Tied dice

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Abstract

A *generalized die* is a list $(x_1, ..., x_n)$ of integers. For integers $n \geq 1$, $a \leq b$ and s let $D(n, a, b, s)$ be the set of all dice with $a \leq x_1 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. Two dice X and Y are *tied* if the number of pairs (i, j) with $x_i < y_j$ equals the number of pairs (i, j) with $x_i > y_j$. We prove the following: with one exception (unique up to isomorphism), if $X \neq Y \in D(n, a, b, s)$ are tied dice neither of which ties all other elements of $D(n, a, b, s)$ then there is a third die $Z \in D(n, a, b, s)$ which ties neither X nor Y.

1. Introduction

If a, b, n, s are integers with $a \leq b$ and $n > 0$ then we denote by $D(n, a, b, s)$ the set of all integer lists $X = (x_1, ..., x_n)$ with $a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. $D(n, a, b, s)$ is the *dice family* containing all *n-sided generalized dice* with integer labels which are bounded by a and b and whose mean is s/n. If $X, Y \in D(n, a, b, s)$ we say X is *stronger* than Y if it happens that among the n^2 choices of an x_i and a y_j, there are more with $x_i > y_j$ than with $x_i < y_j$. The fact that *stronger* can be non-transitive was noted by Gardner [2] and has also been studied by other authors [3, 4]. It has become a popular example of counterintuitive or "paradoxical" behavior of simple mathematical objects, and is mentioned in many textbooks on elementary mathematics and probability; one such book [1] even comes with several generalized dice as accessories.

If neither of $X, Y \in D(n, a, b, s)$ is stronger than the other then we say X and Y are *tied*, and if X is tied with every element of $D(n, a, b, s)$ then we say X is *balanced*. Several recent results indicate that the non-transitivity of the *stronger* relation is not the only counterintuitive property of generalized dice; in addition, ties are rarer than one might expect, considering that every element of $D(n, a, b, s)$ has the same mean label value. Balanced dice are particularly rare; for instance there are only three balanced dice among the 458 elements of the dice families $D(6,1,6,s)$ with $8 \le s \le 34$ [5]. In the present paper we are concerned with ties involving non-balanced dice. Such ties are certainly not uncommon; for instance, the 31 non-balanced elements of $D(6,1,6,21)$ include 94 tied pairs. Nevertheless such ties cannot be considered "typical."

The Tied Dice Theorem. *Suppose $X \neq Y \in D(n,a,b,s)$ are tied, non-balanced dice. Unless X and Y are the dice $(a,a+4,a+8)$ and $(a+2,a+4,a+6)$ in $D(3,a,a+8,3a+12)$, there is a $Z \in D(n,a,b,s)$ which ties neither X nor Y.*

If c is an integer then there is a *stronger*-preserving isomorphism between $D(n,a,b,s)$ and $D(n,a+c,b+c,s+nc)$ under which $(x_1,...,x_n)$ corresponds to $(x_1+c,...,x_n+c)$. Taking these isomorphisms into account, the Tied Dice Theorem tells us that there is essentially only one example of a pair of tied, non-balanced dice $X, Y \in D(n,a,b,s)$ such that every element of $D(n,a,b,s)$ ties at least one of X, Y.

It is natural to represent dice families using graphs. A *dice graph* $G(n,a,b,s)$ has a vertex for each die in $D(n,a,b,s)$; non-tied dice are represented by adjacent vertices. The Tied Dice Theorem implies that up to isomorphism, $G(3,1,9,15)$ is the only dice graph with a component of diameter greater than 2. (The weaker statement $diam(G(n,a,b,s)) \le 6$ was proven in [6].) $G(3,1,9,15)$ is pictured below, with dice assigned vertices lexicographically: a represents $(1,5,9)$, b represents $(1,6,8)$, and so on.

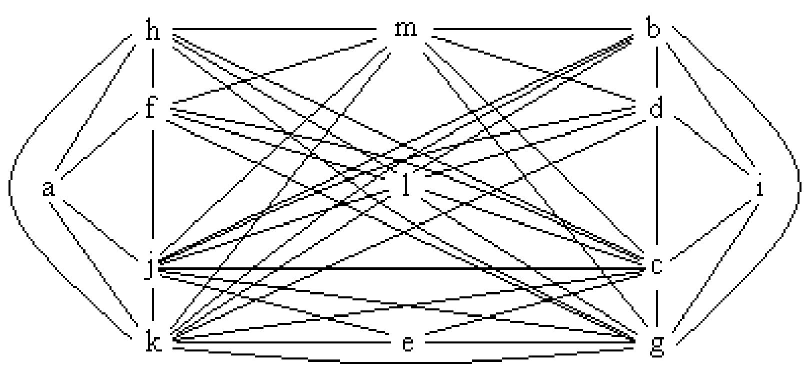

Figure 1: the unique (up to isomorphism) dice family containing two tied, non-balanced dice which do not share a non-tying "neighbor"

In the rest of the paper we outline a proof of the Tied Dice Theorem. The proof is fairly direct for $n \leq 3$ and $n \geq 7$, but requires the consideration of many special cases when $4 \leq n \leq 6$. We present some of these special cases here, and we would be delighted to share the rest with the interested reader.

