

When can you factor a quadratic form?

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1 Introduction

Can you factor either of the polynomials

$$Q_1 = 2X_3^2 + 2X_1X_2 - X_1X_3 - 4X_2X_3 - 6X_3X_4 + 3X_1X_4 \quad \text{or} \\ Q_2 = 2X_3^2 + 2X_1X_2 - X_1X_3 - 4X_2X_3 - 6X_3X_4$$

into a product of polynomials of smaller degree? Feel free to use complex coefficients, but no computers or calculators are allowed.

It turns out that one of Q_1 and Q_2 factors, while the other does not. If you want to check your answer, or do not feel like trying, see the example in Section 2. Of course, we ultimately wish to consider the following general problem:

Given any quadratic form, determine whether it is a product of two linear forms.

If you are uncertain about the meaning of the terms “quadratic form” or “linear form”, definitions will appear in Section 2. Interestingly enough, this problem has a simple solution, and was answered several centuries ago. However, many people we talked to found it intriguing and were surprised they had not thought about or seen this problem before. A solution will be a part of what we present in this article. However, our story begins elsewhere.

Undoubtedly, every reader has tried to clarify a notion in one source by consulting another, only to be frustrated that the presentations are inconsistent in vocabulary or notation. Recently, this happened to us in a study of conics. While reading Peter Cameron’s *Combinatorics: Topics, Techniques, and Algorithms* [8], we encountered a definition of a non-singular quadratic form $Q = Q(x, y, z)$ as one that

...cannot be transformed into a form in less than three variables by any non-singular linear substitution of the variables x, y, z .

Though the definition was clear, it did not appear to translate into simple criteria for determining whether a given quadratic form was non-singular. In searching for such a test, we found that various sources used the word “singular”

to describe quadratic forms in (what seemed to be) completely different ways. Complicating matters further was that terms such as “degenerate” and “reducible” started to appear, and these three words were often used interchangeably. For more details regarding this usage, see Section 5, and in particular Table 1.

In all, we found five criteria related to degeneracy (or non-degeneracy) of quadratic forms $Q = Q(x, y, z)$ in the literature. While we believed them to be equivalent, few sources proved the equivalence of even two of them, and we found only two sources that proved the equivalence of three. Three of the five criteria have immediate generalizations to n dimensions.

Our main motivation for writing this paper was to show once and for all, for ourselves and for the record, that the several conditions that are widely used as definitions are actually equivalent. We found the writing process instructive, and we hope the reader will find what we present to be useful. In particular, many of the proofs we used draw ideas from the basic principles of analysis, algebra, linear algebra, and geometry. We think that some of these equivalences can serve as useful exercises in related undergraduate courses, as they help to stress the unity of mathematics.

2 Notation and an example

The main object of our study below will be a quadratic form and its associated quadric; we will now define these terms. Additional definitions and related results can be found in Hoffman and Kunze [17] or in Shilov [31], for example. Let \mathbb{F} be a field whose characteristic, denoted $\text{char}(\mathbb{F})$, is odd or zero. Examples include the fields of rational numbers, real numbers, and complex numbers, as well as finite fields containing an odd (prime power) number of elements. We view \mathbb{F}^n as the n -dimensional vector space over \mathbb{F} . Any $(n - 1)$ -dimensional subspace of \mathbb{F}^n is called a *hyperplane*. By $\mathbb{F}[X_1, \dots, X_n]$, we will denote the ring of polynomials with (commuting) indeterminants X_1, \dots, X_n , and coefficients in \mathbb{F} . It will often be convenient to view a polynomial of k indeterminants as a polynomial of one of them, with coefficients being polynomials of the other $k - 1$ indeterminants. For instance, a polynomial in $\mathbb{F}[X_1, X_2, X_3]$ may be viewed as an element of $\mathbb{F}[X_2, X_3][X_1]$; i.e. a polynomial of X_1 whose coefficients are polynomials of X_2 and X_3 .

For $f \in \mathbb{F}[X_1, \dots, X_n]$, let $\mathcal{Z}(f) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n : f(\alpha_1, \dots, \alpha_n) = 0\}$. For $f = (f_1, \dots, f_m)$, where all $f_i \in \mathbb{F}[X_1, \dots, X_n]$, we define $\mathcal{Z}(f)$ to be the intersection of all $\mathcal{Z}(f_i)$; note that the “ \mathcal{Z} ” in $\mathcal{Z}(f)$ stands for the zeros of f .

