Journal of Combinatorial Theory, Series A 124 (2014) 97–113

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

# Diagonal forms and zero-sum (mod 2) bipartite Ramsey numbers

Tony W.H. Wong

*Department of Mathematics, Kutztown University of Pennsylvania, United States*

ARTICLE INFO

*Article history:*
Received 18 October 2012
Available online 28 January 2014

*Keywords:*
Diagonal forms
Smith normal forms
Zero-sum Ramsey
Signed bipartite graph designs

ABSTRACT

Let $G$ be a subgraph of a complete bipartite graph $K_{n,n}$. Let $N(G)$ be a 0-1 incidence matrix with edges of $K_{n,n}$ against images of $G$ under the automorphism group of $K_{n,n}$. A diagonal form of $N(G)$ is found for every $G$, and the question as to whether the row space of $N(G)$ over $\mathbb{Z}_p$ contains the vector of all 1's is settled. This implies a new proof of Caro and Yuster's results on zero-sum bipartite Ramsey numbers, and provides necessary and sufficient conditions for the existence of a signed bipartite graph design.



## 1. Introduction

Let $G$ be a nonempty subgraph of the complete bipartite graph $K_{n,n}$ with $2n$ vertices. Note that some of the vertices of $G$ are possibly isolated. Let $\mathbf{h}$ be the characteristic vector of $G$, i.e. $\mathbf{h}$ is a column vector of length $n^2$ indexed by the edges of $K_{n,n}$, with the $i$-th entry equal to 1 if the corresponding edge of $K_{n,n}$ is contained in $G$ and 0 otherwise. Let $\Pi$ be the automorphism group on the vertices of $K_{n,n}$. Let $N = N(G)$ be the matrix with $2(n!)^2$ columns, each column representing an image of $\mathbf{h}$ under the action of $\Pi$ on the vertices.

*E-mail address:* wong@kutztown.edu.

0097-3165/$ – see front matter 
http://dx.doi.org/10.1016/j.jcta.2014.01.007

For any integer matrix $A$, there exist integer square matrices $E$ and $F$ with determinants $\pm 1$ such that $EAF = D$ is a diagonal matrix, i.e. the $(i, j)$-entry of $D$ is 0 unless $i = j$. This $D$ is called a *diagonal form* of $A$. If all diagonal entries $d_1, d_2, \ldots$ are non-negative and $d_i$ divides $d_{i+1}$ for all $i$, then $D$ is called the *Smith normal form* of $A$ which is unique with respect to $A$.

This paper is going to give the diagonal forms of $N(G)$ for all $G$'s (see Theorems 3.2 and 4.2), which can be used to reproduce the results on zero-sum (mod 2) bipartite Ramsey numbers given in [3], and give necessary and sufficient conditions for the existence of a signed bipartite graph design. Some techniques in this paper were introduced in [8] and [9].

## 2. Diagonal forms of $U \cdot N(G)$

Two integer matrices $A$ and $B$ of the same size are $\mathbb{Z}$*-equivalent* if $B$ can be obtained from $A$ by a sequence of integral row and column operations (adding an integer multiple of one row or column to another row or column, or multiplying a row or column by $-1$). Alternatively, $A$ and $B$ are $\mathbb{Z}$-equivalent if there exist integer square matrices $E$ and $F$ with determinants $\pm 1$ such that $EAF = B$. If $A$ is $\mathbb{Z}$-equivalent to a diagonal matrix $D$, i.e. $EAF = D$ where $E$ and $F$ are integer square matrices with determinants $\pm 1$, then $D$ is called a *diagonal form* of $A$, $E$ a *front* and $F$ a *back* of $A$. We will call the set of diagonal entries of $D$ *a set of diagonal factors* of $A$.

If $A$ is $\mathbb{Z}$-equivalent to an identity matrix, then $A$ is said to be *unimodular*. In fact, it is easy to see that a square integer matrix is unimodular if and only if its determinant is $\pm 1$. If there is a submatrix $A'$ of $A$ obtained by deleting some columns such that $A'$ is $\mathbb{Z}$-equivalent to an identity matrix, then $A$ is said to be *row-unimodular*. Any row-unimodular matrix $A$ has a *unimodular extension* $\bar{A}$, which is an extension of $A$ by adding rows below $A$ and is unimodular.

Let $W$ be a $2n \times n^2$ incidence matrix of $K_{n,n}$ with vertices against edges, where the first $n$ rows are indexed by vertices in one partite set, while the last $n$ rows are indexed by vertices in the other. Let $U$ be a $(2n-1) \times n^2$ matrix obtained from $W$ with the first row replaced by $\mathbf{1}$, the vector of all ones, and the last row deleted. It is not difficult to see that $U$ and $W$ have the same row space over $\mathbb{Q}$. The matrix $U$ is row-unimodular since $U$ has a submatrix

$$\begin{array}{|c|c|c|}
\hline
\mathbf{1}_{n-1} & 1 & \mathbf{1}_{n-1} \\
\hline
O_{(n-1)\times(n-1)} & \mathbf{0}_{n-1}^{\top} & I_{(n-1)\times(n-1)} \\
\hline
\multirow{2}{*}{$I_{(n-1)\times(n-1)}$} & \multirow{2}{*}{$\mathbf{0}_{n-1}^{\top}$} & \mathbf{1}_{n-1} \\
\cline{3-3}
 & & O_{(n-2)\times(n-1)} \\
\hline
\end{array}$$

which is $\mathbb{Z}$-equivalent to $I_{(2n-1)\times(2n-1)}$. Here, and throughout the paper, $\mathbf{1}_i$ and $\mathbf{0}_i$ always denote row vectors of all ones and all zeros respectively of length $i$, and $O_{i\times j}$ denotes a zero matrix of dimensions $i$ by $j$.