Before proceeding we should observe that there are many open questions regarding dice families. What can be said about the asymptotic proportion of tied pairs of dice? Is it possible to characterize the dice families which contain elements that are not weaker than any others, or not stronger than any others, or tied with most others? How common are failures of transitivity? We hope an interested reader will find the answers to some of these questions.

2. Generalities

Several notational conventions will be convenient. If a, b, n, s are integers with $a \leq b$ and $n > 0$ then we let $p = p(n, a, b, s) = \min\{x_1 \mid (x_1, ..., x_n) \in D(n, a, b, s)\}$ and $q = q(n, a, b, s) = \max\{x_n \mid (x_1, ..., x_n) \in D(n, a, b, s)\}$; then $D(n, a, b, s) = D(n, p, q, s)$ and both p and q appear on some element of $D(n, a, b, s)$. A die $X = (x_1, ..., x_n) \in D(n, a, b, s)$ is completely described by its *characteristic vector* $v^X = (v_p^X, ..., v_q^X)$, with $v_i^X = |\{j \mid x_j = i\}|$. As was observed in [5], X is balanced if and only if v^X is of the form $(v, w, v, w, ...)$; for instance $(2, 2, 4, 4)$ is balanced in $D(4, 1, 5, 12)$, where its characteristic vector is $(0, 2, 0, 2, 0)$, but it is not balanced in $D(4, 1, 6, 12)$, where its characteristic vector is $(0, 2, 0, 2, 0, 0)$.

If $X \in D(n, a, b, s)$ and $p \leq k \leq q$ then we denote by $f_X(k)$ the win-loss difference of a roll of k against X:

$$f_X(k) = \sum_{i<k} v_i^X - \sum_{i>k} v_i^X.$$

For $Y = (y_1, ..., y_n) \in D(n, a, b, s)$ let $f_X(Y) = \sum_{i=1}^{n} f_X(y_i)$. Then $f_X(Y)$ is positive, negative or 0 according to whether X is weaker than Y, stronger than Y, or tied with Y.

If $v_i^Y, v_j^Y > 0$ then $Y(i \mapsto i+1, j \mapsto j-1)$ denotes the die obtained from Y by replacing a single label i with an $i+1$ and a single label j with a $j-1$. Observe that for any die X, $f_X(Y(i \mapsto i+1, j \mapsto j-1)) = f_X(Y) + v_i^X + v_{i+1}^X - v_{j-1}^X - v_j^X$.

Lemma 2.1. *Suppose $X \in D(n, a, b, s)$ is not balanced. Then there are $i, j \in \{p, ..., q-1\}$ such that $i + j \in \{p+q-1, p+q\}$ and $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$.*

Proof. Suppose not. Then $v_p^X + v_{p+1}^X = v_{q-1}^X + v_q^X = v_{p+1}^X + v_{p+2}^X = v_{q-2}^X + v_{q-1}^X = v_{p+2}^X + v_{p+3}^X$ and so on; hence $v_i^X + v_{i+1}^X = v_j^X + v_{j+1}^X$ for all $i, j \in \{p, ..., q-1\}$. This is a contradiction, for it implies X is balanced. ■

Lemma 2.2. *Suppose $X \in D(n, a, b, s)$ is not balanced. Then there are $i, j \in \{p, ..., q-1\}$ such that $i+j \in \{p+q-1, p+q-2\}$ and $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$.*

Proof. If not then $v_{q-1}^X + v_q^X = v_p^X + v_{p+1}^X = v_{q-2}^X + v_{q-1}^X = v_{p+1}^X + v_{p+2}^X$ etc., so X is balanced. ■

Proposition 2.3. *Suppose $q-p > 2$, $2q + (n-2)p \leq s \leq 2p + (n-2)q$, and $n \geq 5$. If $X, Y \in D(n, a, b, s)$ are tied, non-balanced dice then there is a $Z \in D(n, a, b, s)$ which ties neither X nor Y.*

Proof. Suppose there are $i, j, u, w \in \{p, ..., q-1\}$ such that $i + j = u+w = p+q-1$, $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$ and $v_u^Y + v_{u+1}^Y \neq v_w^Y + v_{w+1}^Y$. We may choose them so that either $u = i$ and $w = j$ or else $v_u^X + v_{u+1}^X = v_w^X + v_{w+1}^X$. Then $(n-4)p \leq s-\ (i + j + 1 + u + w + 1) = s - 2(p+q) \leq (n-4)q$, so there is a die $Z \in D(n, a, b, s)$ for which there are four different indices $\alpha, \beta, \gamma, \delta \in \{1, ..., n\}$ such that $z_\alpha = i$, $z_\beta = j+1$, $z_\gamma = u$ and $z_\delta = w+1$. (If $i = u$ and $j = w$ then $v_i^Z \geq 2$ and $v_{j+1}^Z \geq 2$.) If Z ties neither X nor Y, the proposition is satisfied. Suppose Z ties X, that is, $f_X(Z) = 0$. Then $Z' = Z(i \mapsto i+1, j+1 \mapsto j)$ has $f_X(Z') = f_X(Z) + v_i^X + v_{i+1}^X - v_j^X - v_{j+1}^X = 0 + v_i^X + v_{i+1}^X - v_j^X - v_{j+1}^X \neq 0$, so Z' does not tie X. If Z' does not tie Y, the proposition is satisfied. If Z' does tie Y, $Z'' = Z'(u \mapsto u+1, w+1 \mapsto w)$ has $f_Y(Z'') = f_Y(Z') + v_u^Y + v_{u+1}^Y - v_w^Y - v_{w+1}^Y = 0 + v_u^Y + v_{u+1}^Y - v_w^Y - v_{w+1}^Y \neq 0$ and $f_X(Z'') = f_X(Z') + v_u^X + v_{u+1}^X - v_w^X - v_{w+1}^X = v_i^X + v_{i+1}^X - v_j^X - v_{j+1}^X + v_u^X + v_{u+1}^X - v_w^X - v_{w+1}^X \neq 0$; hence Z'' ties neither X nor Y and the proposition is satisfied. If Z ties Y then a similar argument applies.

