Tied dice ($n = 4$ addendum)

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Abstract

We present a very tedious proof of the $n = 4$ case of the Tied Dice Theorem.

Suppose $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in D(4, a, b, s)$ are distinct, tied, non-balanced dice. Let τ denote the number of pairs (i, j) with $x_i = y_j$; as X and Y are tied, there must be $(16 - \tau)/2$ pairs (x_i, y_j) with $x_i > y_j$, and $(16 - \tau)/2$ pairs (x_i, y_j) with $x_i < y_j$. For each value of τ we list the various possible arrangements of the labels of X and Y by considering the *winning configuration* $c_1c_2c_3c_4$ of X, where $c_i = |\{j \mid x_i > y_j\}|$; the winning configuration is a nondecreasing sequence of non-negative integers whose sum is $(16 - \tau)/2$.

We hope that the layout of the addendum is clear, with sub-cases underlined and sub-sub-cases underlined and indented. In each case we provide an example of a $Z \in D(4, a, b, s)$ which ties neither X nor Y, and we briefly explain the nonzero values of $f_X(Z)$ and $f_Y(Z)$. For instance the first sub-case considered below includes the statement "$Z = (x_2, y_2 + 1, y_3, y_4 + y_1 - x_2 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1$ or $2)$ and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$)." Recalling that $f_X(X) = f_X(Y) = 0 = f_Y(X) = f_Y(Y)$ because X and Y are tied, the quoted statement indicates that if we think of Z as having been obtained from Y by reducing y_1 to x_2, increasing y_2 to $y_2 + 1$ and increasing y_4 to $y_4 + y_1 - x_2 - 1$, then we can see that $f_X(Z) \neq 0 \neq f_Y(Z)$ because reducing y_1 contributes -1 or -2 to $f_X(Z)$ and contributes -1 to $f_Y(Z)$, while increasing y_2 and y_4 contributes at least 2 to $f_Y(Z)$ and does not affect $f_X(Z)$.

1. $\tau = 0$

Suppose $\tau = 0$; we may presume that $x_4 > y_4$. Then there are only four possible arrangements of the x_i and y_j: $x_1 \leq x_2 < y_1 \leq y_2 \leq y_3 \leq y_4 < x_3 \leq x_4$ with winning configuration 0044, $x_1 < y_1 < x_2 < y_2 \leq y_3 < x_3 < y_4 < x_4$ with winning configuration 0134, $x_1 < y_1 \leq y_2 < x_2 \leq x_3 < y_3 \leq y_4 < x_4$ with winning configuration 0224, and $y_1 < x_1 \leq x_2 < y_2 < x_3 < y_3 \leq y_4 < x_4$ with winning configuration 1124.

1.1. $x_1 \leq x_2 < y_1 \leq y_2 \leq y_3 \leq y_4 < x_3 \leq x_4$

<u>$2 \leq y_1 - x_2 \leq x_3 - y_4$</u> If $2 \leq y_1 - x_2 \leq x_3 - y_4$ and $y_1 < y_2$ then $Z = (x_2, y_2 + 1, y_3, y_4 + y_1 - x_2 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1$ or $2)$ and $f_Y(Z) - f_Y(Y) =$ (at least $2) - 1$). If $2 \leq y_1 - x_2 \leq x_3 - y_4$ and $y_1 = y_2$ then again $Z = (x_2, y_2 + 1, y_3, y_4 + y_1 - x_2 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1$ or $2)$ and $f_Y(Z) - f_Y(Y) =$ (at least $m + 1) - m$, where $m = |\{i \,|\, y_i = y_1\}|$).
<u>$2 \leq x_3 - y_4 \leq y_1 - x_2$</u> The preceding paragraph may be applied to $-X = (-x_4, -x_3, -x_2, -x_1)$ and $-Y = (-y_4, -y_3, -y_2, -y_1)$ to find a $Z \in D(4, -b, -a, -s)$ which ties neither $-X$ nor $-Y$. Then $-Z$ satisfies the theorem.
<u>$1 = y_1 - x_2 < x_3 - y_4$</u> If $x_1 < x_2$ then $Z = (x_2 - 1, y_2 + 1, y_3 + 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (2$ or $3)$ and $f_Y(Z) - f_Y(Y) =$ (at least $m + 1) - m$, where $m = |\{i \,|\, y_i = y_1\}|$). If $x_1 = x_2$ and $y_1 = y_2$ then $Z = (y_1, y_1, x_3 - 2, x_4)$ does not tie X or Y ($f_X(Z) - f_X(X) = 4 - (1$ or $2)$, $f_Y(Z) - f_Y(X) =$ (at least 4) $-$ (at most 2) if $y_2 < y_4$ and $f_Y(Z) - f_Y(X) = 8 - 4$ if $y_2 = y_4$). If $x_1 = x_2$ and $y_3 < x_3 - 2$ then again $Z = (y_1, y_1, x_3 - 2, x_4)$ does not tie X or Y ($f_X(Z) - f_X(X) = 4 - 1$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $-$ (at most 1)). If $x_1 = x_2$, $y_1 < y_2$ and $y_3 = y_4 = x_3 - 2$ then $Z = (x_1, y_2, y_3, y_4 + 1)$ does not tie X or Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$).

$\underline{1 = x_3 - y_4 < y_1 - x_2}$ Apply the preceding paragraph to $-X$ and $-Y$.
$\underline{y_1 - x_2 = 1 = x_3 - y_4,\ x_1 < x_2 \text{ and } x_3 = x_4}$ If $x_1 \leq x_2 - 2$ then $Z = (x_1 + 1, x_2, y_4, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 -$ (at least 1)). If $x_1 = x_2 - 1$ then either $y_1 = y_2 < y_3 < y_4$ is true, in which case $Z = (x_1, x_2 + 1, y_4, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 2 - 1$), or $y_1 = y_2 < y_3 < y_4$ is false, in which case $Z = (x_2, x_2 + 1, y_4, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 3 - 4$ and $f_Y(Z) - f_Y(X) = |\{i \,|\, y_i = y_1\}| - 2 \cdot |\{i \,|\, y_i = y_4\}|$).
$\underline{y_1 - x_2 = 1 = x_3 - y_4,\ x_1 = x_2 \text{ and } x_3 < x_4}$ Apply the preceding paragraph to $-X$ and $-Y$.
$\underline{y_1 - x_2 = 1 = x_3 - y_4,\ x_1 < x_2 \text{ and } x_3 < x_4}$ If $x_1 < x_2 - 2$ then $Z = (x_1 + 2, x_2, x_3 - 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 -$ (at least 2) and $f_Y(Z) - f_Y(X) = 0 -$ (at least 1)). If $x_1 = x_2 - 1$ then $Z = (x_1 + 1, x_2, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 0 -$ (at least 1)). If $x_4 > x_3 + 2$ then $Z = (x_1 + 1, x_2 + 1, x_3, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) =$ (at least 1) $- 0$). If $x_4 = x_3 + 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) =$ (at least 1) $- 0$). If $x_1 = x_2 - 2$ and $x_4 = x_3 + 2 > x_2 + 4$ then $Z = (x_1, x_2 + 2, x_3, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $- 0$). If $x_1 = x_2 - 2$, $x_4 = x_3 + 2 = x_2 + 4$ and $p < x_1 - 1$ then $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 8 - 0$). If $x_1 = x_2 - 2$, $x_4 = x_3 + 2 = x_2 + 4$ and $q > x_4 + 1$ then $Z = (x_1, x_2, x_3 - 2, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 8$). X is not balanced, so it cannot be that $x_1 = x_2 - 2$, $x_4 = x_3 + 2 = x_2 + 4$, $p \geq x_1 - 1$ and $q \leq x_4 + 1$.
$\underline{y_1 - x_2 = 1 = x_3 - y_4,\ x_1 = x_2 \text{ and } x_3 = x_4}$

$\underline{y_1 < y_2 < y_3}$ If $y_3 \geq y_4 - 1$ then $Z = (x_1 + 1, x_2 + 1, x_3 - 2, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) = 2 -$ (at least 3)). If $y_3 < y_4 - 2$ then $Z = (x_1 + 1, x_2 + 2, x_3 - 3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) =$ (at least 3) $- 2$). If $y_3 = y_4 - 2$ and $y_1 < y_2 - 2$ then $Z = (x_1, x_2 + 3, x_3 - 2, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 4$ and $f_Y(Z) - f_Y(X) = 2 - 3$). If $y_3 = y_4 - 2$ and $y_1 = y_2 - 1$ then $Z = (y_1 - 1, y_1, y_4, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 3$). If $y_3 = y_4 - 2$, $y_1 = y_2 - 2$ and $y_2 < y_3 - 2$ then $Z = (y_1 - 1, y_2 + 2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_3 = y_4 - 2$, $y_1 = y_2 - 2$ and $y_2 = y_3 - 1$ then $Z = (y_1, y_2, y_3 - 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_3 = y_4 - 2$, $y_1 = y_2 - 2$, $y_2 = y_3 - 2$ and $q > y_4 + 1$ then $Z = (y_1, y_2 - 2, y_3, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 4 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_3 = y_4 - 2$, $y_1 = y_2 - 2$, $y_2 = y_3 - 2$ and $p < y_1 - 1$ then $Z = (y_1 - 2, y_2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 4$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). It cannot be that $y_3 = y_4 - 2$, $y_1 = y_2 - 2$, $y_2 = y_3 - 2$, $q \leq y_4 + 1$ and $p \geq y_1 - 1$, because Y is not balanced.