A polynomial $Q \in \mathbb{F}[X_1, \dots, X_n]$ of the form

$$Q = Q(X_1, \dots, X_n) = \sum_{1 \leq i, j \leq n} a_{ij} X_i X_j,$$

where $a_{ij} = a_{ji}$ for all i, j , is called a *quadratic form*. Clearly, Q defines a polynomial function $Q : \mathbb{F}^n \rightarrow \mathbb{F}$, where $(\alpha_1, \dots, \alpha_n) \mapsto Q(\alpha_1, \dots, \alpha_n)$. Note

that we use the same notation Q for both the algebraic object (i.e. the polynomial) and the corresponding function. $\mathcal{Z}(Q)$ is often referred to as the *quadric* corresponding to Q . For every Q , we define the $n \times n$ matrix of coefficients $M_Q = (a_{ij})$.

Let V denote the set of all degree one polynomials in $\mathbb{F}[X_1, \dots, X_n]$ having zero constant term, called *linear forms*, together with the zero polynomial. V is a vector space over \mathbb{F} , and $\{X_1, \dots, X_n\}$ is a basis of V . For any linear transformation $\varphi : V \rightarrow V$ and any quadratic form Q , we can substitute $\varphi(X_i)$ for X_i in Q for all $i = 1, \dots, n$. After simplifying the result by combining like terms, we again obtain a quadratic form, which we denote by $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ or $\tilde{Q}(X_1, \dots, X_n)$. For $T = (t_1, \dots, t_n) \in \mathbb{F}^n$, $\tilde{Q}(T) = Q(\varphi(X_1)(T), \dots, \varphi(X_n)(T))$.

Next, a comment on derivatives. We treat partial derivatives formally, as is done in algebra. For example, to differentiate Q with respect to X_1 , we view Q as an element of $\mathbb{F}[X_2, \dots, X_n][X_1]$:

$$Q = a_{11}X_1^2 + (2a_{12}X_2 + \dots + 2a_{1n}X_n)X_1 + \sum_{1 < i, j \leq n} a_{ij}X_iX_j$$

and thus

$$\frac{\partial Q}{\partial X_1} = 2a_{11}X_1 + (2a_{12}X_2 + \dots + 2a_{1n}X_n).$$

Partial derivatives with respect to the other indeterminants are defined similarly.

The gradient field ∇Q of Q is then defined as $\nabla Q = \left(\frac{\partial Q}{\partial X_1}, \dots, \frac{\partial Q}{\partial X_n} \right)$.

Before stating our main result in the next section, we wish to first illustrate it with an example.

Example. Let $n = 4$ and \mathbb{F} be the field of real numbers (the following examples are also valid over all fields of characteristic different from 2). Consider the quadratic forms mentioned at the beginning of this article, namely

$$Q_1 = 2X_3^2 + 2X_1X_2 - X_1X_3 - 4X_2X_3 - 6X_3X_4 + 3X_1X_4$$

and

$$Q_2 = 2X_3^2 + 2X_1X_2 - X_1X_3 - 4X_2X_3 - 6X_3X_4.$$

We now examine several properties of Q_1 and Q_2 .

1. We note that

$$M_{Q_1} = \begin{pmatrix} 0 & 1 & -1/2 & 3/2 \\ 1 & 0 & -2 & 0 \\ -1/2 & -2 & 2 & -3 \\ 3/2 & 0 & -3 & 0 \end{pmatrix}$$

has row reduced echelon form

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and thus $\text{rank}(M_{Q_1}) = 2$. In contrast,

$$M_{Q_2} = \begin{pmatrix} 0 & 1 & -1/2 & 0 \\ 1 & 0 & -2 & 0 \\ -1/2 & -2 & 2 & -3 \\ 0 & 0 & -3 & 0 \end{pmatrix}$$

has rank 4 (i.e. is non-singular).

