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**HANDOUT 3**

**THE AXIOMS OF THE REALS**

**MATH 021 FUNDAMENTALS OF MATHEMATICS**

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Some basics on the real line (\( \mathbb{R} \)):

There is *no* centre (e.g.: the nonsense about \( \infty + (-\infty) = 0 \) one may have learnt in high school is a fallacy) so one can reasonably represent the line as:

-4 -3 -2 -1 0 1 2 3 4 or

-10 -9 -8 -7 -6 -5 -4 -3 -2 or

\( e \pi 10 \sqrt{111} 20 40 60 1,000 10^{20} \)

The reals (as all of math) begins with axioms:

**The Field Axioms of \( \mathbb{R} \)**

**Axiom 1** (closure of addition): \( \forall x, y \in \mathbb{R}, x + y \in \mathbb{R} \) and \( (x = w \land y = v) \Rightarrow (x + y = w + v) \)

**Axiom 2** (commutative of addition): \( \forall x, y \in \mathbb{R}, x + y = y + x. \)

**Axiom 3** (associative of addition): \( \forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z) \)

**Axiom 4** (existence of identity of addition): \( \exists \) a unique number 0 \( \ni x + 0 = x \forall x \in \mathbb{R} \)

**Axiom 5** (existence of additive inverse): \( \forall x \in \mathbb{R} \exists \) a unique number \(-x \ni x + (-x) = 0 \)

**Axiom 6** (closure of multiplication): \( \forall x, y \in \mathbb{R}, x \cdot y \in \mathbb{R} \) and \( (x = w \land y = v) \Rightarrow (x \cdot y = w \cdot v) \)

**Axiom 7** (commutative of multiplication): \( \forall x, y \in \mathbb{R}, x \cdot y = y \cdot x. \)

**Axiom 8** (associative of multiplication): \( \forall x, y, z \in \mathbb{R}, (x \cdot y) \cdot z = x \cdot (y \cdot z) \)

**Axiom 9** (existence of identity of multiplication): \( \exists \) a unique number 1 \( \ni x \cdot 1 = x \forall x \in \mathbb{R} \)
Axiom 10 (existence of multiplicative inverse): \( \forall x \in \mathbb{R} \ 3 x \neq 0 \exists \text{ a unique number } x^{-1} \)
\[ \exists x \cdot (x^{-1}) = 1 \]

Axiom 11 (distributive of multiplication over addition): \( \forall x, y, z \in \mathbb{R}, x \cdot (y + z) = (x \cdot y) + (x \cdot z) \)

The Order Axioms of \( \mathbb{R} \)

Axiom 12 (trichotomy): \( \forall x, y \in \mathbb{R}, \) exactly one of the following relationships exists between \( x \) and \( y \):
\[ x < y, \  x = y, \  \vee \ x > y. \ \ [(x < y) \ \text{exor } (x = y) \ \text{exor } (x > y)] \]

Axiom 13 (transitive): \( \forall x, y, z \in \mathbb{R}, [ (x < y) \ \land \ (y < z) ] \Rightarrow (x < z) \)

Axiom 14 (preservation of order under addition): \( \forall x, y, z \in \mathbb{R}, (x < y) \Rightarrow (x + z < y + z) \)

Axiom 15 (preservation of order for positive multiplier): \( \forall x, y \in \mathbb{R}, [(x < y) \ \land \ (0 < z)] \Rightarrow (x \cdot z < y \cdot z) \)

The Completeness Axiom of \( \mathbb{R} \)

Axiom 16 (completeness): \( \forall A \subseteq \mathbb{R} \ 3 A \text{ is bounded above } \exists \text{ a number } m \text{ which is the supremum of the set } \)