$\underline{y_1 < y_2 = y_3}$ If $y_3 < y_4$ then $Z = (y_1 - 1, y_2, y_3 + 1, y_4)$ satisfies the theorem ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) =$ (2 or 3) $- 1$). If $y_3 = y_4$ then $Z = (x_1 + 1, x_2 + 1, x_3 - 2, x_4)$ satisfies the theorem ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) = 2 -$ (6 or 7)).

$\underline{y_1 = y_2 < y_3}$ If $y_3 = y_2 + 1$ then $Z = (y_1 - 1, y_2 + 1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) =$ (3 or 4) $- 2$). If $y_3 > y_2 + 2$ then $Z = (y_1 - 1, y_2 + 2, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 -$ (at least 3)). If $y_3 = y_2 + 2 < y_4$ then $Z = (y_1 - 1, y_2 - 1, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (0 or 2) $- 4$ and $f_Y(Z) - f_Y(Y) =$ (at most 3) $- 4$); note that if $y_3 = y_4 - 1$ then Z is actually $(y_1 - 1, y_2 - 1, y_4, y_3 + 2)$. If $y_3 = y_2 + 2 = y_4$ and $q > y_4 + 1$ then $Z = (y_1 - 1, y_2, y_3 - 1, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 4 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 4$). If $y_3 = y_2 + 2 = y_4$ and $p < y_1 - 1$ then $Z = (y_1 - 2, y_2 + 1, y_3, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 4$ and $f_Y(Z) - f_Y(Y) = 4 - 2$). It cannot be that $y_3 = y_2 + 2 = y_4$, $p \geq y_1 - 1$ and $q \leq y_4 + 1$, because Y is not balanced.

$\underline{y_1 = y_3 < y_4}$ $Z = (y_1 - 1, y_2 - 1, y_3 + 1, y_4 + 1)$ satisfies the theorem ($f_X(Z) - f_X(Y) = 2 - 4$ and $f_Y(Z) - f_Y(Y) = 4 - 6$).

$\underline{y_1 = y_4}$ If $q > x_4 + 1$ then $Z = (x_1, x_2, x_3 - 2, x_4 + 2)$ satisfies the theorem ($f_X(Z) - f_X(X) = 2 - 4$ and $f_Y(Z) - f_Y(X) = 0 - 8$). If $p < x_1 - 1$ then $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) = 8 - 0$). As X is not balanced, it cannot be that $p \geq x_1 - 1$ and $q \leq x_4 + 1$.

1.2. $x_1 < y_1 < x_2 < y_2 \leq y_3 < x_3 < y_4 < x_4$

$\underline{y_1 - x_1 \geq x_4 - y_4 \geq 2}$ If $y_3 > x_2 + 1$ then $Z = (y_1 - (x_4 - y_4) + 1, y_2, y_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = 1 -$ (2 or 3)). If $y_3 = x_2 + 1$ then necessarily $y_2 = y_3$, and

$Z = (y_1, x_2, x_2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 4$).

$x_4 - y_4 \geq y_1 - x_1 \geq 2$ Apply the preceding paragraph to $-X$ and $-Y$.

$x_4 - y_4 > y_1 - x_1 = 1$ If $x_4 - y_4 > 2$ then $Z = (y_1 - 1, y_2 - 1, y_3, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1 \text{ or } 2)$ and $f_Y(Z) - f_Y(Y) = 1 - (\text{at least } 2)$). If $x_4 - y_4 = 2$ and $x_2 - y_1 = 1$ then $Z = (x_1 + 2, x_2, x_3, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 2 - 1$), and if $x_4 - y_4 = 2 \leq x_2 - y_1$ then $Z = (x_1 + 2, x_2 - 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 2 - 0$).

$y_1 - x_1 > x_4 - y_4 = 1$ The preceding paragraph may be applied to $-X$ and $-Y$.

$x_4 - y_4 = y_1 - x_1 = 1$ and $y_4 - x_3 \neq x_3 - y_3$ If $y_4 - x_3 > x_3 - y_3$ then $Z = (y_1, y_2 - 1, x_3 + 1, y_4 - (x_3 - y_3))$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - (0 \text{ or } 1)$ and $f_Y(Z) - f_Y(Y) = \delta - (1 + \delta)$, where $\delta = 2$ or 1 according to whether or not $y_2 = y_3$). If $y_4 - x_3 < x_3 - y_3$ and $y_3 > y_2$ then $Z = (y_1 - 1, y_2, y_3 + y_4 - x_3 + 1, x_3)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$), and if $y_4 - x_3 < x_3 - y_3$ and $y_3 = y_2$ then $Z = (y_1, y_2, y_3 + y_4 - x_3, x_3)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$).

$x_4 - y_4 = 1$, $y_1 - x_1 = 1$ and $y_2 - x_2 \neq x_2 - y_1$ Apply the same arguments to $-X$ and $-Y$.

$x_4 - y_4 = y_1 - x_1 = 1$, $y_4 - x_3 = x_3 - y_3$ and $y_2 - x_2 = x_2 - y_1$

Note that $x_1 + x_4 = y_1 + y_4$, so $x_2 + x_3 = y_2 + y_3$, so $x_3 - y_3 = y_2 - x_2$.

$x_3 - y_3 = y_2 - x_2 > 2$ $Z = (y_1, y_2 - 2, y_3 + 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = (1 + \delta) - \delta$, where $\delta = 2$ or 1 according to whether or not $y_2 = y_3$).

$x_3 - y_3 = y_2 - x_2 = 2$ $Z = (x_1 + 2, x_2 - 1, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 2 - 0$).

$x_3 - y_3 = y_2 - x_2 = 1$ $X = (y_2 - 3, y_2 - 1, y_3 + 1, y_3 + 3)$ and $Y = (y_2 - 2, y_2, y_3, y_3 + 2)$. If $y_2 = y_3$ then the fact that X is not balanced implies that either $p < y_2 - 4$ or $q > y_3 + 4$; if $p < y_2 - 4$ then $Z = (y_2 - 5, y_2, y_3 + 2, y_3 + 3)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 3 - 2$ and $f_Y(Z) - f_Y(Y) = 4 - 1$), and if $q > y_3 + 4$ then $Z = (y_2 - 3, y_2 - 2, y_3, y_3 + 5)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 3$ and $f_Y(Z) - f_Y(Y) = 1 - 4$). If $y_2 = y_3 - 1$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 = y_3 - 2$ then the fact that Y is not balanced implies that either $p < y_2 - 3$ or $q > y_3 + 3$; if $p < y_2 - 3$ then $Z = (y_2 - 4, y_3, y_3, y_3 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$), and if $q > y_3 + 3$ then $Z = (y_2 - 2, y_2, y_2, y_3 + 4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). Finally, if $y_2 < y_3 - 2$ then $Z = (y_2 - 3, y_2 + 2, y_3, y_3 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$).

1.3. $x_1 < y_1 \leq y_2 < x_2 \leq x_3 < y_3 \leq y_4 < x_4$

$y_3 - x_3 \geq 2$ and $x_2 - y_2 \geq 2$

$y_1 - x_1 \geq 2$ If $x_2 - y_2 \geq y_1 - x_1$ then $Z = (y_1, x_2 - (y_1 - x_1) + 1, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - (\text{at least } 2)$ and $f_Y(Z) - f_Y(X) = (1 \text{ or } 2) - 0$). If $y_1 - x_1 \geq x_2 - y_2 \geq y_3 - x_3$ then $Z = (x_1 + x_2 - y_2 - 1, y_2, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = (1 + \delta) - \delta$, where $\delta = 2$ or 1 according to whether or not $x_2 = x_3$, and $f_Y(Z) - f_Y(X) = 0 - (1 \text{ or } 2)$). If $x_2 - y_2 < y_3 - x_3$ and $x_2 - y_2 < y_1 - x_1$ then $Z = (x_1 + 1, y_2, x_3 - y_2 + x_2 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = (1 + \delta) - \delta$, where $\delta = 2$ or 1 according to whether or not $x_2 = x_3$, and $f_Y(Z) - f_Y(X) = 0 - (1 \text{ or } 2)$).