2. Define a linear transformation $\varphi_1 : V \rightarrow V$ by

$$\begin{aligned} \varphi_1(X_1) &= X_1 + 2X_3 \\ \varphi_1(X_2) &= \frac{1}{2}X_2 + \frac{1}{2}X_3 - \frac{3}{2}X_4 \\ \varphi_1(X_3) &= X_3 \\ \varphi_1(X_4) &= X_4. \end{aligned}$$

It is a straightforward verification that φ is non-singular, and that

$$\widetilde{Q}_1 = Q_1(\varphi_1(X_1), \varphi_1(X_2), \varphi_1(X_3), \varphi_1(X_4)) = X_1X_2,$$

which contains only $r = 2$ indeterminants. Furthermore, we comment (without proof) that for any non-singular linear transformation $\varphi : V \rightarrow V$, $Q_1(\varphi(X_1), \varphi(X_2), \varphi(X_3), \varphi(X_4))$ contains *at least* two indeterminants: the argument is the same as in the proof of Theorem 1.

In contrast, for any non-singular linear transformation $\varphi : V \rightarrow V$,

$$\widetilde{Q}_2 = Q_2(\varphi(X_1), \varphi(X_2), \varphi(X_3), \varphi(X_4))$$

contains all four indeterminants. This also follows from the proof of Theorem 1.

3. We now consider the zeros of the gradient fields of Q_1 and Q_2 . We have

$$\nabla Q_1 = \begin{pmatrix} \partial Q_1 / \partial X_1 \\ \partial Q_1 / \partial X_2 \\ \partial Q_1 / \partial X_3 \\ \partial Q_1 / \partial X_4 \end{pmatrix} = \begin{pmatrix} 2X_2 - X_3 + 3X_4 \\ 2X_1 - 4X_3 \\ -X_1 - 4X_2 + 4X_3 - 6X_4 \\ 3X_1 - 6X_3 \end{pmatrix} = 2M_{Q_1}X,$$

where $X = (X_1, X_2, X_3, X_4)^t$ is the transpose of (X_1, X_2, X_3, X_4) . Therefore, $\mathcal{Z}(\nabla Q_1)$ is the null space of the matrix $2M_{Q_1}$. In much the same way,

$$\nabla Q_2 = \begin{pmatrix} 2X_2 - X_3 \\ 2X_1 - 4X_3 \\ -X_1 - 4X_2 + 4X_3 - 6X_4 \\ -6X_3 \end{pmatrix} = 2M_{Q_2}X$$

implies that $\mathcal{Z}(\nabla Q_2)$ is the null space of the matrix $2M_{Q_2}$. However, while $\mathcal{Z}(\nabla Q_1)$ has dimension $2 = 4 - 2 = n - r$, $\mathcal{Z}(\nabla Q_2)$ has dimension 0 (i.e. $\mathcal{Z}(\nabla Q_2)$ contains only the zero vector).

4. $Q_1 = (X_1 - 2X_3)(2X_2 - X_3 + 3X_4)$, a product of two polynomials that are not scalar multiples of each other. Q_2 does not factor into a product of linear polynomials (even over the complex numbers).
5. It is clear from the above factorization of Q_1 that $\mathcal{Z}(Q_1)$ is the union of two hyperplanes whose equations are $X_1 - 2X_3 = 0$ and $2X_2 - X_3 + 3X_4 = 0$. As $X_1 - 2X_3$ and $2X_2 - X_3 + 3X_4$ are not scalar multiples of one another, these hyperplanes are distinct. This contrasts with $\mathcal{Z}(Q_2)$ containing only the zero vector.

3 The main result

The main results of this paper are the following theorems. They lead to equivalence of several definitions of degeneracy (or non-degeneracy) of quadratic forms.

Theorem 1. *Let $n \geq 2$, \mathbb{F} be a field, and $\text{char}(\mathbb{F}) \neq 2$. Let $Q = \sum_{1 \leq i, j \leq n} a_{ij} X_i X_j$ be a non-zero quadratic form in $\mathbb{F}[X_1, \dots, X_n]$. Then the following statements are equivalent.*

1. The matrix $M_Q = (a_{ij})$ has rank r .
2. There exists a non-singular linear transformation $\varphi : V \rightarrow V$ such that $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ contains precisely r of the indeterminants X_1, \dots, X_n ; furthermore, for any other non-singular linear transformation, this number is at least r .
3. $\mathcal{Z}(\nabla Q)$ is a vector space of dimension $n - r$.