Suppose there are $i, j \in \{p, ..., q-1\}$ such that $i + j = p + q - 1$ and $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$, but $v_u^Y + v_{u+1}^Y = v_w^Y + v_{w+1}^Y$ for all $u, w \in \{p, ..., q-1\}$ with $u + w = p + q - 1$. Suppose for the moment that $2q + (n-2)p < s$. Lemma 2.1 implies that there are $u, w \in \{p, ..., q-1\}$ with $u + w = p + q$ and $v_u^Y + v_{u+1}^Y \neq v_w^Y + v_{w+1}^Y$. Then $(n-4)p \leq s - (i + j + 1 + u + w + 1) = s - 2(p+q) - 1 \leq (n-4)q - 1$, so there is a die $Z = (z_1, ..., z_n) \in D(n, a, b, s)$ for which there are four different indices $\alpha, \beta, \gamma, \delta \in \{1, ..., n\}$ such that $z_\alpha = i$, $z_\beta = j+1$, $z_\gamma = u$ and $z_\delta = w+1$. (If i or $j+1$ coincides with u or $w+1$ then $v_i^Z \geq 2$ or $v_{j+1}^Z \geq 2$.) If Z ties neither X nor Y the proposition is satisfied, and if Z ties X but not Y then $Z(i \mapsto i+1, j+1 \mapsto j)$ ties neither X nor Y. If Z ties Y then $Z' = Z(u \mapsto u+1, w+1 \mapsto w)$ and $Z'' = Z(i \mapsto i+1, j+1 \mapsto j, u \mapsto u+1, w+1 \mapsto w)$ do not. As

$f_X(Z'') = f_X(Z') + v_i^X + v_{i+1}^X - v_j^X - v_{j+1}^X \neq f_X(Z')$, at least one of Z', Z'' does not tie X and consequently satisfies the proposition.

A similar argument applies when $2q + (n-2)p = s$; use Lemma 2.2 rather than Lemma 2.1 to find $u, w \in \{p, ..., q-1\}$ with $u + w = p + q - 2$ and $v_u^Y + v_{u+1}^Y \neq v_w^Y + v_{w+1}^Y$.

The preceding arguments verify the proposition if there are $i, j \in \{p, ..., q-1\}$ such that $i + j = p + q - 1$ and $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$; similarly, the proposition holds if there are $i, j \in \{p, ..., q-1\}$ such that $i + j = p + q - 1$ and $v_i^Y + v_{i+1}^Y \neq v_j^Y + v_{j+1}^Y$. It remains to consider the possibility that $v_i^X + v_{i+1}^X = v_j^X + v_{j+1}^X$ and $v_i^Y + v_{i+1}^Y = v_j^Y + v_{j+1}^Y$ for all $i, j \in \{p, ..., q-1\}$ such that $i + j = p + q - 1$.

Suppose for the moment that $s \geq (n-2)p + 2q + 2$. Lemma 2.1 implies that there are $i, j, u, w \in \{p, ..., q-1\}$ such that $i + j = u + w = p + q$, $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$ and $v_u^Y + v_{u+1}^Y \neq v_w^Y + v_{w+1}^Y$; we may choose them so that either $u = i$ and $w = j$ or else $v_u^X + v_{u+1}^X = v_w^X + v_{w+1}^X$. Then $(n-4)p \leq s - (i + j + 1) - (u + w + 1) = s - 2(p + q) - 2 \leq (n-4)q$ so there is a die $Z \in D(n, a, b, s)$ for which there are four different indices $\alpha, \beta, \gamma, \delta \in \{1, ..., n\}$ such that $z_\alpha = i$, $z_\beta = j + 1$, $z_\gamma = u$ and $z_\delta = w + 1$. The argument of the first paragraph of the proof applies in this case.