$x_4 - y_4 \geq 2$ Apply the preceding paragraph to $-X$ and $-Y$.

$x_4 - y_4 = 1 = y_1 - x_1$ and $y_3 - x_3 \geq x_2 - y_2$ If both $y_1 = y_2$ and $y_3 < y_4 - 1$ are true then $Z = (x_1, y_2, y_3 + 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). Otherwise $Z = (y_1, x_2, y_3 - (x_2 - y_2) + 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (1 \text{ or } 2) - 0$, $f_Y(Z) - f_Y(Y) = 1 - (\text{at least } 2)$ if $y_1 < y_2$, and $f_Y(Z) - f_Y(Y) = 2 - (\text{at least } 3)$ if $y_1 = y_2$ and $y_3 \geq y_4 - 1$).

$x_4 - y_4 = 1 = y_1 - x_1$ and $y_3 - x_3 \leq x_2 - y_2$ Apply the preceding paragraph to $-X$ and $-Y$.

$x_2 - y_2 = 1 < y_3 - x_3$

$x_4 - y_4 \geq y_3 - x_3$ If $y_1 = y_2$ and $y_3 = y_4$ then $Z = (y_1, y_2 + 2, y_3 - 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (\text{at least } 2) - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 4$). If $y_1 < y_2$ and $y_3 = y_4$ then $Z = (y_1, y_2 + 1, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (\text{at least } 1) - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$); the same Z satisfies the theorem if $y_1 = y_2$ and $y_3 < y_4$ ($f_X(Z) - f_X(Y) = (\text{at least } 1) - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_1 < y_2$ and $y_3 + 1 = y_4$ then $Z = (y_1, x_2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (\text{at least } 1) - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $2 \leq y_4 - y_3 \leq y_2 - y_1$ then $Z = (y_1 + y_4 - y_3 - 1, y_2 + 2, y_3 - 1, y_3)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (\text{at least } 2) - 0$ and

$f_Y(Z)-f_Y(Y)=2-3$). If $y_4-y_3>y_2-y_1>0$ then $Z=(y_2,x_2,y_3,y_4-(y_2-y_1)-1)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=$ (at least 1) $-\,0$ and $f_Y(Z)-f_Y(Y)=3-$ (1 or 2)).

$\underline{x_4-y_4<y_3-x_3}$ $Z=(y_1,x_2,y_3-(x_4-y_4)-1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=(\delta+1)-$ (0 or δ), where $\delta=2$ or 1 according to whether or not $x_2=x_3$, and $f_Y(Z)-f_Y(Y)=$ (at least $\varepsilon+1$) $-\,\varepsilon$, where $\varepsilon=2$ or 1 according to whether or not $y_3=y_4$).

$\underline{x_2-y_2>1=y_3-x_3}$ Apply the preceding arguments to $-X$ and $-Y$.

$\underline{x_2-y_2=1=y_3-x_3 \text{ and } y_1-x_1\geq 2}$ If $y_1-x_1>2$ then $Z=(x_1+2,x_2-1,x_3-1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=1-$ (at least 2) and $f_Y(Z)-f_Y(X)=0-$ (at least 1)). If $y_1-x_1=2$ and $y_3<y_4$ then $Z=(x_1+1,x_2-2,x_3+1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=(1+\delta)-\delta$, where $\delta=2$ or 1 according to whether or not $x_2=x_3$, and $f_Y(Z)-f_Y(X)=$ (at most 1) $-$ (at least 2)). If $y_1-x_1=2$ and $y_1=y_2$ then again $Z=(x_1+1,x_2-2,x_3+1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=(1+\delta)-\delta$, where $\delta=2$ or 1 according to whether or not $x_2=x_3$, and $f_Y(Z)-f_Y(X)=$ (at most 2) $-$ (at least 4)). If $y_1-x_1=2$, $y_3=y_4$ and $y_1<y_2$ then $Z=(x_1,x_2-1,x_3+2,x_4-1)$ ties neither X nor Y ($f_X(Z)-f_X(X)=\delta-(1+\delta)$, where $\delta=2$ or 1 according to whether or not $x_2=x_3$, and $f_Y(Z)-f_Y(X)=4-$ (1 or 3)).

$\underline{x_2-y_2=1=y_3-x_3 \text{ and } x_4-y_4\geq 2}$ Apply the preceding paragraph to $-X$ and $-Y$.

$\underline{x_2-y_2=1=y_3-x_3 \text{ and } y_1-x_1=1=x_4-y_4}$

$\underline{x_3-x_2>2}$ $Z=(x_1+1,x_2+1,x_3-2,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$ and $f_Y(Z)-f_Y(X)=$ (1 or 2) $-\,0$).

$\underline{x_3-x_2=2}$ If $x_3<x_4-2$, $Z=(x_1,x_2+2,x_3,x_4-2)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$ and $f_Y(Z)-f_Y(X)=0-$ (at least 2)). If $x_2>x_1+2$, $Z=(x_1+2,x_2,x_3-2,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=1-2$ and $f_Y(Z)-f_Y(X)=$ (at least 2)-0). If $x_3=x_4-2$, $x_1=x_2-2$ and $p<x_1-1$ then $Z=(x_1-2,x_2+1,x_3+1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$ and $f_Y(Z)-f_Y(X)=2-0$). If $x_3=x_4-2$, $x_1=x_2-2$ and $q>x_4+1$ then $Z=(x_1,x_2-1,x_3-1,x_4+2)$ ties neither X nor Y ($f_X(Z)-f_X(X)=1-2$ and $f_Y(Z)-f_Y(X)=0-2$). If $x_3=x_4-2$, $x_1=x_2-2$, $p\geq x_1-1$ and $q\leq x_4+1$ then X is balanced, contrary to hypothesis.

$\underline{x_3-x_2=1}$ $Z=(x_1,x_2+1,x_3,x_4-1)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$ and $f_Y(Z)-f_Y(X)=0-$ (1 or 2)).

$\underline{x_3=x_2 \text{ and } y_2-y_1\notin\{0,2\}}$ If $y_1<y_2-2$, then $Z=(y_1+2,y_2,y_3-1,y_4-1)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=0-$ (at least 2) and $f_Y(Z)-f_Y(Y)=1-$ (at least 2)). If $y_1=y_2-1$ and $y_3<y_4$ then $Z=(y_1+1,y_2,y_3-1,y_4)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=0-2$ and $f_Y(Z)-f_Y(Y)=2-1$). If $y_1=y_2-1$ and $y_3=y_4$ then $Z=(y_1+1,y_2+1,y_3-1,y_4-1)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=1-4$ and $f_Y(Z)-f_Y(Y)=3-4$).

$\underline{x_3=x_2 \text{ and } y_4-y_3\notin\{0,2\}}$ Apply the preceding paragraph to $-X$ and $-Y$.

$\underline{x_3=x_2,\ y_4-y_3\in\{0,2\} \text{ and } y_2-y_1=2}$ If $y_3=y_4$ then $Z=(y_1+1,y_2,y_3-1,y_4)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=0-2$ and $f_Y(Z)-f_Y(Y)=1-2$). If $y_3=y_4-2$ and $p<x_1$ then $Z=(x_1-1,x_2,x_3+1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$) and $f_Y(Z)-f_Y(X)=1-0$); if $y_3=y_4-2$ and $q>x_4$ then $Z=(x_1,x_2-1,x_3,x_4+1)$ ties neither X nor Y ($f_X(Z)-f_X(X)=1-2$ and $f_Y(Z)-f_Y(X)=0-1$). As Y is not balanced, it cannot be that $y_3=y_4-2$, $p\geq x_1$ and $q\leq x_4$.

$\underline{x_3=x_2,\ y_2-y_1\in\{0,2\} \text{ and } y_4-y_3=2}$ Apply the preceding paragraph to $-X$ and $-Y$.

$\underline{x_3=x_2,\ y_2=y_1 \text{ and } y_4=y_3}$ The fact that Y is not balanced implies that $p<x_1$ or $q>x_4$. If $p<x_1$ then $Z=(x_1-1,x_2,x_3+1,x_4)$ ties neither X nor Y ($f_X(Z)-f_X(X)=2-1$ and $f_Y(Z)-f_Y(X)=2-0$). If $q>x_4$ then $Z=(x_1,x_2-1,x_3,x_4+1)$ ties neither X nor Y ($f_X(Z)-f_X(X)=1-2$ and $f_Y(Z)-f_Y(X)=0-2$).