Define r , the *rank* of a quadratic form Q , as in any of the three equivalent statements listed above. Clearly, r is an integer such that $0 \leq r \leq n$. If $r = 1$ or 2 , then one can supplement the above three statements with two more, which appear frequently in the context of conics or quadratic surfaces. In the following, let \mathbb{K} denote a field that is either equal to \mathbb{F} , or is a quadratic extension $\mathbb{F}(m)$ of \mathbb{F} for some $m \in \mathbb{K} \setminus \mathbb{F}$ such that $m^2 \in \mathbb{F}$.

Theorem 2. *Let n , \mathbb{F} , and Q be as in Theorem 1, and let $r = 1$ or 2 . Then the following statements are equivalent.*

1. The matrix $M_Q = (a_{ij})$ has rank r .
2. There exists a non-singular linear transformation $\varphi : V \rightarrow V$ such that $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ contains precisely r of the indeterminants X_1, \dots, X_n ; furthermore, for any other non-singular linear transformation, this number is at least r .
3. $\mathcal{Z}(\nabla Q)$ is a vector space of dimension $n - r$.
4. Q is a product of two linear forms with coefficients in \mathbb{K} . These forms are scalar multiples of one another if $r = 1$, and are not if $r = 2$.

5. $\mathcal{Z}(Q)$ is a hyperplane in \mathbb{K}^n for $r = 1$. $\mathcal{Z}(Q)$ is the union of two distinct hyperplanes in \mathbb{K}^n for $r = 2$.

Note that statements 1, 2, 3, and 5 primarily use terms of linear algebra, with statement 3 having an analysis flavor and statement 5 a geometric one. Statement 4 is algebraic. Furthermore, take note of how the statements in Theorems 1 and 2 correspond to the properties of Q_1 and Q_2 discussed in the above example.

We are now ready for our main definition.

Definition. Let Q be as in Theorem 1, and suppose Q has rank r .

1. If $1 \leq r < n$, we call Q *degenerate* and *singular*. If instead $n = r$, we call Q *non-degenerate* and *non-singular*.
2. If $r = 1$ or 2 , we call Q *reducible*. If instead $r \geq 3$, we call Q *irreducible*.

We now make several comments about this definition.

- As a non-zero quadratic form can be factored only into the product of two linear forms, our definition of irreducibility corresponds to the one in algebra for polynomials. Indeed, a quadratic form factors into linear forms (over some extension \mathbb{K} of \mathbb{F}) for $n \geq 2$ if and only if $1 \leq r \leq 2$.
- Similar definitions can be applied to the case $n = 1$. Note that the only non-zero quadratic forms in one indeterminate are $Q = aX_1^2$, where $a \neq 0$. Clearly, $Q = (aX_1)X_1$ factors, and therefore we say that Q is *reducible*. Furthermore, as $n = r = 1$, we call Q *non-degenerate* and *non-singular*.
- We wish to emphasize that $n = 3$ is the only case in which the notions of degeneracy, singularity, and reducibility are equivalent.
- Consider the example from Section 2. Note that since $n = 4$ and $r = 2$, our definition implies that Q_1 is degenerate, singular, and reducible. By contrast, Q_2 is non-degenerate, non-singular, and irreducible.

4 Proof of Theorems 1 and 2

First we remind the reader of a few well-known results that we will need.

Let R^t denote the transpose of a matrix R , and let $X = (X_1, \dots, X_n)^t$ be the column vector of indeterminants. Then, in terms of matrix multiplication, we have $[Q] = X^t M_Q X$, a 1×1 matrix. From now on, we will view $[Q]$ as the polynomial Q , and simply write $Q = X^t M_Q X$.

Let $\varphi : V \rightarrow V$ be a linear transformation, $Y_i = \varphi(X_i)$ for all i , and $Y = (Y_1, \dots, Y_n)^t$. Then we define

$$\varphi(X) = (\varphi(X_1), \dots, \varphi(X_n))^t,$$

and therefore

$$\varphi(X) = (Y_1, \dots, Y_n)^t = Y.$$

Letting B_φ be the matrix of φ with respect to a basis $\{X_1, \dots, X_n\}$ of V , we have

$$Y = B_\varphi X.$$

This allows us to use matrix multiplication in order to determine $M_{\tilde{Q}}$. Indeed, we have

$$\tilde{Q} = \tilde{Q}(X_1, \dots, X_n) = X^t M_{\tilde{Q}} X$$

and

$$\begin{aligned} \tilde{Q} &= Q(\varphi(X_1), \dots, \varphi(X_n)) = Q(Y_1, \dots, Y_n) \\ &= Y^t M_Q Y \\ &= (B_\varphi X)^t M_Q (B_\varphi X) \\ &= X^t (B_\varphi^t M_Q B_\varphi) X. \end{aligned}$$

