^T = B^T A^T$ (if the multiplication can be performed). Therefore,

$$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2 \underset{\substack{A \text{ is} \\ \text{symmetric}}}{=} A^2$$

which shows that A^2 is symmetric.

(b) $(2A^2-3A+I)^T \underset{\substack{\text{Th.}\\ 1.4.8\\ \text{(b-d)}}}{=} 2(A^2)^T-3A^T+I^T \underset{\substack{\text{Th.}\\ 1.4.8\\ \text{(e)}}}{=} 2(A^T)^2-3A^T+I^T \underset{\substack{A \text{ and } I\\ \text{are}\\ \text{symmetric}}}{=} 2A^2-3A+I$

which shows that $2A^2-3A+I$ is symmetric.

34. All 3×3 diagonal matrices have a form $\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$.

$$\begin{aligned}
A^2-3A-4I &= \begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix} - 3\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix} a^2 & 0 & 0\\ 0 & b^2 & 0\\ 0 & 0 & c^2\end{bmatrix} - \begin{bmatrix} 3a & 0 & 0\\ 0 & 3b & 0\\ 0 & 0 & 3c\end{bmatrix} - \begin{bmatrix} 4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}\\
&= \begin{bmatrix} a^2-3a-4 & 0 & 0\\ 0 & b^2-3b-4 & 0\\ 0 & 0 & c^2-3c-4\end{bmatrix}\\
&= \begin{bmatrix} (a-4)(a+1) & 0 & 0\\ 0 & (b-4)(b+1) & 0\\ 0 & 0 & (c-4)(c+1)\end{bmatrix}
\end{aligned}$$

This is a zero matrix whenever the value of a, b, and c is either 4 or -1. We conclude that the following are all 3×3 diagonal matrices that satisfy the equation:

$$\begin{bmatrix} 4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}, \begin{bmatrix} 4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix} 4 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 4\end{bmatrix}, \begin{bmatrix} -1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix},$$
$$\begin{bmatrix} 4 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix} -1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 4\end{bmatrix}, \begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

38. $\begin{bmatrix} 0 & 0 & 4\\ 0 & 0 & 1\\ -4 & -1 & 0\end{bmatrix}$

40. The condition $A^T=-A$ is equivalent to the linear system

$$\begin{array}{rcrcrcrcl}
2a & - & 3b & + & c & & & = & 2\\
3a & - & 5b & + & 5c & & & = & 3\\
5a & - & 8b & + & 6c & & & = & 5\\
 & & & & & & d & = & 0
\end{array}$$

The augmented matrix $\begin{bmatrix} 2 & -3 & 1 & 0 & 2 \\ 3 & -5 & 5 & 0 & 3 \\ 5 & -8 & 6 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -10 & 0 & 1 \\ 0 & 1 & -7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

If we assign c the arbitrary value t, the general solution is given by the formulas

$$a = 1 + 10t, \ b = 7t, \ c = t, \ d = 0.$$

42. (a) Step 1. Solve $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

The first equation is $y_1 = 1$.

The second equation $(-2)(1) + 3y_2 = -2$ yields $y_2 = 0$.

The third equation $(2)(1) + (4)(0) + 1y_3 = 0$ yields $y_3 = -2$.

Step 2. Solve $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ using back-substitution:

The third equation $4x_3 = -2$ yields $x_3 = -\frac{1}{2}$.

The second equation $1x_2 + (2)\left(-\frac{1}{2}\right) = 0$ yields $x_2 = 1$.

The first equation $2x_1 + (-1)(1) + (3)\left(-\frac{1}{2}\right) = 1$ yields $x_1 = \frac{7}{4}$.

(b) Step 1. Solve $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

The first equation $2y_1 = 4$ yields $y_1 = 2$.

The second equation $(4)(2) + 1y_2 = -5$ yields $y_2 = -13$.

The third equation $(-3)(2) + (-2)(-13) + 3y_3 = 2$ yields $y_3 = -6$.

Step 2. Solve $\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -13 \\ -6 \end{bmatrix}$ using back-substitution:

The third equation $2x_3 = -6$ yields $x_3 = -3$.

The second equation $4x_2 + (1)(-3) = -13$ yields $x_2 = -\frac{5}{2}$.

The first equation $3x_1 + (-5)\left(-\frac{5}{2}\right) + (2)(-3) = 2$ yields $x_1 = -\frac{3}{2}$.