1.4. $y_1<x_1\leq x_2<y_2<x_3<y_3\leq y_4<x_4$

$\underline{x_4-y_4\geq 2}$

$\underline{y_3-x_3=1}$ If $x_3-y_2=1$ then $Z=(y_1,y_2,y_2,y_4+2)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=$ (at most 1) $-\,2$ and $f_Y(Z)-f_Y(Y)=\delta-(1+\delta)$, where $\delta=2$ or 1 according to whether or not $y_4=y_3$). If $x_3-y_2>1$ then $Z=(y_1,y_2+1,x_3-1,y_4+1)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=0-2$ and $f_Y(Z)-f_Y(Y)=(1+\delta)-\delta$, where $\delta=2$ or 1 according to whether or not $y_4=y_3$).

$\underline{2\leq y_3-x_3\leq x_4-y_4}$ If $x_3-y_2>1$ then $Z=(y_1,y_2+1,x_3,y_4+y_3-x_3-1)$ ties neither X nor Y ($f_X(Z)-f_X(Y)=0-1$ and $f_Y(Z)-f_Y(Y)=(1+\delta)-\delta$, where $\delta=2$ or 1 according to whether or not $y_4=y_3$). If $x_3-y_2=1$ and $y_4-y_3\leq 2$ then $Z=(y_1,y_2+2,y_3,y_4-2)$ ties neither X nor Y

$(f_X(Z)-f_X(Y) = 2-$(at most 1) and $f_Y(Z)-f_Y(Y) = 1-$(at least 2)). If $x_3-y_2 = 1$ and $y_4-y_3 > 2$ then $Z = (y_1, y_2+1, y_3+1, y_4-2)$ ties neither X nor Y $(f_X(Z) - f_X(Y) = 1-0$ and $f_Y(Z) - f_Y(Y) = 2-1)$.

$y_3 - x_3 = 1 + x_4 - y_4$ If $x_3-y_2 = 1$ then $Z = (y_1, x_3, x_3, x_4)$ ties neither X nor Y $(f_X(Z)-f_X(Y) = 2-1$ and $f_Y(Z) - f_Y(Y) = (1+\delta)-\delta$, where $\delta = 2$ or 1 according to whether or not $y_4 = y_3$). If $x_3-y_2 = 2$ then $Z = (y_1, y_2+2, y_3-1, y_4-1)$ ties neither X nor Y $(f_X(Z) - f_X(Y) = 1-0$ and $f_Y(Z) - f_Y(Y) = 1-$ (at least 2)). If $x_3 - y_2 > 2$ then $Z = (y_1, y_2+2, x_3, x_4-1)$ ties neither X nor Y $(f_X(Z) - f_X(Y) = 0-1$ and $f_Y(Z) - f_Y(Y) = (1+\delta)-\delta$, where $\delta = 2$ or 1 according to whether or not $y_4 = y_3$).

$y_3 - x_3 > 1 + x_4 - y_4$ $Z = (y_1+1, y_2, y_3-(x_4-y_4)-1, x_4)$ ties neither X nor Y $(f_X(Z) - f_X(Y) =$ (at least 1) -0 and $f_Y(Z) - f_Y(Y) = (1+\delta)-\delta$, where $\delta = 2$ or 1 according to whether or not $y_4 = y_3$).

$x_1 - y_1 \geq 2$ Apply the preceding argument to $-Y$ in place of X and $-X$ in place of Y.

$x_1 - y_1 = 1 = x_4 - y_4$

$0 \neq x_2 - x_1 \neq 2$ If $x_2 = x_1+1$ then $Z = (x_2, x_2, x_3, y_4)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 0-$ (at least 1)). If $x_2 > x_1+2$ then $Z = (x_1+2, x_2, x_3-1, x_4-1)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 1-2$ and $f_Y(Z) - f_Y(X) = 0-$ (at least 1)).

$0 \neq y_4 - y_3 \neq 2$ Apply the preceding paragraph to $-Y$ in place of X and $-X$ in place of Y.

$0 = x_2 - x_1 = y_4 - y_3$ $Z = (x_1, x_2+1, x_3, x_4-1)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) =$ (at most 1) -2).

$2 = x_2 - x_1 = y_4 - y_3$ $Z = (x_2, x_2, x_3, x_4-2)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 0-2)$.

$0 = x_2 - x_1$ and $2 = y_4 - y_3$ If $y_2 > x_2+1$ then $Z = (x_1, x_2+1, x_3, y_4)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 0-1)$. If $y_3 > x_3+1$ then $Z = (y_1, x_2, x_3+1, x_4)$ ties neither X nor Y $(f_X(Z)-f_X(X) = 1-2$ and $f_Y(Z)-f_Y(X) = 0-1)$. If $y_2 = x_2+1 < x_3-1$ then $Z = (x_1, x_2+1, x_3-1, x_4)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 1-0)$. If $y_2 = x_2+1 = x_3-1$ and $y_3 = x_3+1$ then the fact that Y is not balanced implies that $p < y_1 - 1$ or $q > x_4$. If $p < y_1 - 1$ then $Z = (y_1-2, y_2+1, y_3, y_4+1)$ ties neither X nor Y $(f_X(Z) - f_X(Y) = 2-0$ and $f_Y(Z) - f_Y(Y) = 2-1)$, and if $q > x_4$ then $Z = (x_1-1, x_2, x_3, x_4+1)$ ties neither X nor Y $(f_X(Z) - f_X(X) = 1-2$ and $f_Y(Z) - f_Y(X) = 0-1)$.

$2 = x_2 - x_1$ and $0 = y_4 - y_3$ Apply the preceding paragraph to $-Y$ in place of X and $-X$ in place of Y.

2. $\tau = 2$

Suppose $\tau = 2$. We may presume that $x_4 \geq y_4$, that $x_4 = y_4 \Rightarrow x_3 \geq y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 > y_2$. The possible winning configurations for X are then 0034, 0124, 1114, 0133, 0223 and 1123. The last one is incompatible with the presumption $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 > y_2$, i.e., if one of a pair of tied dice has winning configuration 1123 then that die must be Y.

2.1. $x_1 \leq x_2 \leq y_1 \leq y_2 \leq y_3 < x_3 \leq y_4 < x_4$

If the winning configuration of X is 0034, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq y_1 \leq y_2 \leq y_3 < x_3 \leq y_4 < x_4$, with equalities among the inequalities to produce $\tau = 2$. It turns out that there are three different sets of such equalities.

2.1.1. $x_1 = x_2 = y_1 < y_2 \leq y_3 < x_3 < y_4 < x_4$

Apply the argument of 2.4.3 below to $-X$ and $-Y$.

2.1.2. $x_1 < x_2 = y_1 = y_2 < y_3 < x_3 < y_4 < x_4$

Apply the argument of 2.2.4 below to $-X$ and $-Y$.

2.1.3. $x_1 < x_2 = y_1 < y_2 \leq y_3 < x_3 = y_4 < x_4$

$x_2 - x_1 > 2$ $Z = (x_1+2, x_2-1, x_3-1, x_4)$ ties neither X nor Y $(f_X(Z)-f_X(X) = 1-2$ and $f_Y(Z)-f_Y(X) = 0-$ (at least 2)).

$x_4 - y_4 > 2$ $Z = (x_1, x_2+1, x_3+1, x_4-2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $- 0$).
$x_2 - x_1 \neq x_4 - y_4$ If $x_2 - x_1 > x_4 - y_4$ then $Z = (x_1 + x_4 - y_4, x_2, x_3, x_3)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1-2$ and $f_Y(Z) - f_Y(X) = 0-1$). If $x_2 - x_1 < x_4 - y_4$ then $Z = (x_2, x_2, x_3, x_4 - (x_2 - x_1))$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(Y) - f_Y(X) = 1-0$).
$x_2 - x_1 = 2 = x_4 - y_4$ If $x_3 - y_2 > 1$ then $Z = (x_1, y_2+1, y_3+1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0$ or $1) - 2$ and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$). If $x_3 - y_2 = 1 < y_2 - y_1$ then $Z = (y_1+1, y_2-1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1-0$ and $f_Y(Z) - f_Y(Y) = 1-2$). If $x_3 - y_2 = 1 = y_2 - y_1$ then $X = (x_1, x_1+2, x_1+4, x_1+6)$ and the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1-2, x_2+1, x_3+1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 4-0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2-1, x_3-1, x_4+2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1-2$ and $f_Y(Z) - f_Y(X) = 0-4$)).
$x_2 - x_1 = 1 = x_4 - y_4$ and $y_3 \neq y_2 + 2$ If $y_3 > y_2 + 2$ then $Z = (y_1 - 1, y_2 + 2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0-3$ and $f_Y(Z) - f_Y(Y) = 1-$(at least 2)). If $y_3 = y_2+1$ then $Z = (y_1-1, y_2+1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0-2$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_3 = y_2$ then $Z = (y_1-1, y_2, y_3+1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at most 1) $- 2$ and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$).
$x_2 - x_1 = 1 = x_4 - y_4$ and $y_3 = y_2 + 2$ If $y_2 > y_1 + 2$ then $Z = (y_1 + 2, y_2, y_2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1-0$ and $f_Y(Z) - f_Y(Y) = 1-2$). If $y_2 = y_1 + 1$ then $Z = (y_1, y_1, y_3, y_4+1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-1$ and $f_Y(Z) - f_Y(Y) = 1-2$). If $y_4 > y_3 + 2$ then $Z = (y_1, y_3, y_3, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0-1$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_4 = y_3+1$ then $Z = (y_1-1, y_2, y_4, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1-2$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_2 - y_1 = 2 = y_4 - y_3$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0-3$ and $f_Y(Z) - f_Y(Y) = 2-1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 3-0$ and $f_Y(Z) - f_Y(Y) = 1-2$)).