The equality $X^t M_{\tilde{Q}} X = X^t (B_\varphi^t M_Q B_\varphi) X$, viewed as an equality of 1×1 matrices with polynomial entries, implies that

$$M_{\tilde{Q}} = B_\varphi^t M_Q B_\varphi.$$

Recall that for any square matrix M and any non-singular matrix N of the same dimensions, $\text{rank}(MN) = \text{rank}(M) = \text{rank}(NM)$. Therefore, φ non-singular implies that so is B_φ . Thus

$$\text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}).$$

Finally, we remind the reader of the fundamental fact (see, e.g. [31], Section 7.33(b)) that given any quadratic form Q with $\text{rank}(M_Q) = r$, there exists a non-singular linear transformation ψ of V such that if $Q' = Q(\psi(X_1), \dots, \psi(X_n))$, then $M_{Q'}$ is diagonal. In other words,

$$Q' = d_1 X_1^2 + \dots + d_r X_r^2,$$

where all d_i are non-zero elements of \mathbb{F} . This implies that for any non-singular linear transformation φ of V , $\tilde{Q} = Q(\varphi(X_1), \dots, \varphi(X_n))$ contains at least r variables X_i . Indeed, if \tilde{Q} contained $r' < r$ variables, then $r = \text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}) \leq r'$, a contradiction.

We are now ready to prove Theorems 1 and 2. Let (k) stand for statement k of Theorem 1 or Theorem 2, $k = 1, \dots, 5$.

(1) \Leftrightarrow (2). The statement follows immediately from our remarks above on the diagonal form of Q , as the rank of a diagonal matrix is the number of its non-zero entries.

(1) \Leftrightarrow (3). It is straightforward to verify that $(2M_Q)X = \nabla Q$. Since $\text{char}(\mathbb{F}) \neq 2$, $\mathcal{Z}(\nabla Q)$ coincides with the null-space of M_Q . Hence, $\dim(\mathcal{Z}(\nabla Q)) = n - \text{rank}(M_Q) = n - r$.

This proves the equivalence of statements (1), (2), and (3), and thus concludes the proof of Theorem 1. We remind the reader that this equivalence leads to the notion of the rank of a quadratic form Q , which was defined to be r from any of these equivalent statements.

We now suppose that the rank of Q is $r = 1$ or 2 .

(2) \Leftrightarrow (4). We have $\text{rank}(M_Q) = r = 1$ or 2 . As explained above, there exists a non-singular transformation ψ of V such that $Q' = Q(\psi(X_1), \dots, \psi(X_n)) = d_1 X_1^2 + \dots + d_r X_r^2$ with all d_i being non-zero elements of \mathbb{F} , and $\text{rank}(M_{Q'}) = \text{rank}(M_Q) = r$. If $r = 1$, then $Q' = d_1 X_1^2$, where $d_1 \neq 0$. Then $Q = d_1(\psi^{-1}(X_1))^2$ factors as claimed. If instead $r = 2$, then $Q' = d_1 X_1^2 + d_2 X_2^2$, where both $d_i \neq 0$. Let $m \in \mathbb{K}$ such that $m^2 = -d_2/d_1$. Then

$$\begin{aligned} Q' &= d_1 X_1^2 + d_2 X_2^2 \\ &= d_1 \left(X_1^2 - \frac{-d_2}{d_1} X_2^2 \right) \\ &= d_1 (X_1^2 - m^2 X_2^2) \\ &= (d_1 X_1 - d_1 m X_2)(X_1 + m X_2). \end{aligned}$$

Clearly, the two factors of Q' are independent in the vector space of linear forms in $\mathbb{K}[X_1, \dots, X_n]$; otherwise, $m = -m$, which is equivalent to $m = 0$ because $\text{char}(\mathbb{F}) \neq 2$. This implies $d_2 = 0$, a contradiction. Hence,

$$Q = (d_1 \psi^{-1}(X_1) - d_1 m \psi^{-1}(X_2)) \cdot (\psi^{-1}(X_1) + m \psi^{-1}(X_2)).$$

Since ψ^{-1} is non-singular, $\psi^{-1}(X_1)$ and $\psi^{-1}(X_2)$ are linearly independent over \mathbb{F} . Hence, the factors of Q are linearly independent over \mathbb{K} . This proves that (2) \Rightarrow (4).