Let $G$ be a nonempty subgraph of the complete bipartite graph $K_{n,n}$ with degrees $a_1,\dots,a_n,b_1,\dots,b_n$, where the $a_i$'s are the degrees of the vertices in one partite set and the $b_i$'s are those in the other one. Note that some of the $a_i$'s and $b_i$'s are possibly zeroes. The case where $n=1$ is uninteresting, so throughout the paper, we only consider $n\geqslant 2$.

Let $\mathbf{h}$ be the characteristic vector of $G$, i.e. $\mathbf{h}$ is a column vector of length $n^2$ indexed by the edges of $K_{n,n}$, with 1 if the edge is in $G$ and 0 otherwise. Let $\Pi\cong S_n\wr_{\{a,b\}} S_2$ be the permutation group on the vertices of $K_{n,n}$ which permutes vertices within each partite set and interchange the two partite sets. Let $N=N(G)$ be the matrix with $2(n!)^2$ columns, each column representing an image of $\mathbf{h}$ under the action of $\Pi$ on the vertices.

In $U\cdot N(G)$, each column is either $(m,a_{i_2},a_{i_3}\dots,a_{i_n},b_{j_2},b_{j_3},\dots,b_{j_n})^\top$ or $(m,b_{i_2},b_{i_3}\dots,b_{i_n},a_{j_2},a_{j_3},\dots,a_{j_n})^\top$, where $m$ is the number of edges of $G$ and $\{i_1,i_2,\dots,i_n\}=\{j_1,j_2,\dots,j_n\}=\{1,2,\dots,n\}$. Pick two columns in $UN$ that are identical except one entry, e.g. $(m,a_1,a_3\dots,a_n,b_1,\dots,b_{n-1})^\top$ and $(m,a_2,a_3,\dots,a_n,b_1,\dots,b_{n-1})^\top$. Taking the difference of these two columns, we get $(0,a_1-a_2,0,\dots,0)^\top$ in the column module of $UN$ over $\mathbb{Z}$. Hence, the column module of $UN$ contains $g\mathbf{e}_k^\top$, $k=2,\dots,2n-1$, where $g=\mathrm{GCD}\{a_i-a_j,b_i-b_j\}$ over $1\leqslant i,j\leqslant n$ and $\mathbf{e}_k$ denotes the $k$-th standard unit vector of length $2n-1$. So the matrix

$$\begin{array}{|c|c|c|}\hline m & m & \mathbf{0}_{2n-2} \\ \hline a_1\mathbf{1}_{n-1}^\top & b_1\mathbf{1}_{n-1}^\top & \\ \cline{1-2} b_1\mathbf{1}_{n-1}^\top & a_1\mathbf{1}_{n-1}^\top & gI_{(2n-2)\times(2n-2)} \\ \hline\end{array}$$

has the same column module as $UN$. After some integral row and column operations, we can see that the matrix

$$\begin{array}{|c|c|c|}\hline m & 0 & \mathbf{0}_{2n-3} \\ \hline a_1 & \tilde{g} & \mathbf{0}_{2n-3} \\ \hline \mathbf{0}_{n-2}^\top & \mathbf{0}_{n-2}^\top & \\ \cline{1-2} (a_1+b_1)\mathbf{1}_{n-1}^\top & \mathbf{0}_{n-1}^\top & gI_{(2n-3)\times(2n-3)} \\ \hline\end{array} \qquad (1)$$

has the same diagonal form as $UN$, where

$$\tilde{g} = \mathrm{GCD}\{a_i - b_j, a_i - a_j, b_i - b_j\} = \mathrm{GCD}\{a_1 - b_1, g\}.$$

By computing the determinantal divisors (see [6] for details), we have the following theorem.

**Theorem 2.1.** *A set of diagonal factors of $UN$ is*

$$\left(\frac{mg}{hc}\right)^1, \quad (\tilde{g}c)^1, \quad (h)^1, \quad (g)^{2n-4},$$

*where $h = \mathrm{GCD}\{a_i, b_j\} = \mathrm{GCD}\{a_1, \tilde{g}\}$ and $c = \mathrm{GCD}\{\frac{m}{h}, \frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\}$. Here, $(x)^y$ means the diagonal entry $x$ occurs with multiplicity $y$. (Note that $h \neq 0$ since $G$ is nonempty. Note also that $\tilde{g} = 0$ if and only if $g = 0$, and in this case, we define $\frac{g}{\tilde{g}} = 1$.)*

**Proof.** The GCD of the determinants of $1 \times 1$ submatrices in (1) is $\mathrm{GCD}\{a_1, \tilde{g}\} = h$. The GCD of the determinants of $i \times i$ submatrices in (1), $2 \leqslant i \leqslant 2n-2$, is $\mathrm{GCD}\{m\tilde{g}g^{i-2}, \tilde{g}g^{i-1}, a_1 g^{i-1}, \tilde{g}g^{i-2}(a_1+b_1)\} = \tilde{g}g^{i-2}hc$. The determinant of the full matrix in (1) is $m\tilde{g}g^{2n-3}$.