2.2. $x_1 \leq y_1 < x_2 \leq y_2 < x_3 \leq y_3 \leq y_4 < x_4$

If the winning configuration of X is 0124, then the labels of X and Y must be arranged as $x_1 \leq y_1 < x_2 \leq y_2 < x_3 \leq y_3 \leq y_4 < x_4$, with equalities among the inequalities to produce $\tau = 2$. It turns out that there are four different sets of such equalities.

2.2.1. $x_1 = y_1 < x_2 = y_2 < x_3 < y_3 \leq y_4 < x_4$

Apply the argument of 2.5 below to $-X$ and $-Y$.

2.2.2. $x_1 = y_1 < x_2 < y_2 < x_3 = y_3 < y_4 < x_4$

Apply the argument of 2.4.2 below to $-X$ and $-Y$.

2.2.3. $x_1 < y_1 < x_2 = y_2 < x_3 = y_3 < y_4 < x_4$

Observe that $\sum x_i = s = \sum y_i$ implies that $x_4 - y_4 = y_1 - x_1$.
$x_4 - y_4 = y_1 - x_1 \geq 2$ $Z = (x_1, x_2+1, x_3+1, x_4-2)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $-$ (at most 1)).
$x_4 - y_4 = y_1 - x_1 = 1 < y_4 - y_3$ If $y_4 - y_3 > 2$ then $Z = (y_1, y_2+1, y_3+1, y_4-2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at least 2) $- 0$ and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$). If $y_4 - y_3 = 2$ and $y_3 - y_2 > 2$ then $Z = (y_1, y_2+1, y_3-2, y_4+1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-1$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_4 - y_3 = 2$ and $y_3 - y_2 = 1$ then $Z = (y_1, y_2+1, y_3, y_4-1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-0$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_4 - y_3 = 2 = y_3 - y_2$ and $y_2 - y_1 > 2$ then $Z = (y_1+2, y_2, y_3-2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0-2$ and $f_Y(Z) - f_Y(Y) = 1-2$). If $y_4 - y_3 = 2 = y_3 - y_2$ and $y_2 - y_1 = 1$ then $Z = (y_1+1, y_2, y_3, y_4-1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1-0$ and $f_Y(Z) - f_Y(Y) = 2-1$). If $y_4 - y_3 = 2 = y_3 - y_2 = y_2 - y_1$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1-2$ and $f_Y(Z) - f_Y(Y) = 2-1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2 - 2, y_3, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-1$ and $f_Y(Z) - f_Y(Y) = 1-2$)).

$\underline{x_4 - y_4 = y_1 - x_1 = 1 = y_4 - y_3}$ If $x_3 - x_2 > 2$ then $Z = (y_1, y_2 + 1, y_3 - 2, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_3 - x_2 = 1$ then $Z = (y_1, y_2, y_3 - 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $x_3 - x_2 = 2 \le x_2 - y_1$ then $Z = (y_1 + 1, y_2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $x_3 - x_2 = 2$ and $x_2 - y_1 = 1$ then $X = (x_1, x_1+2, x_1+4, x_1+6)$ and the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 1, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 3 - 0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2 - 1, x_3 - 1, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 3$)).

2.2.4. $x_1 < y_1 < x_2 < y_2 < x_3 = y_3 = y_4 < x_4$

$\underline{x_4 - y_4 \ne 2}$ If $x_4 - y_4 > 2$ then $Z = (x_1, x_2 + 1, x_3 + 1, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $- 0$). If $x_4 - y_4 = 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) =$ (at most 1) $- 2$).
$\underline{x_4 - y_4 = 2 < x_3 - x_2}$ If $x_3 - y_2 \ge 2$ then $Z = (y_1, y_2 + 1, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_2 - x_2 \ge 2$ then $Z = (y_1, y_2 - 1, y_3, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$).
$\underline{x_4 - y_4 = 2 = x_3 - x_2}$ If $x_2 - y_1 \ge 2$ then $Z = (y_1 + 1, y_2, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 -$ (at least 2)). If $y_1 - x_1 \ge 2$ then $Z = (y_1 - 1, y_2, y_3, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_2 - y_1 = 1 = y_1 - x_1$ then $X = (x_1, x_1+2, x_1+4, x_1+6)$ and the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 1, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 3 - 0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2 - 1, x_3 - 1, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 4$)).

2.3. $y_1 < x_1 \le x_2 \le x_3 \le y_2 \le y_3 \le y_4 < x_4$

If the winning configuration of X is 1114, then the labels of X and Y must be arranged as $y_1 < x_1 \le x_2 \le x_3 \le y_2 \le y_3 \le y_4 < x_4$, with equalities among the inequalities to produce $\tau = 2$. It turns out that there are two different sets of such equalities.

2.3.1. $y_1 < x_1 < x_2 = x_3 = y_2 < y_3 \le y_4 < x_4$

$\underline{x_4 - y_4 \ge 2}$ $Z = (y_1, y_2 - 1, y_3 - 1, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at most 1) $-$ (at least 2) and $f_Y(Z) - f_Y(Y) = \delta -$ (at least $\delta + 1$), where $\delta = 2$ or 1 according to whether or not $y_3 = y_4$).
$\underline{y_4 - y_3 \ne 2}$ If $y_4 - y_3 > 2$ then $Z = (y_1, y_2 - 1, y_3 + 2, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 -$ (at least 2) and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_4 - y_3 \le 1$ then $Z = (y_1, y_2 - 1, y_3 + 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (no more than 1) $-$ (at least 2) and $f_Y(Z) - f_Y(Y) = 2 - 1$).
$\underline{x_4 - y_4 = 1 \text{ and } y_4 - y_3 = 2}$ If $y_3 - y_2 > 2$ then $Z = (y_1, y_2 + 1, y_3 - 2, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 3 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_3 - y_2 = 1$ then $Z = (y_1, y_2, y_3 - 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_2 - y_1 > 2$ then $Z = (y_1, y_2 - 2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 -$ (at least 2) and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 - y_1 = 2 = y_3 - y_2$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2 - 2, y_3, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 4$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)).

2.3.2. $y_1 < x_1 \le x_2 < x_3 = y_2 = y_3 < y_4 < x_4$

Apply the preceding paragraph to $-Y$ in place of X and $-X$ in place of Y.

2.4. $x_1 \le y_1 < x_2 \le y_2 \le y_3 < x_3 \le x_4 = y_4$

If the winning configuration of X is 0133, then the presumption that $x_4 \ge y_4$ implies that the labels of X and Y must be arranged as $x_1 \le y_1 < x_2 \le y_2 \le y_3 < x_3 \le x_4 = y_4$, with equalities among the inequalities to produce $\tau = 2$. It turns out that there are three different sets of such equalities.

2.4.1. $x_1 = y_1 < x_2 < y_2 \le y_3 < x_3 < x_4 = y_4$

$\underline{x_4 - x_3 > 2 \text{ or } x_2 - x_1 > 2}$ If $x_4 - x_3 > 2$ then $Z = (x_1, x_2 - 1, x_3 + 2, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 -$ (at least 2) and $f_Y(Z) - f_Y(X) = 0 -$ (1 or 2)). If $x_2 - x_1 > 2$ then $Z = (x_1 + 1, x_2 - 2, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) =$ (at least 1) $- 0$).
$\underline{x_4 - x_3 \ne x_2 - x_1}$ If $x_4 - x_3 > x_2 - x_1$ then $Z = (x_1, x_1, x_3 + x_2 - x_1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 1$). If $x_4 - x_3 < x_2 - x_1$ then $Z = (x_1, x_2 - (x_4 - x_3), x_4, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 1 - 0$).
$\underline{x_4 - x_3 = 2 = x_2 - x_1}$ If $y_2 < y_3$ then $Z = (y_1, y_2 + 1, y_3 + 1, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at most 1) $- 2$ and $f_Y(Z) - f_Y(X) =$ (at least 2) $- 1$). If $y_2 = y_3$ and $x_3 - y_3 > 1$ then $Z = (y_1, y_2, y_3 + 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 = y_3$ and $x_3 - y_3 = 1 < y_2 - x_2$ then $Z = (x_1, x_2 + 2, x_3 - 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) =$ (0 or 2) $- 3$). If $y_2 = y_3$ and $x_3 - y_3 = 1 = y_2 - x_2$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 1$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2, x_3 - 2, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 1 - 4$)).
$\underline{x_4 - x_3 = 1 = x_2 - x_1}$ If $x_3 - y_3 > 1$ then $Z = (x_1 + 1, x_2, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 1 - 0$). If $y_2 - x_2 > 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 1$). If $x_3 - y_3 = 1 = y_2 - x_2$ and $y_3 - y_2 > 2$ then $Z = (y_1, y_2 + 2, y_3 - 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $x_3 - y_3 = 1 = y_2 - x_2 \ge y_3 - y_2$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at most 1) $- 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_3 - y_3 = 1 = y_2 - x_2$ and $y_3 - y_2 = 2$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)).