A similar argument applies when $s < 2q + (n-2)p + 2$; use Lemma 2.2 rather than Lemma 2.1 to find $i, j, u, w \in \{p, ..., q-1\}$ with $i + j = u + w = p + q - 2$, $v_i^X + v_{i+1}^X \neq v_j^X + v_{j+1}^X$ and $v_u^Y + v_{u+1}^Y \neq v_w^Y + v_{w+1}^Y$. ■

Proposition 2.4. *Suppose $q - p > 2$, $s \leq 2q + (n-2)p$ or $s \geq 2p + (n-2)q$, and $n \geq 7$. If $X \neq Y \in D(n, a, b, s)$ are tied, non-balanced dice then there is a $Z \in D(n, a, b, s)$ which ties neither X nor Y.*

Proof. Suppose first that $s \leq 2q + (n-2)p$. Let $C = (c_1, ..., c_n), D = (d_1, ..., d_n) \in D(n, a, b, s)$ have $c_{n-2} = p < d_{n-2}$. Then every pair (i, j) with $i \leq n-2$ and $j \geq n-2$ has $c_i < d_j$, and every pair (i, j) with $c_i > d_j$ has $i > n-2$. Consequently there are at least $3(n-2)$ pairs (i, j) with $c_i < d_j$ and at most $2n$ pairs (i, j) with $c_i > d_j$; as $n \geq 7$, C is weaker than D. The proposition follows because every element of $D(n, a, b, s)$ is of type C or type D, and $D(n, a, b, s)$ contains both dice of both types. If $s \geq 2p + (n-2)q$ a similar argument applies. ■

Theorem 2.5. *Suppose $q - p \leq 2$ or $n \leq 2$ or $n \geq 7$. If $X \neq Y \in D(n, a, b, s)$ are tied, non-balanced dice then there is a $Z \in D(n, a, b, s)$ which ties neither X nor Y.*

Proof. If $q-p=1$ then $D(n,a,b,s)$ only contains one die, which must be balanced as it certainly ties itself. If $q-p=2$ then $D(n,a,b,s)$ has one of two structures: either all its elements are tied or else it is linearly ordered by the *stronger* relation [6, 7]. In the first case all its elements are balanced, and in the second case none are tied. If $n=1$ then $D(n,a,b,s)$ has only one element. If $n=2$ then any two distinct elements of $D(n,a,b,s)$ are (c_1,c_2) and (d_1,d_2) with $c_1 < d_1 \leq d_2 < c_2$; clearly then all elements of $D(n,a,b,s)$ are tied, and hence all are also balanced.

In sum, if $q-p \leq 2$ or $n \leq 2$ then there are no distinct, tied, non-balanced dice in $D(n,a,b,s)$. If $n \geq 7$, on the other hand, then one of the two preceding propositions applies. ■

3. $n=6$

If $n=6$ then Proposition 2.3 and Theorem 2.5 show that the Tied Dice Theorem is satisfied if $q-p \leq 2$ or $4p+2q \leq s \leq 2p+4q$. We may ignore the possibility that $s > 2p+4q$ because in this case $X \mapsto -X$ defines a *stronger*-reversing bijection between $D(n,a,b,s)$ and $D(n,-b,-a,-s)$. Moreover, if $s=5p+q$ then the Tied Dice Theorem holds because (p,p,p,p,p,q) is weaker than every other element of $D(n,a,b,s)$. Consequently we may proceed assuming that $q-p>2$ and $4p+2q > s > 5p+q$. We partition the dice in $D(n,a,b,s)$ into three types: dice of type C are $C=(p,p,p,p,c_5,c_6)$ with $c_5 > p$; dice of type D are $D=(p,p,p,d_4,d_5,d_6)$ with $d_4 > p$; and dice of type E are $E=(e_1,e_2,e_3,e_4,e_5,e_6)$ with $e_3 > p$. As $s > 5p+q$ and $q > p+2$, there are dice of all three types. Observe that a die of type C cannot tie any die of type E, and a die of type C ties a die of type D if and only if $c_5 > d_6$.

Suppose X and Y are distinct, tied, non-balanced elements of $D(n,a,b,s)$. If both are of type C (resp. E) then the Tied Dice Theorem is satisfied by any die Z of type E (resp. C); hence we may as well suppose that X is of type D. As $s > 5p+q > 6p+2$, $x_6 \geq p+2$.

If Y is of type C then $Z = Y(p \mapsto p+1, y_6 \mapsto y_6-1)$ does not tie Y, because $v^Y_p + v^Y_{p+1} \geq 4 > v^Y_{y_6} + v^Y_{y_6-1}$. If Z does not tie X then the theorem is satisfied. If Z ties X then it must be that $v^X_p + v^X_{p+1} = v^X_{y_6} + v^X_{y_6-1}$. As $v^X_p = 3$ and $y_6 \geq y_5 > x_6 > p+1$, this can only happen if $y_6 - 1 = x_6$ and $v^X_{y_6-1} = 3$. Then $s = 3p+3(y_6-1) = 4p+2y_6$, so $y_6 = p+3$. As $s = 6p+6 < 4p+2q$, $q \geq p+4$; then $(p,p,p,p+1,p+1,p+4)$ is an element of $D(n,a,b,s)$ which ties neither $X=(p,p,p,p+2,p+2,p+2)$ nor $Y=(p,p,p,p,p+3,p+3)$.