As a result, we are done by computing the determinantal divisors, since the $i$-th diagonal entry of a diagonal form is given by the quotient of the GCD of the determinants of $i \times i$ submatrices divided by the GCD of the determinants of $(i-1)\times(i-1)$ submatrices. Here, a $0 \times 0$ matrix is defined to have determinant 1. □

**Lemma 2.2.** *For any $x, y \in \mathbb{Z}$, there exists $a, b \in \mathbb{Z}$ such that $\mathrm{GCD}\{a, x, y\} = 1$ and $ax + by = \mathrm{GCD}\{x, y\}$.*

**Proof.** Let $\mathrm{GCD}\{x, y\} = d$, $x = x'd$ and $y = y'd$. Let $a_0, b_0 \in \mathbb{Z}$ be such that $a_0 x' + b_0 y' = 1$. Note that $\mathrm{GCD}\{a_0, y'\} = 1$. Our goal is to find an integer $a$ such that $a \equiv a_0 \pmod{y'}$ and $\mathrm{GCD}\{a, d\} = 1$.

Without loss of generality, we can assume that all prime factors of $d$ are of degree 1. Let $d = d_1 d_2$ such that $d_1 \mid y'$ and $\mathrm{GCD}\{d_2, y'\} = 1$. As $\mathrm{GCD}\{a_0, y'\} = 1$, it suffices to find $a$ such that $a \equiv a_0 \pmod{y'}$ and $a \equiv 1 \pmod{d_2}$, which is possible by the Chinese remainder theorem. □

**Theorem 2.3.** *By Lemma 2.2, let $\alpha, \sigma \in \mathbb{Z}$ be such that $\mathrm{GCD}\{\alpha, \frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} = 1$ and $-\alpha\frac{a_1+b_1}{h} + \sigma\frac{g}{\tilde{g}} = \mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\}$. Then there exist $\ell, \ell' \in \mathbb{Z}$ such that $\frac{\ell'}{c}\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} - \ell\alpha\frac{m}{hc} = 1$. Let $\beta, \tau \in \mathbb{Z}$ be such that $-\beta a_1 + \tau\tilde{g} = h$. Let $\ell'' = \beta\frac{\ell m + \ell'(a_1+b_1)}{h}$. A front $E$ of $UN$ is*

| | | | | |
|---|---|---|---|---|
| $\frac{1}{c}\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\}$ | $\alpha\frac{m}{hc} + \sigma\beta\frac{mg}{hc\tilde{g}}$ | $\mathbf{0}_{n-2}$ | $\alpha\frac{m}{hc}$ | $\mathbf{0}_{n-2}$ |
| $\ell$ | $\ell' + \ell''$ | $\mathbf{0}_{n-2}$ | $\ell'$ | $\mathbf{0}_{n-2}$ |
| $0$ | $1$ | $\mathbf{0}_{n-2}$ | $0$ | $\mathbf{0}_{n-2}$ |
| $\mathbf{0}^\top_{n-2}$ | $-\mathbf{1}^\top_{n-2}$ | $I_{(n-2)\times(n-2)}$ | $\mathbf{0}^\top_{n-2}$ | $O_{(n-2)\times(n-2)}$ |
| $\mathbf{0}^\top_{n-2}$ | $\mathbf{0}^\top_{n-2}$ | $O_{(n-2)\times(n-2)}$ | $-\mathbf{1}^\top_{n-2}$ | $I_{(n-2)\times(n-2)}$ |

,

*where the first three rows correspond to the diagonal entries $\frac{mg}{hc}$, $\tilde{g}c$ and $h$ respectively, and the other rows correspond to the diagonal entries $g$.*

**Proof.** We first show that $E$ is unimodular. Given that $\mathrm{GCD}\{\alpha, \frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} = 1$, we have $\mathrm{GCD}\{\alpha\frac{m}{h}, \frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} = c$, or $\mathrm{GCD}\{\alpha\frac{m}{hc}, \frac{1}{c}\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\}\} = 1$. Hence, there exist $\ell, \ell' \in \mathbb{Z}$, $\mathrm{GCD}\{\ell, \ell'\} = 1$, such that $\frac{\ell'}{c}\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} - \ell\alpha\frac{m}{hc} = 1$. So the submatrix

| | | |
|---|---|---|
| $\frac{1}{c}\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\}$ | $\alpha\frac{m}{hc} + \sigma\beta\frac{mg}{hc\tilde{g}}$ | $\alpha\frac{m}{hc}$ |
| $\ell$ | $\ell' + \ell''$ | $\ell'$ |
| $0$ | $1$ | $0$ |

has determinant $-1$, which implies $E$ is unimodular.

From the derivation of (1), we note that the column module of $UN$ is the same as that of

$$M = \left[\begin{array}{c|c|c} m & 0 & \mathbf{0}_{2n-3} \\ \hline a_1\mathbf{1}^\top_{n-1} & \tilde{g}\mathbf{1}^\top_{n-1} & \begin{array}{c} \mathbf{0}_{2n-3} \\ \hline \\ gI_{(2n-3)\times(2n-3)} \end{array} \\ \cline{1-2} b_1\mathbf{1}^\top_{n-1} & -\tilde{g}\mathbf{1}^\top_{n-1} & \end{array}\right].$$

The product of the first row of $E$ with the first column of $M$ is

$$\begin{aligned} &\frac{m}{c}\left(\mathrm{GCD}\left\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\right\} + \alpha\frac{a_1+b_1}{h} + \sigma\beta\frac{a_1 g}{h\tilde{g}}\right) \\ &= \frac{m}{c}\left(-\alpha\frac{a_1+b_1}{h} + \sigma\frac{g}{\tilde{g}} + \alpha\frac{a_1+b_1}{h} + \sigma\beta\frac{a_1 g}{h\tilde{g}}\right) \\ &= \sigma\frac{mg}{hc\tilde{g}}(h + \beta a_1) = \sigma\frac{mg}{hc\tilde{g}}(-\beta a_1 + \tau\tilde{g} + \beta a_1) \\ &= \sigma\tau\frac{mg}{hc}. \end{aligned}$$

The product of the first row of $E$ with the second column of $M$ is $\sigma\beta\frac{mg}{hc}$, and the product of the first row of $E$ with the $(n+1)$-th column of $M$ is $\alpha\frac{mg}{hc}$. From the definition, it is clear that $\mathrm{GCD}\{\beta,\tau\} = \mathrm{GCD}\{\alpha,\sigma\} = 1$, so $\mathrm{GCD}\{\sigma\tau,\sigma\beta,\alpha\} = 1$, and the first row of $E$ corresponds to the diagonal entry $\frac{mg}{hc}$.