Suppose Q factors over \mathbb{K} such that $Q = a(a_1 X_1 + \dots + a_n X_n)^2$ with $a, a_i \in \mathbb{K}$ and $a \neq 0$. Permuting indices as necessary, suppose $a_1 \neq 0$; then apply the linear substitution φ defined by

$$\varphi(X_1) = \frac{1}{a_1} X_1 - \sum_{i=2}^n \frac{a_i}{a_1} X_i \quad \text{and} \quad \varphi(X_i) = X_i \quad \text{for all } 2 \leq i \leq n.$$

This transformation is non-singular, and the resulting quadratic form $\tilde{Q} = aX_1^2$ has $r = 1$ indeterminate, implying (2).

Suppose instead that Q factors over \mathbb{K} with $Q = (a_1 X_1 + \dots + a_n X_n)(b_1 X_1 + \dots + b_n X_n)$, where $a_i, b_i \in \mathbb{K}$ such that the factors do not differ by a scalar multiple. This is equivalent to the vectors (a_1, \dots, a_n) and (b_1, \dots, b_n) being linearly independent. Therefore, permuting indices as necessary, the vectors

$(a_1, a_2), (b_1, b_2) \in \mathbb{F}^2$ are linearly independent with $a_1 \neq 0$ and $b_2 \neq 0$. Apply the linear substitution φ defined by

$$\begin{aligned}\varphi(X_1) &= \alpha_{11}X_1 + \alpha_{12}X_2 + \cdots + \alpha_{1n}X_n \\ \varphi(X_2) &= \alpha_{21}X_1 + \alpha_{22}X_2 + \cdots + \alpha_{2n}X_n \\ \varphi(X_i) &= X_i \text{ for all } i = 3, \dots, n,\end{aligned}$$

where (α_{ij}) is the inverse of the matrix

$$\left(\begin{array}{ccccc|cccc} a_1 & a_2 & a_3 & \cdots & a_n & & & & & \\ b_1 & b_2 & b_3 & \cdots & b_n & & & & & \\ 0 & 0 & & & & & & & & \\ \vdots & \vdots & & & & & & & & \\ 0 & 0 & & & & & & & & \end{array} \right),$$

and I_{n-2} is the $(n-2) \times (n-2)$ identity matrix. This non-singular transformation produces the quadratic form $\tilde{Q} = X_1X_2$, which contains $r = 2$ indeterminants. Note that if there existed a linear transformation $\phi: V \rightarrow V$ such that $\hat{Q} = Q(\phi(X_1), \dots, \phi(X_n))$ had only one indeterminant, then $1 = \text{rank}(M_{\hat{Q}}) = \text{rank}(M_Q) = \text{rank}(M_{\tilde{Q}}) = 2$, a contradiction. This proves that (4) \Rightarrow (2).

(4) \Leftrightarrow (5). The implication (4) \Rightarrow (5) is obvious, and we concentrate on the converse.

Let $a_1X_1 + \cdots + a_nX_n = 0$ be an equation of a hyperplane W of \mathbb{K}^n such that $W \subseteq \mathcal{Z}(Q)$. Then for every solution $(\alpha_1, \dots, \alpha_n)$ of $a_1X_1 + \cdots + a_nX_n = 0$, $Q(\alpha_1, \dots, \alpha_n) = 0$. As not all a_i are zero, we may assume by permuting the indices as necessary that $a_1 \neq 0$. Dividing by a_1 , we rewrite the equation $a_1X_1 + \cdots + a_nX_n = 0$ as $X_1 + a'_2X_2 + \cdots + a'_nX_n = 0$. Viewing Q as an element of $\mathbb{K}[X_2, \dots, X_n][X_1]$ and dividing it by $X_1 + (a'_2X_2 + \cdots + a'_nX_n)$ with remainder, we obtain

$$Q = q \cdot (X_1 + (a'_2X_2 + \cdots + a'_nX_n)) + t,$$

with quotient $q \in \mathbb{K}[X_2, \dots, X_n][X_1]$ and remainder $t \in \mathbb{K}[X_2, \dots, X_n]$. Now, $Q(w_1, \dots, w_n) = 0 = w_1 + (a'_2w_2 + \cdots + a'_nw_n)$ for every $(w_1, \dots, w_n) \in W$. Furthermore, for every $(w_2, \dots, w_n) \in \mathbb{K}^{n-1}$, there exists $w_1 \in \mathbb{K}$ such that $(w_1, \dots, w_n) \in W$. This implies that $t(w_2, \dots, w_n) = 0$ for every $(w_2, \dots, w_n) \in \mathbb{K}^{n-1}$. The following lemma will allow us to conclude that $t = 0$:

Lemma. Let $n \geq 1$, \mathbb{K} be a field such that $\text{char}(\mathbb{K}) \neq 2$, and $f \in \mathbb{K}[X_1, \dots, X_n]$ be a quadratic form that vanishes on \mathbb{K}^n . Then $f = 0$.

Proof. We proceed by induction. For $n = 1$, $f = f(X_1) = aX_1^2$ for some $a \in \mathbb{K}$. Then $0 = f(1) = a$, and so $f = 0$.

Suppose the statement holds for all quadratic forms containing fewer than n indeterminants. We write f as

$$f(X_1, \dots, X_n) = aX_1^2 + f_2(X_2, \dots, X_n)X_1 + f_3(X_2, \dots, X_n),$$

where $a \in \mathbb{K}$, $f_2 \in \mathbb{K}[X_2, \dots, X_n]$ is a linear form, and $f_3 \in \mathbb{K}[X_2, \dots, X_n]$ is a quadratic form. As $\text{char}(\mathbb{K}) \neq 2$, we know that 0, 1, and -1 are distinct elements of \mathbb{K} . As f vanishes on \mathbb{K}^n , f vanishes at every point of the form $(\alpha, \alpha_2, \dots, \alpha_n) \in \mathbb{K}^n$ with $\alpha \in \{0, 1, -1\}$. Therefore, we obtain the three equations

$$\begin{aligned} 0 &= f(0, \alpha_2, \dots, \alpha_n) = f_3(\alpha_2, \dots, \alpha_n) \\ 0 &= f(1, \alpha_2, \dots, \alpha_n) = a + f_2(\alpha_2, \dots, \alpha_n) + f_3(\alpha_2, \dots, \alpha_n) \\ 0 &= f(-1, \alpha_2, \dots, \alpha_n) = a - f_2(\alpha_2, \dots, \alpha_n) + f_3(\alpha_2, \dots, \alpha_n) \end{aligned}$$

for all $(\alpha_2, \dots, \alpha_n) \in \mathbb{K}^{n-1}$. By induction hypothesis, the first equation implies that $f_3 = 0$. Then the second and third equations imply that $a = 0$ and that $f_2(\alpha_2, \dots, \alpha_n) = 0$ for all $(\alpha_2, \dots, \alpha_n) \in \mathbb{K}^{n-1}$. Let $f_2 = a_2X_2 + \dots + a_nX_n$. If $f_2 \neq 0$, then there exists $a_i \neq 0$. Setting $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$, we obtain $0 = f_2(0, \dots, 0, 1, 0, \dots, 0) = a_i \neq 0$, a contradiction. Thus, $f_2 = 0$, and so $f = 0$. The lemma is proved.

Indeed, this lemma implies that $t = 0$. Therefore, Q factors into a product of two linear forms, each defining a hyperplane in \mathbb{K}^n . We now proceed based on whether $r = 1$ or 2. If $r = 1$, then $\mathcal{Z}(Q)$ is a hyperplane of \mathbb{K}^n . Thus, both factors of Q must define $\mathcal{Z}(Q)$, and so they are non-zero scalar multiples of one another. If instead $r = 2$, then $\mathcal{Z}(Q)$ is the union of two hyperplanes of \mathbb{K}^n , corresponding to the two factors of Q . As the hyperplanes are distinct, the factors are not scalar multiples of one another. Therefore (5) \Rightarrow (4).

This concludes the proof of the theorem.

5 Related terminology in the literature

We now present a table of sources that utilize the statements from Theorems 1 and 2. The letters used in the table are as follows: D for *degenerate*, S for *singular*, and R for *reducible*. The columns labeled (1) to (5) refer to the corresponding statements from the theorems. Entries with a * indicate that the source mentions the statement, but does not use a particular term to describe it. In addition to providing the reader with additional materials, we use this table to illustrate the variety of ways in which the words *degenerate*, *reducible*, and *singular* are used to refer to these statements. In particular, we encourage the reader to note that each word is used to describe multiple statements, and that statements (1), (2), (4), and (5) are each referred to by multiple words!