Suppose X and Y are distinct dice of type D; rename them if necessary so that $x_6 \le y_6$. If $s \le 6p+4$ then it is not possible for X and Y to be distinct, and if $s = 6p+5$ then it must be that $X = (p,p,p,p+1,p+2,p+2)$ and $Y = (p,p,p,p+1,p+1,p+3)$; but these are not tied. Hence it must be that $s \ge 6p+6$. If $s = 6p+6$ then $X = (p,p,p,p+2,p+2,p+2)$, $Y = (p,p,p,p+1,p+2,p+3)$, and neither ties $(p,p,p,p,p+2,p+4)$; note that $q \ge p+4$ because $q \le p+3$ would imply X is balanced. If $s > 6p+6$ then $y_6 \ge x_6 \ge p+3$. If $x_6 > p+3$ then let $Z = (p+1,p+1,p+1,x_4,x_5,x_6-3)$; Z should be renamed $(p+1,p+1,p+1,x_4,x_6-3,x_5)$ or $(p+1,p+1,p+1,x_6-3,x_4,x_5)$ if necessary. As $x_6 \le y_6$, $x_5 < y_4$ is impossible; hence Z ties neither X nor Y. Suppose $s > 6p+6$ and $x_6 = p+3$; then $x_5 \ge p+2$. Let $Z = (p+1,p+1,p+1,p+1,x_4,x_5-1)$; Z should be renamed $(p+1,p+1,p+1,p+1,x_5-1,x_4)$ if necessary. Z does not tie X, and Z ties Y if and only if $y_4 > x_4$ and $y_4 > x_5-1$. As $s > 6p+6$ and $x_6 = p+3$, x_5-1 and x_4 cannot both be as small as $p+1$; hence if y_4 is bigger than both then $y_4 \ge p+3 = x_6$. This cannot occur if X and Y are distinct, so Z ties neither X nor Y.

Finally, suppose X is of type D and Y is of type E. $Z = (p,p,p,p,s-4p-q,q)$ does not tie Y; if Z does not tie X then the Tied Dice Theorem is satisfied. If Z does tie X then $s-4p-q > x_6$; as $x_6 \ge p+2$, it follows that $s-4p-q \ge p+3$. Then $s \ge 5p+q+3$, so $s < 4p+2q$ implies $q \ge p+4$ and consequently $s \ge 5p+q+3 \ge 6p+7$. As $s \le 3x_6+3p$, $s \ge 6p+7$ implies that $x_6 \ge p+3$; then $s-4p-q > x_6$ implies $s > 5p+q+3$, so $s < 4p+2q$ implies $q > p+4$, and then $s > 5p+q+3$ implies $s \ge 6p+9$.

Choose the smallest r with $v^Y_p + v^Y_{p+1} \ne v^Y_r + v^Y_{r+1}$. Then $2p < p+r+4 \le p+q+3$, so $4p+2q > s > 5p+q+3$ implies $2p+2q > s-(p+r+3) > 4p$; consequently there is a $Z' \in D(n,a,b,s)$ with $v^{Z'}_p \ge 2$ and either $v^{Z'}_{p+2} \ge 2$ (if $r = p+1$) or $v^{Z'}_{p+2} \ne 0 \ne v^{Z'}_{r+1}$ (if $r > p+1$). If Z' ties neither X nor Y then the Tied Dice Theorem is satisfied. If Z' ties Y then $Z'' = Z'(p \mapsto p+1, r+1 \mapsto r)$ does not; if Z'' does not tie X then the theorem is satisfied. If Z'' does tie X then $Z''' = Z''(p+2 \mapsto p+1, p \mapsto p+1)$ does not, because $x_6 \ge p+3$ implies $v^X_p + v^X_{p+1} > v^X_{p+1} + v^X_{p+2}$. Moreover Z''' does not tie Y, for $f_Y(Z''') = f_Y(Z') + 2(v^Y_p + v^Y_{p+1}) - (v^Y_{p+1} + v^Y_{p+2} + v^Y_r + v^Y_{r+1}) = 0 + v^Y_p + v^Y_{p+1} - (v^Y_{p+1} + v^Y_{p+2}) + v^Y_p + v^Y_{p+1} - (v^Y_r + v^Y_{r+1}) = \delta(v^Y_p + v^Y_{p+1} - v^Y_r - v^Y_{r+1}) \ne 0$, where δ is 2 or 1 according to whether $r = p+1$ or $r > p+1$. If Z' ties X and does not tie Y then $W = Z'(p \mapsto p+1, p+2 \mapsto p+1)$ does not tie X, because $v^X_p + v^X_{p+1} > v^X_{p+1} + v^X_{p+2}$ and hence $f_X(W) > f_X(Z') = 0$. If W does not tie Y, the theorem is satisfied. If W does tie Y then $W' = W(p \mapsto p+1, r+1 \mapsto r)$ does not, because $v^Y_p + v^Y_{p+1} \ne v^Y_r + v^Y_{r+1}$; moreover W' does not tie X because $v^X_p + v^X_{p+1} \ge v^X_i + v^X_{i+1}$ for all i and hence $f_X(W') \ge f_X(W) > 0$.