2.4.2. $x_1 < y_1 < x_2 = y_2 < y_3 < x_3 < x_4 = y_4$

$\underline{x_4 - x_3 > 2}$ $Z = (x_1, x_2 - 1, x_3 + 2, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 -$ (2 or 3)).
$\underline{x_4 - x_3 = 2 < x_3 - x_2}$ If $x_3 - y_3 \ge 2$ then $Z = (y_1, y_2 - 1, y_3 + 2, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (0 or 1) $- 2$ and $f_Y(Z) - f_Y(Y) = 1 -$ (2 or 3)). If $y_3 - x_2 > 2$ and $x_3 - y_3 = 1$ then $Z = (x_1, x_2 + 2, x_3 - 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 1 - 2$). If $y_3 - x_2 = 2$ and $x_3 - y_3 = 1$ then $Z = (x_1, x_2 + 2, x_3, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 2 - 1$).
$\underline{x_4 - x_3 = 2 = x_3 - x_2}$ If $x_2 - y_1 > 1$ then $Z = (y_1 + 1, y_2, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $x_2 - y_1 = 1 < y_1 - x_1$ then $Z = (y_1 - 1, y_2 + 1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_2 - y_1 = 1 = y_1 - x_1$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 3 - 0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2, x_3 - 2, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 1 - 3$)).
$\underline{x_4 - x_3 = 1 \text{ and either } y_4 - y_3 \ne 2 \text{ or } y_3 - y_2 \ne 2}$ If $x_3 - y_3 > 1$ then $Z = (y_1, y_2 - 1, y_3 + 2, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at most 1) $- 3$ and $f_Y(Z) - f_Y(Y) = 1 -$ (at least 2)). If $x_3 - y_3 = 1$ and $y_3 - y_2 > 2$ then $Z = (x_1, x_2 + 2, x_3 - 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 3$ and $f_Y(Z) - f_Y(X) = 1 - 2$). If $x_3 - y_3 = 1 = y_3 - y_2$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$).
$\underline{y_4 - y_3 = 2 = y_3 - y_2}$ If $y_2 - y_1 > 2$ then $Z = (y_1 + 2, y_2 - 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 3$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_2 - y_1 = 1$ then $Z = (y_1 + 1, y_2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 - y_1 = 2 > y_1 - x_1$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4 + 1$ (in which case $Z = (y_1 - 1, y_2 - 1, y_3, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)). If $y_2 - y_1 = 2 \le y_1 - x_1$ then $Z = (x_1 + 1, x_2, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 1$).

2.4.3. $x_1 < y_1 < x_2 < y_2 \leq y_3 < x_3 = x_4 = y_4$

$\underline{y_3 - y_2 \neq 2}$ If $y_3 - y_2 > 2$ then $Z = (y_1, y_2 + 2, y_3 - 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) - f_Y(X) = 1 - (2 \text{ or } 3)$). If $y_3 - y_2 \leq 1 < x_3 - y_3$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_3 - y_2 \leq 1 = x_3 - y_3$ then $Z = (y_1 - 1, y_2, y_3 + 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 -$ (at most 1) and $f_Y(Z) - f_Y(Y) =$ (at least 2) $- 1$).
$\underline{y_3 - y_2 = 2}$ If $y_4 - y_3 \neq 2$ then $Z = (y_1, y_2 + 2, y_3, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - (1 \text{ or } 3)$). If $y_4 - y_3 = 2$ and $y_2 - x_2 \geq 2$ then $Z = (y_1, y_2 - 2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - (0 \text{ or } 1)$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_4 - y_3 = 2$ and $x_2 - y_1 \geq 2$ then $Z = (y_1 + 2, y_2, y_3, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_4 - y_3 = 2$, $y_2 - x_2 = x_2 - y_1 = 1$ and $y_1 - x_1 \leq 2$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1 \text{ or } 2)$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)). If $y_4 - y_3 = 2$, $y_2 - x_2 = x_2 - y_1 = 1$ and $y_1 - x_1 > 2$ then $Z = (y_1 - 2, y_2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$).

2.5. $x_1 < y_1 \leq y_2 < x_2 < x_3 = y_3 < x_4 = y_4$

If the winning configuration of X is 0223, then the presumptions $x_4 \geq y_4$ and $x_4 = y_4 \Rightarrow x_3 \geq y_3$ imply that the labels of X and Y must be arranged as $x_1 < y_1 \leq y_2 < x_2 < x_3 = y_3 < x_4 = y_4$.
$\underline{y_2 - y_1 > 2}$ $Z = (y_1 + 2, y_2 - 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1 \text{ or } 2)$ and $f_Y(Z) - f_Y(Y) = 1 - (2 \text{ or } 3)$).
$\underline{y_2 - y_1 = 2}$ If $y_4 - y_3 \neq 2$ then $Z = (y_1 + 2, y_2, y_3, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1 \text{ or } 3 \text{ or } 4)$ and $f_Y(Z) - f_Y(Y) = 2 - (1 \text{ or } 3)$). If $y_4 - y_3 = 2$ and $y_3 - x_2 \geq 2$ then $Z = (y_1 + 2, y_2, y_3 - 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - (1 \text{ or } 2)$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_4 - y_3 = 2$ and $x_2 - y_2 \geq 2$ then $Z = (y_1, y_2 + 2, y_3, y_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y)) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_4 - y_3 = 2$ and $y_1 - x_1 \geq 2$ then $Z = (y_1 - 2, y_2, y_3 + 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - (0 \text{ or } 1)$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_4 - y_3 = 2$ and $x_2 - y_2 = y_3 - x_2 = y_1 - x_1 = 1$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (x_1 - 1, x_2 + 1, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 1 - 0$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 3$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)).
$\underline{y_2 - y_1 = 1}$ If $y_4 - y_3 > 1$ then $Z = (y_1 + 1, y_2, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_4 - y_3 = 1$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$).
$\underline{y_2 = y_1}$ If $x_3 - x_2 = 1$ then $Z = (y_1, y_2 + 1, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_3 - x_2 > 2$ then $Z = (y_1 + 1, y_2 + 1, y_3 - 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 2) - 1$ and $f_Y(Z) - f_Y(Y) = 4 - 1$). If $x_3 - x_2 = 2 \leq x_2 - y_2$ then $Z = (y_1, y_2 + 2, y_3 - 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = (0 \text{ or } 1) - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $x_3 - x_2 = 2 \leq y_1 - x_1$ then $Z = (x_1 + 2, x_2, x_3 - 2, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = (0 \text{ or } 2) - 1$). If $x_3 - x_2 = 2$, $y_1 - x_1 = 1 = x_2 - y_2$ and $x_4 - x_3 \neq 2$ then $Z = (x_1 + 2, x_2, x_3, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - (1 \text{ or } 3 \text{ or } 4)$ and $f_Y(Z) - f_Y(X) = 4 -$ (at most 3)). If $x_3 - x_2 = 2$, $y_1 - x_1 = 1 = x_2 - y_2$ and $x_4 - x_3 = 2$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 1 - 0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2 - 2, x_3, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 1 - 4$)).

3. $\tau = 4$

Suppose $\tau = 4$. We may presume that $x_4 \geq y_4$, that $x_4 = y_4 \Rightarrow x_3 \geq y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \geq y_2$. These presumptions are incompatible with the winning configuration 1122, which requires that the labels of X and Y be arranged as $y_1 < x_1 \leq x_2 \leq y_2 < x_3 \leq x_4 \leq y_3 \leq y_4$; any set of equalities that satisfies the presumptions will produce $\tau > 4$. Consequently the possible winning configurations for X are 0024, 0114, 0033, 0123, and 0222.