The product of the second row of $E$ with the first column of $M$ is

$$\ell m + \ell'(a_1+b_1) + \ell'' a_1 = \frac{\ell m + \ell'(a_1+b_1)}{h}(h+\beta a_1) = \frac{\ell m + \ell'(a_1+b_1)}{hc}\tau\tilde{g}c.$$

The product of the second row of $E$ with the second column of $M$ is $\ell''\tilde{g} = \beta\frac{\ell m+\ell'(a_1+b_1)}{hc}\tilde{g}c$, and the product of the second row of $E$ with the $(n+1)$-th column of $M$ is $\ell' g = \ell'\frac{g}{\tilde{g}c}\tilde{g}c$. Recall that $\mathrm{GCD}\{\tau,\beta\} = 1$. Note that $\mathrm{GCD}\{\frac{\ell m+\ell'(a_1+b_1)}{h}, \ell'\frac{g}{\tilde{g}}\}$ divides

$$-\ell\alpha\frac{m}{h} + \ell'\left(-\alpha\frac{a_1+b_1}{h} + \sigma\frac{g}{\tilde{g}}\right) = -\ell\alpha\frac{m}{h} + \ell'\mathrm{GCD}\left\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\right\} = c,$$

so $\mathrm{GCD}\{\tau\frac{\ell m+\ell'(a_1+b_1)}{hc}, \beta\frac{\ell m+\ell'(a_1+b_1)}{hc}, \ell'\frac{g}{\tilde{g}c}\} = 1$, and the second row of $E$ corresponds to the diagonal entry $\tilde{g}c$.

Finally, it is obvious that the third row of $E$ corresponds to the diagonal entry $h$, and all the other rows correspond to the diagonal entries $g$. □

## 3. Primitivity

Let $u$, $u'$, $v$, $v'$ be four distinct vertices of $K_{n,n}$ such that $u$ and $u'$ are in the same partite set while $v$ and $v'$ are in the other one. Let $\mu_{\{u,v\}}$ be a row vector of length $n^2$, indexed by the edges of $K_{n,n}$, such that the entry corresponding to the edge $\{u,v\}$ is 1 and all other entries are 0. Let $\mathbf{v}_{u,u',v,v'} = \mu_{\{u,v\}} + \mu_{\{u',v'\}} - \mu_{\{u,v'\}} - \mu_{\{u',v\}}$. Such a vector is called a 2-*pod* over the tuple $(u,u',v,v')$. If $\mathbf{h}$ is a characteristic vector of a nonempty subgraph $G$ of $K_{n,n}$, then we say $G$ or $\mathbf{h}$ is *primitive* if $\mathrm{GCD}(\mathbf{v}N(G)) = 1$, where $\mathbf{v}$ is any 2-pod.

**Proposition 3.1.** *The collection of* 2*-pods* $\mathbf{v}_{u,u',v,v'}$ *over all tuples* $(u,u',v,v')$ *spans over* $\mathbb{Z}$ *all the integer vectors in the null space of* $U$.

**Proof.** Let $\mathbf{w}$ be an integer vector in the null space of $U$, or equivalently $W$. Let $\mathring{K}_{i,j}$ denote a graph with integer multiplicities on each edge of $K_{i,j}$ such that the degree at each vertex is 0. Then $\mathbf{w}$ corresponds to a $\mathring{K}_{n,n}$, and $\mathring{K}_{2,2}$ corresponds to a scalar multiple of some $\mathbf{v}_{u,u',v,v'}$. We will prove that any $\mathring{K}_{i,j}$, $i,j \geqslant 2$, can be decomposed into $\mathring{K}_{2,2}$'s by induction on $i+j$.

Let the two partite sets of $\mathring{K}_{i,j}$ be $\{u_1,\dots,u_i\}$ and $\{v_1,\dots,v_j\}$. Let $w(u_s,v_t)$ denote the multiplicity of the edge $\{u_s,v_t\}$. When $i=j=2$, the decomposition is trivial. Assume that any $\mathring{K}_{i,j}$ can be decomposed into $\mathring{K}_{2,2}$'s for some $i,j \geqslant 2$. In $\mathring{K}_{i+1,j}$, $u_1$ is incident to

$j$ edges with multiplicities $w(u_1, v_1), \dots, w(u_1, v_j)$. Then $\mathbf{w} + \sum t = 2^j w(u_1, v_t)\mathbf{v}_{u_1,u_2,v_1,v_t}$ will still correspond to a $\mathring{K}_{i+1,j}$, but all the edges incident to $u_1$ will have multiplicity 0, so this $\mathring{K}_{i+1,j}$ is equivalent to a $\mathring{K}_{i,j}$. By induction hypothesis, $\mathring{K}_{i+1,j}$ can be decomposed into $\mathring{K}_{2,2}$'s. A similar proof will work for $\mathring{K}_{i,j+1}$. □

**Theorem 3.2.** *If* $\mathbf{h}$ *is primitive, a set of diagonal factors of* $N$ *is*

$$\left(\frac{mg}{hc}\right)^1, \quad (\tilde{g}c)^1, \quad (h)^1, \quad (g)^{2n-4}, \quad (1)^{(n-1)^2},$$

*and a corresponding front can be any unimodular extension* $\tilde{E}$ *of* $EU$*, where* $E$ *is defined in Theorem 2.3.*

We proceed to prove Theorem 3.2 by first introducing a number of lemmas. The proofs of Lemmas 3.3 and 3.4 can be found in [8] and [4] respectively.