4. $n = 5$

If $n = 5$ then Proposition 2.3 and Theorem 2.5 show that the Tied Dice Theorem is satisfied if $q-p \leq 2$ or $3p+2q \leq s \leq 2p+3q$. We may ignore the possibility that $s > 2p + 3q$ because $X \mapsto -X$ defines a *stronger*-reversing bijection between $D(n, a, b, s)$ and $D(n, -b, -a, -s)$.

If $s = 4p + q$ then the Tied Dice Theorem holds because (p, p, p, p, q) is weaker than every other element of $D(n, a, b, s)$. Otherwise $s > 4p + q > 5p+2$ and hence $s \geq 5p+4$. If $s = 5p+4$ then $(p, p+1, p+1, p+1, p+1)$ is stronger than every other element of $D(n, a, b, s)$, so the Tied Dice Theorem is satisfied. If $s = 5p + 5$ and $q = p + 3$ then $(p, p, p, p + 2, p + 3)$ is weaker than every other element of $D(n, a, b, s)$. If $s = 5p + 5$ and $q = p + 4$ then $(p, p, p, p + 1, p + 4)$ and $(p, p, p, p + 2, p + 3)$ are tied, and each is weaker than every other element of $D(n, a, b, s)$. In any case, if $s = 5p + 5$ the theorem is satisfied. Consequently we may proceed assuming that $q-p > 2$, $3p + 2q > s > 4p + q$ and $s > 5p + 5$.

We partition the dice in $D(n, a, b, s)$ into three types: dice of type C are $C = (p, p, p, c_4, c_5)$ with $c_4 > p$; dice of type D are $D = (p, p, d_3, d_4, d_5)$ with $d_3 > p$; and dice of type E are $E = (e_1, e_2, e_3, e_4, e_5)$ with $e_2 > p$. As $s > 4p + q$ and $q > p + 2$, there are dice of all three types. Observe that a die of type C cannot tie any die of type E, and a die of type C ties a die of type D if and only if $c_5 > c_4 = d_5 > d_4$.

Suppose X and Y are distinct, tied, non-balanced elements of $D(n, a, b, s)$; we may as well suppose that X is of type D. As $s > 5p + 5$, $x_5 \geq p + 2$.

If Y is of type C then $y_5 > y_4 = x_5 > x_4$, so $Y = (p, p, p, y_4, y_5)$ and $X = (p, p, x_3, x_4, y_4)$ with $v^Y_{y_4} = 1 = v^X_{y_4}$. As $y_5 > x_5$ and $y_4 > x_4$, it must be that $x_3 > p + 1$; then $(p, p + 1, x_3 - 1, x_4, y_4)$ ties neither X nor Y.

Suppose X and Y are distinct dice of type D. They are tied, so there must be an odd number of pairs (i, j) with $i, j \geq 3$ and $x_i = y_j$. If this number is more than 3 then $X = Y$. If this number is precisely 3 then changing names if necessary, we may presume $y_3 < x_3 = x_4 = x_5 = y_4 < y_5$ and $y_4-y_3 = y_5-y_4$. If $y_4-y_3 > 1$ then $(p, p+1, y_3, y_4-1, y_5)$ ties neither X nor Y. If $y_4 - y_3 = 1$ and $y_3 > p+1$ then $(p, p+1, y_3 - 1, y_4, y_5)$ ties neither X nor Y. If $y_4 - y_3 = 1$ and $y_3 = p + 1$ then $X = (p, p, p + 2, p + 2, p + 2)$ and $Y = (p, p, p + 1, p + 2, p + 3)$; neither ties $(p, p, p, p + 3, p + 3)$.

Suppose the number of pairs (i, j) with $i, j \geq 3$ and $x_i = y_j$ is 1. If $x_3 = y_3$ then we may presume $X = (p, p, x_3, x_4, x_5)$ and $Y = (p, p, x_3, y_4, y_5)$ with $x_4 < y_4 < y_5 < x_5$ and $y_4 - x_4 = x_5 - y_5$; then $(p, p+1, x_3, x_4, x_5 - 1)$

ties neither X nor Y. If $x_4 = y_4$ then we may presume $X = (p, p, x_3, x_4, x_5)$ and $Y = (p, p, x_3, y_4, y_5)$ with $x_3 < y_3 < x_4 = y_4 < y_5 < x_5$ and $y_3 - x_3 = x_5 - y_5$; then $(p, p+1, x_3, x_4, x_5 - 1)$ ties neither X nor Y. If $x_5 = y_5$ then we may presume $X = (p, p, x_3, x_4, x_5)$ and $Y = (p, p, x_3, y_4, y_5)$ with $x_3 < y_3 < y_4 < x_4 < x_5 = y_5$ and $y_3 - x_3 = x_4 - y_4$; then $(p, p+1, x_3, x_4 - 1, x_5)$ ties neither X nor Y.

Suppose X is of type D and Y is of type E. Y ties no die of type C. As observed above the only type C die which might tie X is $(p, p, p, x_5, s - x_5 - 3p)$, so the theorem is satisfied unless this is the unique die of type C and it ties X. We proceed assuming this uniqueness, which implies $s - x_5 - 3p = q$ and $x_5 = q - 1$, and assuming $C = (p, p, p, q-1, q)$ ties X, which implies $p < x_3 \leq x_4 = p + q - x_3 < x_5 = q - 1$.