3.1. $x_1 \le x_2 \le y_1 \le y_2 < x_3 \le y_3 \le y_4 < x_4$

If the winning configuration of X is 0024, then the labels of X and Y must be arranged as $x_1 \le x_2 \le y_1 \le y_2 < x_3 \le y_3 \le y_4 < x_4$, with equalities among the inequalities to produce $\tau = 4$. It turns out that there are three different sets of such equalities.

3.1.1. $x_1 = x_2 = y_1 = y_2 < x_3 < y_3 \le y_4 < x_4$

$\underline{y_4 > y_3}$ $Z = (y_1, y_2 + 1, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (2 or 3) $-$ (0 or 1) and $f_Y(Z) - f_Y(Y) = 2 - 1$).
$\underline{y_4 = y_3}$ If $y_3 - x_3 > 1$ then $Z = (y_1 + 1, y_2 + 1, y_3 - 2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) =$ (at least 4) $-$ (0 or 1) and $f_Y(Z) - f_Y(Y) = 4 - 1$). If $x_3 - y_2 > 1$ then $Z = (x_1, x_2 + 1, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 2 - 0$). If $x_4 - y_4 > 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) = 2 - 0$). If $x_4 - y_4 = y_3 - x_3 = x_3 - y_2 = 1$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 1, y_3, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 4 - 2$ and $f_Y(Z) - f_Y(Y) = 4 - 2$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 4$ and $f_Y(Z) - f_Y(Y) = 2 - 4$)).

3.1.2. $x_1 = x_2 = y_1 < y_2 < x_3 = y_3 = y_4 < x_4$

If $x_4 > y_4 + 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) =$ (1 or 2) $- 0$). If $x_3 > y_2 + 1$ then $Z = (y_1, y_2 + 1, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 1$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $x_4 - y_4 = 1 = x_3 - y_2$ then $Z = (y_1, y_2, y_3 - 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 3$).

3.1.3. $x_1 < x_2 = y_1 = y_2 < x_3 = y_3 = y_4 < x_4$

$\underline{x_4 - x_3 > 2 \text{ or } x_2 - x_1 > 2}$ If $x_4 - x_3 > 2$ then $Z = (x_1, x_2 + 1, x_3 + 1, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) =$ (at least 4) $- 0$). If $x_2 - x_1 > 2$ then $Z = (x_1 + 2, x_2 - 1, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 -$ (at least 2) and $f_Y(Z) - f_Y(X) = 0 -$ (at least 4)).
$\underline{2 \ge x_4 - x_3 \ne x_2 - x_1 \le 2}$ If $x_4 - x_3 = 2$ then $Z = (x_1 + 1, x_2, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 2 - 0$). If $x_4 - x_3 = 1$ then again $Z = (x_1 + 1, x_2, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 2$).
$\underline{x_4 - x_3 = 2 = x_2 - x_1}$ If $x_3 - x_2 > 2$ then $Z = (y_1, y_2 + 2, y_3 - 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 4$). If $x_3 - x_2 = 2$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 0$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2 - 2, x_3, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 2$)). If $x_3 - x_2 = 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 0$).
$\underline{x_4 - x_3 = 1 = x_2 - x_1}$ If $x_3 - x_2 > 2$ then $Z = (y_1, y_2 + 2, y_3 - 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 1 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 4$). If $x_3 - x_2 = 2$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (x_1 - 1, x_2, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 0$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 3 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 4$)). If $x_3 - x_2 = 1$ then the fact that X is not balanced implies that either $p < x_1$ (in which case $Z = (x_1 - 1, x_2 + 1, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 0$)) or $q > x_4$ (in which case $Z = (x_1, x_2, x_3 - 1, x_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 4$)).

3.2. $x_1 \le y_1 < x_2 \le x_3 \le y_2 \le y_3 \le y_4 < x_4$

If the winning configuration of X is 0114, then the labels of X and Y must be arranged as $x_1 \le y_1 < x_2 \le x_3 \le y_2 \le y_3 \le y_4 < x_4$, with equalities among the inequalities to produce $\tau = 4$. It turns out that there are two different sets of such equalities.

3.2.1. $x_1 = y_1 < x_2 < x_3 = y_2 = y_3 = y_4 < x_4$

$\underline{x_4 - x_3 > 2 \text{ or } x_3 - x_2 > 2 \text{ or } x_2 - x_1 > 2}$ If $x_4 - x_3 > 2$ then $Z = (x_1, x_2 + 1, x_3 + 1, x_4 - 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) =$ (at least 2) $- 1$ and $f_Y(Z) - f_Y(X) =$ (at least 3) $- 0$). If $x_3 - x_2 > 2$ then $Z = (x_1, x_2+2, x_3-1, x_4-1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1-$(at least 2) and $f_Y(Z) - f_Y(X) = 0-$(at least 3)). If $x_2 - x_1 > 2$ then $Z = (x_1 + 2, x_2 - 1, x_3 - 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 -$ (at least 2) and $f_Y(Z) - f_Y(X) = 1 -$ (at least 3)).
$\underline{2 \geq x_4 - x_3 \neq x_2 - x_1 \leq 2}$ If $x_4 - x_3 = 2$ then $Z = (x_1 + 1, x_2, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 1 - 0$). If $x_4 - x_3 = 1$ then again $Z = (x_1 + 1, x_2, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 1 - 3$).
$\underline{x_4 - x_3 = x_2 - x_1 = 2 \geq x_3 - x_2}$ If $x_3 - x_2 = 1$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 3 - 0$). If $x_3 - x_2 = 2$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2-1$ and $f_Y(Z) - f_Y(X) = 3-1$)) or $q > x_4+1$ (in which case $Z = (x_1, x_2, x_3-2, x_4+2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 3$)).
$\underline{x_4 - x_3 = x_2 - x_1 = 1 \text{ and } 2 \geq x_3 - x_2}$ If $x_3 - x_2 = 2$ then $Z = (x_1, x_2 + 1, x_3, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 3$). If $x_3 - x_2 = 1$ then the fact that X is not balanced implies that either $p < x_1$ (in which case $Z = (x_1 - 1, x_2 + 1, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 3 - 1$)) or $q > x_4$ (in which case $Z = (x_1, x_2, x_3 - 1, x_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 2$ and $f_Y(Z) - f_Y(X) = 0 - 3$)).

3.2.2. $x_1 < y_1 < x_2 = x_3 = y_2 = y_3 < y_4 < x_4$

$Z = (x_1, x_2, x_3 + 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) =$ (2 or 3) $-$ (0 or 1)).

3.3. $x_1 \leq x_2 \leq y_1 \leq y_2 \leq y_3 < x_3 \leq y_4 = x_4$

If the winning configuration of X is 0033, then the presumption that $x_4 \geq y_4$ implies that the labels of X and Y must be arranged as $x_1 \leq x_2 \leq y_1 \leq y_2 \leq y_3 < x_3 \leq y_4 = x_4$, with equalities among the inequalities to produce $\tau = 4$. It turns out that there are three different sets of such equalities.

3.3.1. $x_1 < x_2 = y_1 = y_2 = y_3 < x_3 < y_4 = x_4$

Apply the argument of 3.2.1 above to $-X$ and $-Y$.

3.3.2. $x_1 < x_2 = y_1 = y_2 < y_3 < x_3 = y_4 = x_4$

Apply the argument of 3.1.2 above to $-X$ and $-Y$.

3.3.3. $x_1 = x_2 = y_1 < y_2 \leq y_3 < x_3 = y_4 = x_4$

$\underline{y_3 - y_2 > 2}$ $Z = (y_1+1, y_2+1, y_3-2, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-0$ and $f_Y(Z) - f_Y(Y) =$ (2 or 3) $- 1$).
$\underline{y_3 - y_2 = 2}$ If $y_2 - y_1 > 2$ then $Z = (y_1+2, y_2-1, y_3-1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2-0$ and $f_Y(Z) - f_Y(Y) = 1-2$). If $y_2 - y_1 = 1$ then $Z = (y_1+1, y_2, y_3-1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 - y_1 = 2$ and $y_4 - y_3 > 2$ then $Z = (x_1 + 1, x_2 + 1, x_3 - 2, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) = 2 - 1$). If $y_2 - y_1 = 2$ and $y_4 - y_3 = 1$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_2 - y_1 = 2 = y_4 - y_3$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (y_1 - 2, y_2 + 2, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4 + 1$ (in which case $Z = (y_1, y_2, y_3 - 2, y_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)).
$\underline{y_3 - y_2 = 1}$ If $y_2 - y_1 > 1$ then $Z = (y_1 + 1, y_2, y_3 - 1, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_4 - y_3 > 1$ then $Z = (y_1, y_2 + 1, y_3, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 - y_1 = y_4 - y_3 = 1$ then the fact that Y

is not balanced implies that either $p < y_1$ (in which case $Z = (y_1 - 1, y_2 + 1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 0 - 2$ and $f_Y(Z) - f_Y(Y) = 2 - 1$)) or $q > y_4$ (in which case $Z = (y_1, y_2, y_3 - 1, y_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$)).
$\underline{y_3 - y_2 = 0}$ If $y_2 - y_1 > 1$ then $Z = (y_1 + 1, y_2 - 1, y_3, y_4)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 0$ and $f_Y(Z) - f_Y(Y) = 1 - 2$). If $y_4 - y_3 > 1$ then $Z = (y_1, y_2, y_3 + 1, y_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(Y) = 2 - 1$ and $f_Y(Z) - f_Y(Y) = 2 - 1$). If $y_2 - y_1 = y_4 - y_3 = 1$ then the fact that X is not balanced implies that either $p < x_1 - 1$ (in which case $Z = (x_1 - 2, x_2 + 2, x_3, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 4 - 2$ and $f_Y(Z) - f_Y(X) = 6 - 1$)) or $q > x_4 + 1$ (in which case $Z = (x_1, x_2, x_3 - 2, x_4 + 2)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 4$ and $f_Y(Z) - f_Y(X) = 1 - 6$)).