**Lemma 3.3.** *Let* $A$ *be an* $r \times m$ *integer matrix. Suppose* $EA = DA'$ *for some unimodular* $E$*, diagonal* $D$ *and integral* $A'$*. Let* $E_i$ *be the i-th row of* $E$ *and* $d_i$ *be the i-th diagonal entry of* $D$*. If the conditions* $E_i\mathbf{b} \equiv 0 \pmod{d_i}$ *for* $i = 1, \dots, r$ *are sufficient for the existence of an integer solution* $\mathbf{b}$ *of* $A\mathbf{x} = \mathbf{b}$*, then there exists a unimodular matrix* $F$ *such that* $EAF = D$*.*

**Lemma 3.4.** *Given a rational matrix* $A$ *and a column vector* $\mathbf{b}$*, the system* $A\mathbf{x} = \mathbf{b}$ *has an integer solution* $\mathbf{x}$ *if and only if for any rational row vector* $\mathbf{y}$*,*

$$\mathbf{y}A \equiv \mathbf{0} \pmod 1 \quad \textit{implies} \quad \mathbf{y}\mathbf{b} \equiv 0 \pmod 1.$$

**Lemma 3.5.** *If* $\mathbf{h}$ *is primitive, then any rational row vector* $\mathbf{y}$ *satisfying* $\mathbf{y}N \equiv \mathbf{0} \pmod 1$ *can be expressed as* $\mathbf{y} \equiv \mathbf{z}U \pmod 1$ *for some rational vector* $\mathbf{z}$*.*

**Proof.** Let $\mathbf{h}_\pi$ be the image of $\mathbf{h}$ under the permutation $\pi \in \Pi$, the automorphism group on the vertices of $K_{n,n}$. For every 2-pod $\mathbf{v} = \mathbf{v}_{u,u',v,v'}$, let $\mathbf{h}_{\pi(u,u')}$, $\mathbf{h}_{\pi(v,v')}$ and $\mathbf{h}_{\pi(u,u')(v,v')}$ be the images of $\mathbf{h}_\pi$ under the permutations $(u, u')$, $(v, v')$ and $(u, u')(v, v')$ respectively. By direct computation, $(\mathbf{v} \cdot \mathbf{h}_\pi)\mathbf{v}^\top = \mathbf{h}_\pi + \mathbf{h}_{\pi(u,u')(v,v')} - \mathbf{h}_{\pi(u,u')} - \mathbf{h}_{\pi(v,v')}$, i.e. $(\mathbf{v} \cdot \mathbf{h}_\pi)\mathbf{v}^\top$ is in the column module of $N$ over $\mathbb{Z}$. As $\mathbf{h}$ is primitive, $\mathrm{GCD}\{\mathbf{v} \cdot \mathbf{h}_\pi\colon \pi \in \Pi\} = 1$. Hence, $\mathbf{v}^\top$ will be in the column module of $N$ over $\mathbb{Z}$ by Euclidean algorithm.

Consequently, $\mathbf{v}\mathbf{y}^\top = \mathbf{y}\mathbf{v}^\top \equiv 0 \pmod 1$ since $\mathbf{y}N \equiv \mathbf{0} \pmod 1$. Note that $\mathbf{v}$ can run through all 2-pods, so $V\mathbf{y}^\top \equiv \mathbf{0} \pmod 1$, where $V$ is a matrix whose rows are all the 2-pods $\mathbf{v}$. By Proposition 3.1, the row space of $V$ is the same as the null space of $U$.

We claim that there is an integer solution $\mathbf{x}$ to $V\mathbf{x} = V\mathbf{y}^\top$. For any rational row vector $\mathbf{w}$ such that $\mathbf{w}V \equiv \mathbf{0} \pmod 1$, $\mathbf{w}V$ is an integer vector in the null space of $U$.

By Proposition 3.1, $\mathbf{w}V = \mathbf{w}'V$ for some integer vector $\mathbf{w}'$, so $\mathbf{w}V\mathbf{y}^\top = (\mathbf{w}'V)\mathbf{y}^\top \equiv \mathbf{w}'(V\mathbf{y}^\top) \equiv 0 \pmod 1$. Our claim then follows from Lemma 3.4.

Let $\mathbf{x}$ be our integer solution to $V\mathbf{x} = V\mathbf{y}^\top$, or $V(\mathbf{y}^\top - \mathbf{x}) = \mathbf{0}$. This implies that $\mathbf{y} - \mathbf{x}^\top$ is in the null space of $V$. In other words, $\mathbf{y} - \mathbf{x}^\top$ is in the row space of $U$, so $\mathbf{y} = \mathbf{z}U + \mathbf{x}^\top$ for some rational vector $\mathbf{z}$, i.e. $\mathbf{y} \equiv \mathbf{z}U \pmod 1$. □

**Lemma 3.6.** *If* $\mathbf{h}$ *is primitive,* $N\mathbf{x} = \mathbf{b}$ *has an integer solution* $\mathbf{x}$ *if and only if* $UN\mathbf{x}' = U\mathbf{b}$ *has an integer solution* $\mathbf{x}'$*.*