Suppose for the moment that $x_4 = q-2$; then $X = (p, p, p+2, q-2, q-1)$. If $q - p = 3$ then $s = 3p + 2q - 1 = 5p + 5$; but $s > 5p + 5$, so $q - p \geq 4$. If $q = p + 4$ then $X = (p, p, p+2, p+2, p+3)$. X does not tie $X(p+3 \mapsto p+4, p+2 \mapsto p+1)$ or $X(p+3 \mapsto p+2, p \mapsto p+1)$, so the theorem is satisfied unless both of these dice tie Y; consequently we may presume $v^Y_{p+3} + v^Y_{p+4} = v^Y_{p+2} + v^Y_{p+1}$ and $v^Y_{p+3} + v^Y_{p+2} = v^Y_p + v^Y_{p+1}$. As Y is not balanced, it must be that $v^Y_{p+2} + v^Y_{p+1} \neq v^Y_{p+3} + v^Y_{p+2} = v^Y_p + v^Y_{p+1}$; then neither X nor Y ties $X(p \mapsto p+1, p \mapsto p+1, p+3 \mapsto p+1)$. If $q = p+5$ then $X = (p, p, p+2, p+3, p+4)$. X does not tie $X(p \mapsto p+1, p+2 \mapsto p+1)$ or $X(p+3 \mapsto p+4, p+2 \mapsto p+1)$ or $X(p+4 \mapsto p+5, p+3 \mapsto p+2)$, so the theorem is satisfied unless Y ties all three of these dice; consequently we may presume $v^Y_p + v^Y_{p+1} = v^Y_{p+2} + v^Y_{p+1} = v^Y_{p+3} + v^Y_{p+4}$ and $v^Y_{p+4} + v^Y_{p+5} = v^Y_{p+2} + v^Y_{p+3}$. As Y is not balanced, it must be that $v^Y_p + v^Y_{p+1} = v^Y_{p+2} + v^Y_{p+1} \neq v^Y_{p+2} + v^Y_{p+3}$; then neither X nor Y ties $X(p \mapsto p+1, p \mapsto p+1, p+3 \mapsto p+1)$. If $q > p+5$ then $X = (p, p, p+2, q-2, q-1)$ and $p + 2 < q - 3$. X does not tie $X(p \mapsto p+1, q-2 \mapsto q-3)$, so the theorem is satisfied unless Y does; hence we may presume $v^Y_p + v^Y_{p+1} = v^Y_{q-3} + v^Y_{q-2}$. By Lemma 2.2 there are $i, j \in \{p, ..., q-1\}$ such that $i + j \in \{p+q-1, p+q-2\}$ and $v^Y_i + v^Y_{i+1} \neq v^Y_j + v^Y_{j+1}$. Then $s - (i + j + 1 + p + q - 2) = 3p + 2q - 1 - (i + j + p + q - 1) = 2p + q - (i+j) \in \{p+1, p+2\}$, so there is a die $Z \in D(n, a, b, s)$ with one label equal to i, another label equal to $j+1$, another label equal to p, another label equal to $q-2$, and the fifth label equal to $p+1$ or $p+2$. If Z ties neither X nor Y, the theorem is satisfied. If Z ties X but not Y then $Z(p \mapsto p+1, q-2 \mapsto q-3)$ ties neither, for $v^X_p + v^X_{p+1} \neq v^X_{q-3} + v^X_{q-2}$ and $v^Y_p + v^Y_{p+1} = v^Y_{q-3} + v^Y_{q-2}$. If Z ties Y then $Z' = Z(j+1 \mapsto j, i \mapsto i+1)$ doesn't; $Z'' = Z(p \mapsto p+1, q-2 \mapsto q-3)$ has $f_X(Z'') \neq f_X(Z')$ and $f_Y(Z'') = f_Y(Z')$, so at least one of Z', Z'' ties neither X nor Y.

Suppose now that $x_4 < q-2$; recall $X = (p, p, x_3, p+q-x_3, q-1)$. If $v_p^Y + v_{p+1}^Y \neq v_{q-2}^Y + v_{q-1}^Y$ then $X(p \mapsto p+1, q-1 \mapsto q-2)$ ties neither X nor Y; hence we may presume $v_p^Y + v_{p+1}^Y = v_{q-2}^Y + v_{q-1}^Y$. By Lemma 2.2 there are $i, j \in \{p, ..., q-1\}$ such that $i + j \in \{p+q-1, p+q-2\}$ and $v_i^Y + v_{i+1}^Y \neq v_j^Y + v_{j+1}^Y$. Then $s - (i+j+1+p+q-1) = 3p+2q-1-(i+j+p+q) = 2p+q-1-(i+j) \in \{p+1, p\}$, so there is a die $Z \in D(n, a, b, s)$ with one label equal to i, another label equal to $j+1$, another label equal to p, another label equal to $q-2$, and the fifth label equal to $p+1$ or p. As in the closing sentences of the preceding paragraph, at least one of Z, $Z(p \mapsto p+1, q-1 \mapsto q-2)$, $Z(j+1 \mapsto j, i \mapsto i+1)$, $Z(j+1 \mapsto j, i \mapsto i+1, p \mapsto p+1, q-1 \mapsto q-2)$ ties neither X nor Y.