3.4. $x_1 \le y_1 < x_2 \le y_2 < x_3 \le y_3 < x_4 \le y_4$

If the winning configuration of X is 0123, then the presumptions $x_4 \ge y_4$, $x_4 = y_4 \Rightarrow x_3 \ge y_3$ and $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \ge y_2$ imply that the labels of X and Y must be arranged as $x_1 \le y_1 < x_2 \le y_2 < x_3 \le y_3 < x_4 \le y_4$, with equalities among the inequalities to produce $\tau = 4$. It turns out that these equalities imply $X = Y$, contrary to the hypothesis that X and Y are distinct.

3.5. $x_1 < y_1 \le y_2 < x_2 < x_3 = x_4 = y_3 = y_4$

If the winning configuration of X is 0222, then the presumptions $x_4 \ge y_4$, $x_4 = y_4 \Rightarrow x_3 \ge y_3$ and $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \ge y_2$ imply that the labels of X and Y must be arranged as $x_1 < y_1 \le y_2 < x_2 < x_3 = x_4 = y_3 = y_4$. We may apply the argument of 3.1.1 above to $-X$ and $-Y$.

4. $\tau = 6$

Suppose $\tau = 6$. We presume that $x_4 \ge y_4$, that $x_4 = y_4 \Rightarrow x_3 \ge y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \ge y_2$. The possible winning configurations for X are then 0014, 0023, 0113, and 0122.

If the winning configuration of X is 0014, then the labels of X and Y must be arranged as $x_1 \le x_2 \le y_1 < x_3 \le y_2 \le y_3 \le y_4 < x_4$. There is no set of equalities that yields $\tau = 6$.

If the winning configuration of X is 0023, then the labels of X and Y must be arranged as $x_1 \le x_2 \le y_1 \le y_2 < x_3 \le y_3 < x_4 = y_4$. The only arrangement of equalities that yields $\tau = 6$ is $x_1 = x_2 = y_1 = y_2 < x_3 = y_3 < x_4 = y_4$; this contradicts the hypothesis that X and Y are distinct.

If the winning configuration of X is 0113, then the labels of X and Y must be arranged as $x_1 \le y_1 < x_2 \le x_3 \le y_2 \le y_3 < x_4 = y_4$. The only arrangement of equalities that yields $\tau = 6$ is $x_1 = y_1 < x_2 = y_2 = x_3 = y_3 < x_4 = y_4$; this contradicts the hypothesis that X and Y are distinct.

If the winning configuration of X is 0122, then the labels of X and Y must be arranged as $x_1 \le y_1 < x_2 \le y_2 < x_3 = y_3 = x_4 = y_4$. The only arrangement of equalities that yields $\tau = 6$ is $x_1 = y_1 < x_2 = y_2 < x_3 = y_3 = x_4 = y_4$; this contradicts the hypothesis that X and Y are distinct.

5. $\tau = 8$

Suppose $\tau = 8$. We presume that $x_4 \ge y_4$, that $x_4 = y_4 \Rightarrow x_3 \ge y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \ge y_2$. The possible winning configurations for X are then 0004, 0013, 0022, 0112 and 1111.

If the winning configuration of X is 0004, then the labels of X and Y must be arranged as $x_1 \le x_2 \le x_3 \le y_1 \le y_2 \le y_3 \le y_4 < x_4$. The only set of equalities that yields $\tau = 8$ is $x_1 < x_2 = x_3 = y_1 = y_2 = y_3 = y_4 < x_4$. Observe that $x_4 - x_3 = x_2 - x_1$ because $\sum x_i = \sum y_i$. If $x_4 - x_3 > 1$ then $Z = (x_1, x_2, x_3 + 1, x_4 - 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 2 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 0$). If $x_4 - x_3 = 1$ then the fact that Y is not balanced implies that either $p < y_1 - 1$ (in which case $Z = (x_1 - 1, x_2, x_3 + 1, x_4)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 3 - 1$ and $f_Y(Z) - f_Y(X) = 4 - 0$)) or $q > y_4 + 1$ (in which case $Z = (x_1, x_2 - 1, x_3, x_4 + 1)$ ties neither X nor Y ($f_X(Z) - f_X(X) = 1 - 3$ and $f_Y(Z) - f_Y(X) = 0 - 4$)).

If the winning configuration of X is 0013, then the labels of X and Y must be arranged as $x_1 \le x_2 \le y_1 < x_3 \le y_2 \le y_3 < x_4 \le y_4$; no set of equalities will yield $\tau = 8$.

If the winning configuration of X is 0022, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq y_1 \leq y_2 < x_3 \leq x_4 \leq y_3 \leq y_4$. The only arrangement of equalities which yields $\tau = 8$ is $x_1 = y_1 = x_2 = y_2 < x_3 = y_3 = x_4 = y_4$; this contradicts the hypothesis that X and Y are distinct.

If the winning configuration of X is 0112, then the labels of X and Y must be arranged as $x_1 \leq y_1 < x_2 \leq x_3 \leq y_2 < x_4 \leq y_3 \leq y_4$; no set of equalities will yield $\tau = 8$.

If the winning configuration of X is 1111, then the labels of X and Y must be arranged as $y_1 < x_1 \leq x_2 \leq x_3 \leq x_4 \leq y_2 \leq y_3 \leq y_4$. The presumptions that $x_4 \geq y_4$, that $x_4 = y_4 \Rightarrow x_3 \geq y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \geq y_2$ cannot be satisfied with $\tau = 8$.

6. $\tau = 10$

Suppose $\tau = 10$. We presume that $x_4 \geq y_4$, that $x_4 = y_4 \Rightarrow x_3 \geq y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \geq y_2$. The possible winning configurations for X are then 0003, 0012, and 0111.

If the winning configuration is 0003, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq x_3 \leq y_1 \leq y_2 \leq y_3 < x_4 \leq y_4$. The only set of equalities that yields $\tau = 10$ is $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 < x_4 = y_4$; this violates the hypothesis that X and Y are distinct.

If the winning configuration is 0012, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq y_1 < x_3 \leq y_2 < x_4 \leq y_3 \leq y_4$; no set of equalities will yield $\tau = 10$.

If the winning configuration is 0111, then the labels of X and Y must be arranged as $x_1 \leq y_1 < x_2 \leq x_3 \leq x_4 \leq y_2 \leq y_3 \leq y_4$. The only set of equalities that yields $\tau = 10$ is $x_1 = y_1 < x_2 = x_3 = x_4 = y_2 = y_3 = y_4$; this violates the hypothesis that X and Y are distinct.

7. $\tau = 12$

Suppose $\tau = 12$. We presume that $x_4 \geq y_4$, that $x_4 = y_4 \Rightarrow x_3 \geq y_3$, and that $((x_4 = y_4) \wedge (x_3 = y_3)) \Rightarrow x_2 \geq y_2$. The possible winning configurations for X are then 0002 and 0011.

If the winning configuration is 0002, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq x_3 \leq y_1 \leq y_2 < x_4 \leq y_3 \leq y_4$. No set of equalities yields $\tau = 12$.

If the winning configuration is 0011, then the labels of X and Y must be arranged as $x_1 \leq x_2 \leq y_1 < x_3 \leq x_4 \leq y_2 \leq y_3 \leq y_4$. No set of equalities yields $\tau = 12$.

8. $\tau = 14$

The only possible winning configuration for X is 0001, corresponding to the label arrangement $x_1 \leq x_2 \leq x_3 \leq y_1 < x_4 \leq y_2 \leq y_3 \leq y_4$. No set of equalities yields $\tau = 14$.

9. $\tau = 16$

The only arrangement of the labels of X and Y that yields $\tau = 16$ is $x_i = y_j \ \forall i, j$. This contradicts the hypothesis that X and Y are distinct.