**Proof.** The direction "only if" is trivial. Assume that $\mathbf{x}'$ is an integer solution of $UN\mathbf{x}' = U\mathbf{b}$. Let $\mathbf{y}$ be a rational row vector such that $\mathbf{y}N \equiv \mathbf{0} \pmod 1$. By Lemma 3.5, $\mathbf{y} \equiv \mathbf{z}U \pmod 1$ for some rational $\mathbf{z}$. Then $\mathbf{y}\mathbf{b} \equiv \mathbf{z}U\mathbf{b} = \mathbf{z}UN\mathbf{x}' \equiv \mathbf{y}N\mathbf{x}' \equiv 0 \pmod 1$. By Lemma 3.4, we are done. □

**Proof of Theorem 3.2.** Let $d_1 = \frac{mg}{hc}$, $d_2 = \tilde{g}c$, $d_3 = h$ and $d_i = g$ for $i = 4, 5, \dots, 2n-1$. As $E$ is a front of $UN$, there exists a back $F$ such that $EUNF = D$ where $D$ is a diagonal matrix with the $d_i$'s as its diagonal entries.

As $EU$ is row-unimodular, let $\tilde{E}$ be a unimodular extension of $EU$ with rows $\tilde{E}_i$, $i = 1, \dots, n^2$. Suppose $\tilde{E}_i\mathbf{b} \equiv 0 \pmod{d_i}$ for $i = 1, \dots, 2n-1$ and $\tilde{E}_i\mathbf{b} \equiv 0 \pmod 1$ for $i = 2n, \dots, n^2$. Note that the first $2n-1$ congruences are equivalent to $E_iU\mathbf{b} \equiv 0 \pmod{d_i}$ for $i = 1, \dots, 2n-1$, which implies $EU\mathbf{b} = D\mathbf{x}'' = EUNF\mathbf{x}''$ for some integer vector $\mathbf{x}''$. Hence, $UN\mathbf{x}' = U\mathbf{b}$ has an integer solution $\mathbf{x}' = F\mathbf{x}''$ since $E$ is invertible. By Lemma 3.6, $N\mathbf{x} = \mathbf{b}$ has an integer solution $\mathbf{x}$. By Lemma 3.3, $\tilde{E}$ is a front of $N$ and $d_1, \dots, d_{2n-1}, (1)^{(n-1)^2}$ are the corresponding diagonal factors. □

Let $\mathbf{h}$ be primitive. Let $\mathbf{x}$ be a row vector of length $n^2$, with $\ell'$ in the first entry, $-\alpha\frac{m}{hc}$ in the second, $\ell''\alpha\frac{m}{hc} - \ell'\sigma\beta\frac{mg}{hc\tilde{g}}$ in the third, and 0's in the rest. Consider $\mathbf{x}\tilde{E}N = \mathbf{x}DF$, where $\tilde{E}$ is the front and $D$ is the diagonal form in Theorem 3.2, and $F$ is a unimodular matrix. Then L.H.S. $= \mathbf{1}N = m\mathbf{1}$, and

$$\begin{aligned}
\text{R.H.S.} &= \ell'\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 + \left(\ell''\alpha\frac{m}{hc} - \ell'\sigma\beta\frac{mg}{hc\tilde{g}}\right)hF_3 \\
&= \ell'\frac{mg}{hc}F_1 - \alpha\frac{m\tilde{g}}{h}F_2 + \beta\frac{m}{c}\left(\frac{\ell m + \ell'(a_1+b_1)}{h}\alpha - \ell'\sigma\frac{g}{\tilde{g}}\right)F_3 \\
&= \ell'\frac{mg}{hc}F_1 - \alpha\frac{m\tilde{g}}{h}F_2 + \beta\frac{m}{c}\left(\ell\alpha\frac{m}{h} - \ell'\operatorname{GCD}\left\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\right\}\right)F_3 \\
&= \ell'\frac{mg}{hc}F_1 - \alpha\frac{m\tilde{g}}{h}F_2 - \beta mF_3,
\end{aligned}$$

where $F_i$ represents the $i$-th row of $F$. As L.H.S. = R.H.S., we have $\ell'\frac{g}{hc}F_1 - \alpha\frac{\tilde{g}}{h}F_2 - \beta F_3 = \mathbf{1}$, which is the unique way to get $\mathbf{1}$ in the row space of $F$ over any $\mathbb{F}_p$ since $F$ is unimodular. From now on, we use $\mathrm{row}_p(A)$ to denote the row space of an integer matrix $A$ over $\mathbb{F}_p$.

**Theorem 3.7.** *If* $\mathbf{h}$ *is primitive and* $p$ *is a prime such that* $p \mid m$, *where* $m$ *is the number of edges in* $G$, *then* $\mathbf{1}$ *is in* $\mathrm{row}_p(N)$ *if and only if one of the following holds*:

(i) $p \nmid h$ *and* $p \mid \tilde{g}$,
(ii) $p = 2$, $p \nmid \tilde{g}$ *and* $p \mid g$,
(iii) $p \neq 2$, $p \nmid \tilde{g}$, $p \mid g$ *and* $p \nmid a_1 + b_1$.

**Proof.** Note that $\mathrm{row}_p(N)$ is equal to $\mathrm{row}_p(\tilde{E}N)$, which in turn is the same as $\mathrm{row}_p(DF)$, so $\mathbf{1}$ is in $\mathrm{row}_p(N)$ if and only if $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ is in $\mathrm{row}_p(DF)$. Again, let $d_1 = \frac{mg}{hc}$, $d_2 = \tilde{g}c$ and $d_3 = h$, which are the first three diagonal entries of $D$.