5. $n = 4$

Our proof of The Tied Dice Theorem for 4-sided dice is much longer than the arguments above. Notice that the proof of Proposition 2.3 involves manipulating four labels (one each of i, j, u and w) to produce an element $Z \in D(n, a, b, s)$ which ties neither X nor Y. The other $n-4$ labels of Z are important only in that they give $\sum z_k = s$. When $n = 4$, however, we cannot freely choose four labels on Z – the fourth label is determined by s and the first three. For this reason we have not found anything better than an extremely ungainly argument when $n = 4$ and $q - p > 2$; our proof is simply an exhaustive list of the various ways X and Y may be related, with verification that an appropriate Z exists in every case. We would be glad to provide an interested reader with a full account.

6. $n = 3$

Let X and Y be distinct, tied, non-balanced elements of $D(3, a, b, s)$.

Suppose that $s = 3k+1$, where $k \in \mathbb{Z}$, and let $A = (a_1, a_2, a_3) \in D(3, a, b, s)$. If A ties $(k, k, k+1)$ there must be an odd number of indices i with $a_i = k+1$. This odd number cannot be 3, as $s \neq 3k+3$; consequently $A = (k-a, k+1, k+a)$ for some a. If $a \neq 0$ then $(k-a, k+1, k+a)$ does not tie $(k, k, k+1)$, so we conclude that the only element of $D(3, a, b, s)$ which ties $(k, k, k+1)$ is $(k, k, k+1)$ itself. Consequently neither X nor Y is $(k, k, k+1)$, and the theorem is satisfied by $Z = (k, k, k+1)$. A similar argument applies if $s = 3k-1$.

Suppose now that $s = 3k$. If $p \geq k-1$ and $q \leq k+1$ then $D(3, a, b, s)$ actually contains no non-balanced dice. If $p = k-1$ and $q > k+1$ then $q = k+2$ and $(k-1, k-1, k+2)$ doesn't tie either of the other elements of $D(3, a, b, s)$; similarly, if $q = k + 1$ and $p < k - 1$ then $p = k - 2$ and $(k - 2, k + 1, k + 1)$ doesn't tie any other element of $D(3, a, b, s)$.

Suppose $p \leq k - 2$ and $q \geq k + 2$. If (k, k, k) ties neither X nor Y the theorem is satisfied, so we may presume that at least one of X, Y ties (k, k, k); necessarily then k appears as a label on at least one of X, Y. Similarly, we may presume that at least one of X, Y ties $(k-1, k-1, k+2)$ and hence has $k + 2$ as one of its labels, and at least one of X, Y ties $(k - 2, k + 1, k + 1)$ and hence has $k - 2$ as one of its labels. Consequently each of $k - 2, k, k + 2$ appears on at least one of X, Y, so at least one of X, Y must involve two of these labels and hence must equal $(k-2, k, k+2)$; say $X = (k - 2, k, k + 2)$. As X is not balanced, $p \leq k - 4$ or $q \geq k + 4$.

In order to tie, X and Y must share an odd number of labels. They cannot share all three labels of X because $X \neq Y$, so they share precisely one label of X; suppose the one label they share is $k - 2$. If $Y = (k - 2, k - 1, k + 3)$ or $Y = (k - 2, k + 1, k + 1)$ then $Z = (k - 2, k - 2, k + 4)$ or $Z = (k-4, k+2, k+2)$ satisfies the theorem. The other dice on which $k-2$ appears exactly once are of the form $(k-a, k-2, k+a+2)$ for some $a \geq 2$; Y cannot be of this form because X does not tie such a die. Similarly, if $k + 2$ is the one label shared by X and Y then $Y = (k - 3, k + 1, k + 2)$ or $Y = (k-1, k-1, k+2)$, and the theorem is satisfied by $Z = (k-4, k+2, k+2)$ or $Z = (k - 2, k - 2, k + 4)$.

Suppose the one label of X shared by Y is k; then $Y = (k - a, k, k + a)$ for some $a \geq 0$, $a \neq 2$. If $p \leq k - 4$ then the theorem is satisfied by $Z = (k - 4, k + 2, k + 2)$ unless Y ties this die. In order to tie it, Y would have to share exactly one of its labels; hence $k - 4$ would have to appear on Y. Similarly, if $q \geq k + 4$ then the theorem is satisfied by $Z = (k - 2, k - 2, k + 4)$ unless Y ties this die, and in order to tie it, Y would have to share exactly one of its labels; hence $k + 4$ would have to appear on Y. We conclude that if the theorem is not satisfied then $k - 4$ or $k + 4$ must appear on Y; as the label k is shared by X and Y, it follows in either case that $Y = (k - 4, k, k + 4)$. As both $(k - 3, k - 2, k + 5)$ and $(k-5, k+2, k+3)$ tie neither $X = (k-2, k, k+2)$ nor $Y = (k-4, k, k+4)$, it follows that if there is no $Z \in D(3, a, b, s)$ which ties neither X nor Y then $p = k - 4$ and $q = k + 4$.

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