A few additional notes will prove useful before studying the table. While some sources consider forms (often in the $n = 3$ case), others focus on conics in the classical projective plane $PG(2, q)$. As conics are simply quadrics in projective space, it is not surprising that the characterizations of degenerate quadrics and degenerate conics are nearly identical. Thus, we will not make any further attempt to distinguish them. In addition, we found many sources that

discuss equivalent notions, but not in the context of quadratic forms or conics; they have therefore been excluded from this table.

Finally, we emphasize that we intend this table to illustrate vocabulary usage in some of the sources that are available to us; the decision to exclude (or include) a source should not be interpreted as a criticism (or endorsement).

Table 1: This table illustrates the variety of ways in which the words degenerate (D), reducible (R), and singular (S) are used in the literature to refer to the statements in Theorems 1 and 2. A * indicates that a source refers to the statement, but does not use a particular term to describe it.

| | (1) | (2) | (3) | (4) | (5) |
|------|-----|--------|-----|--------|-----|
| [1] | | D or S | | | |
| [2] | | | | R | * |
| [3] | | | | D | D |
| [4] | S | | | | |
| [6] | | | | | R |
| [8] | | S | | | |
| [9] | * | | | D | |
| [10] | * | | S | R | R |
| [11] | S | * | | | |
| [15] | * | * | S | | |
| [16] | S | | | | |
| [19] | * | | * | | * |
| [20] | D | | | | |
| [23] | * | | | D | D |
| [24] | * | | S | | |
| [25] | * | | | D | D |
| [26] | | | | * | R |
| [28] | D | | | | |
| [29] | * | | * | D or R | * |
| [30] | D | * | | | |
| [31] | S | | | | |
| [33] | * | | | | D |
| [35] | * | | S | R | |

6 Concluding remarks

The discussion in previous sections leads to many interesting questions. We will briefly describe some of them.

What about similar studies of higher order forms? Let $1 \leq d \leq n$, $f \in \mathbb{F}[X_1, \dots, X_n]$ such that $f = \sum_{(i_1, \dots, i_d)} a_{i_1 \dots i_d} X_{i_1} \cdots X_{i_d}$, where all $a_{i_1 \dots i_d} \in$

\mathbb{F} , and summation is taken over all integer sequences (i_1, \dots, i_d) , $1 \leq i_1 \leq i_2 \leq \dots \leq i_d$. Then f is called a d -form of n indeterminants. 1-forms and 2-forms were discussed in the previous sections as linear and quadratic forms, respectively. When $d \geq 3$, the coefficients $a_{i_1 \dots i_d}$ can be considered as entries of a d -dimensional matrix, which has n^d entries total. Such matrices and their determinants have been studied for more than three hundred years, and by many mathematicians, including Cayley, Sylvester, Weierstrass, Garbieri, Gegenbauer, Dedekind, Lecat, Oldenburger, and Sokolov. The time and space necessary to define related notions and results is much more than this article allows, and we refer the reader to a monograph by Sokolov [32]. It contains a complete classification of forms over the fields of real and complex numbers for $d = 3$ and $n = 2$ or 3 , but discussion even of these cases is far from short. (The classification is with respect to the non-singular linear transformation of variables). While the monograph is in Russian, it contains 231 references in a variety of languages.

The question of factorization of a general d -form with n indeterminants, over a field for $d \geq 3$, seems to be highly non-trivial, and we could not find any useful criteria for this.

The question of classification of quadratic forms over the integers with respect to the unimodular linear transformation of variables is classical, and significant progress was made in this direction in the 18th and 19th century, culminating with the work of Gauss. Its source is the problem of representing integers by a given quadratic form with integer coefficients. For example:

Let $Q(X, Y) = X^2 - XY + 5Y^2$. Describe all ordered triples of integers (n, a, b) such that $n = Q(a, b)$.

For related results and their extensions for forms over other rings, see O'Meara [27], Buell [7], and Conway [12]. For new directions and results related to multi-dimensional determinants, d -forms, see Gelfand, Kapranov, and Zelevinsky [13]. For cubic forms in algebra, geometry and number theory, see Manin [22].

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