*Case* 1: $p \mid h$.

By Lemma 2.2, $\beta$ can be chosen in Theorem 2.3 such that $\mathrm{GCD}\{\beta, h\} = 1$, so the coefficient of $F_3$ in $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ is non-zero in $\mathbb{F}_p$. However, $F_3$ is not in $\mathrm{row}_p(DF)$ since $d_3 = 0$ in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is not in $\mathrm{row}_p(N)$.

*Case* 2: $p \nmid h$ and $p \mid \tilde{g}$.

In this case, $F_3$ is in $\mathrm{row}_p(DF)$. Note that $\frac{\tilde{g}}{h} = \frac{g}{h(g/\tilde{g})}$ which divides $\frac{g}{hc}$, so the coefficients of $F_1$ and $F_2$ in $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ are 0's in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is in $\mathrm{row}_p(N)$.

*Case* 3: $p \nmid \tilde{g}$ and $p \mid g$.

Again, $F_3$ is in $\mathrm{row}_p(DF)$. As $\tilde{g} = \mathrm{GCD}\{a_1 - b_1, g\}$, we have $p \nmid a_1 - b_1$. If $p = 2$, then $p \nmid a_1 + b_1$, so $p \nmid c$, and hence the coefficient of $F_1$ in $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ is 0 in $\mathbb{F}_p$. Also, $F_2$ is in $\mathrm{row}_p(DF)$ since $d_2 = \tilde{g}c$ which is non-zero in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is in $\mathrm{row}_p(N)$.

Next, consider the case where $p \neq 2$. If $p \nmid a_1 + b_1$, then by the same argument, $\mathbf{1}$ is in $\mathrm{row}_p(N)$. If $p \mid a_1 + b_1$, then $p \mid c$, so $F_2$ is not in $\mathrm{row}_p(DF)$. However, $\mathrm{GCD}\{\alpha, \frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} = 1$ implies $p \nmid \alpha$, so the coefficient of $F_2$ in $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ is non-zero in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is not in $\mathrm{row}_p(N)$.

*Case* 4: $p \nmid g$.

As $p \nmid c$, $\ell' \mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} - \ell\alpha \frac{m}{h} = c$ implies $p \nmid \ell'$, so the coefficient of $F_1$ in $\ell' \frac{g}{hc} F_1 - \alpha \frac{\tilde{g}}{h} F_2 - \beta F_3$ is non-zero in $\mathbb{F}_p$. However, $F_1$ is not in $\mathrm{row}_p(DF)$ since $d_1 = 0$ in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is not in $\mathrm{row}_p(N)$. $\square$

## 4. The non-primitive case

**Proposition 4.1.** *Let $G$ be a nonempty subgraph of the complete bipartite graph $K_{n,n}$ with $2n$ vertices. Then $G$ is non-primitive if and only if $G$ is $K_{n,n}$, $K_{s,n} \sqcup \{n-s$ isolated vertices$\}$ or $K_{s,t} \sqcup K_{n-s,n-t}$ for some $1 \leqslant s,t \leqslant n-1$.*

**Proof.** Let $\{u_1,\dots,u_n\}$ and $\{v_1,\dots,v_n\}$ be two partite sets of $G$. We use $u \leftrightarrow v$ to denote $\{u,v\}$ is an edge in $G$, and use $u \not\leftrightarrow v$ to denote $\{u,v\}$ is not.

If $G$ is $K_{n,n}$, then it is obviously non-primitive. If $G$ is non-primitive but not $K_{n,n}$, then without loss of generality, $u_1$, $u_2$, $v_1$, $v_2$ satisfy one of the following two cases:

(a) $u_1 \leftrightarrow v_1$ and $u_1 \leftrightarrow v_2$, while $u_2 \not\leftrightarrow v_1$ and $u_2 \not\leftrightarrow v_2$;
(b) $u_1 \leftrightarrow v_1$ and $u_2 \leftrightarrow v_2$, while $u_1 \not\leftrightarrow v_2$ and $u_2 \not\leftrightarrow v_1$.

If (a) occurs, then for any $u \neq u_1,u_2$, either $u \leftrightarrow v_i$ for both $i \in \{1,2\}$ or $u \not\leftrightarrow v_i$ for both $i \in \{1,2\}$, so $\Gamma(v_1)=\Gamma(v_2)$, where $\Gamma(x)$ denotes all the neighbors of vertex $x$. Now, for any $v \neq v_1,v_2$, either $v \leftrightarrow u$ for all $u \in \Gamma(v_1)$ and $v \not\leftrightarrow u'$ for all $u' \notin \Gamma(v_1)$, or $v \not\leftrightarrow u$ for all $u \in \Gamma(v_1)$ and $v \leftrightarrow u'$ for all $u' \notin \Gamma(v_1)$. Hence, $G$ is either $K_{s,n} \sqcup \{n-s$ isolated vertices$\}$ or $K_{s,t} \sqcup K_{n-s,n-t}$ for some $1 \leqslant s,t \leqslant n-1$.

If (b) occurs, then for any $u \neq u_1,u_2$, exactly one of $u \leftrightarrow v_1$ and $u \leftrightarrow v_2$ occurs. As a result, $\Gamma(v_1)$ and $\Gamma(v_2)$ form a partition of $\{u_1,\dots,u_n\}$. Now, for any $v \neq v_1,v_2$, either $v \leftrightarrow u$ for all $u \in \Gamma(v_1)$ and $v \not\leftrightarrow u'$ for all $u' \in \Gamma(v_2)$, or $v \not\leftrightarrow u$ for all $u \in \Gamma(v_1)$ and $v \leftrightarrow u'$ for all $u' \in \Gamma(v_2)$. Hence, $G$ is $K_{s,t} \sqcup K_{n-s,n-t}$ for some $1 \leqslant s,t \leqslant n-1$. □

**Theorem 4.2.** *If $G$ is non-primitive, then diagonal forms of $G$ are given by the following*:

| | $G$ | A set of diagonal factors of $N(G)$ |
|---|---|---|
| (a) | $K_{n,n}$ | $(1)^1$, $(0)^{n^2-1}$ |
| (b) | $K_{s,n} \sqcup \{n-s$ isolated vertices$\}$<br>$1 \leqslant s \leqslant n-1$ | $(s)^1$, $(h)^1$, $(1)^{2n-3}$, $(0)^{(n-1)^2}$<br>$h=\mathrm{GCD}\{n,s\}$ |
| (c) | $K_{s,t} \sqcup K_{n-s,n-t}$<br>$1 \leqslant s \leqslant t \leqslant n-1$ | $(\frac{2((n-s)(n-t)+st)\tilde{g}}{h\delta})^1$, $(h)^1$, $(\delta\tilde{g})^1$, $(2\tilde{g})^{2n-4}$, $(2)^{(n-2)^2}$, $(1)^{2n-3}$<br>$\tilde{g}=\mathrm{GCD}\{n-2s,t-s\}$, $h=\mathrm{GCD}\{n,s,t\}$, $\delta=\mathrm{GCD}\{\frac{n-t+s}{h},2\}$ |

**Proof.** (a) is trivial.

(b) and (c). The diagonal forms in these cases are obtained by explicit integral row and column operations. Please refer to [10] for the details. □

**Theorem 4.3.** *If* **h** *is non-primitive,* **1** *is in* $\mathrm{row}_p(N)$ *if and only if one of the following holds, where $p \mid m$ in each case*:

(i) *$G$ is $K_{n,n}$,*

(ii) *$G$ is $K_{s,n} \sqcup \{n-s$ isolated vertices$\}$ with $p \nmid s$,*
(iii) *$G$ is $K_{s,t} \sqcup K_{n-s,n-t}$ with $p \nmid h$ and $p \mid \frac{2\tilde{g}}{\delta}$, where $h = \mathrm{GCD}\{n,s,t\}$, $\tilde{g} = \mathrm{GCD}\{n-2s, t-s\}$, $\delta = \mathrm{GCD}\{\frac{n-t+s}{h}, 2\}$.*

**Proof.** (i) Every row of $N$ is $\mathbf{1}$, so $\mathbf{1}$ is in $\mathrm{row}_p(N)$.

(ii) By keeping track of the row operations in the proof of part (b) in Theorem 4.2, a front $E$ of $N$ is

$$\left[\begin{array}{c|c|c|c} P & Q & \cdots & Q \\ \hline R & S & & \\ \hline \vdots & & \ddots & \\ \hline R & & & S \end{array}\right],$$

where

$$P = \left[\begin{array}{c|c|c} 1 & \mathbf{1}_{n-2} & -(n-2) \\ \hline \mathbf{0}_{n-2}^{\top} & I_{(n-2)\times(n-2)} & -\mathbf{1}_{n-2}^{\top} \\ \hline 0 & \mathbf{0}_{n-2} & 1 \end{array}\right], \qquad Q = \left[\begin{array}{c|c} & 1 \\ \cline{2-2} O_{n\times(n-1)} & \mathbf{0}_{n-2}^{\top} \\ \cline{2-2} & 1 \end{array}\right],$$

$$R = \left[\begin{array}{c|c} -I_{(n-1)\times(n-1)} & \mathbf{1}_{n-1}^{\top} \\ \hline \mathbf{0}_{n-1} & 0 \end{array}\right], \qquad S = \left[\begin{array}{c|c} I_{(n-1)\times(n-1)} & -\mathbf{1}_{n-1}^{\top} \\ \hline \mathbf{0}_{n-1} & 1 \end{array}\right],$$

and there are $n$ horizontal sections. The first row of $E$ corresponds to the diagonal entry $s$, the $n$-th row corresponds to the diagonal entry $h$, the second to the $(n-1)$-th row and the $(in)$-th row, $2 \leqslant i \leqslant n$, correspond to the diagonal entries 1, and the rest correspond to 0.

Let $EN = DF$. Note that the first row of $EN$ is $s\mathbf{1}$, so $F_1 = \mathbf{1}$, where $F_1$ is the first row of $F$. Therefore, $\mathbf{1}$ is in $\mathrm{row}_p(N)$ if and only if $p \nmid s$ since the first entry of $D$ is $s$.

(iii) Let $\theta, \phi, \xi \in \mathbb{Z}$ be such that $\theta n + \phi s + \xi t = h = \mathrm{GCD}\{n,s,t\}$. A front $E$ of $N$ in part (c) of Theorem 4.2 is

$$\begin{array}{|c|c|c|c|c|}
\hline
P' & P' & \dots & P' & Q' \\
\hline
 & R' & & & S' \\
\hline
 & & \ddots & & \vdots \\
\hline
 & & & R' & S' \\
\hline
T' & T' & \dots & T' & U' \\
\hline
\end{array},$$

where

$$P' = \begin{array}{|c|c|c|}
\hline
1 & \mathbf{1}_{n-2} & \omega_1 \\
\hline
\mathbf{0}_{n-2}^\top & I_{(n-2)\times(n-2)} & -\mathbf{1}_{n-2}^\top \\
\hline
0 & \mathbf{0}_{n-2} & 1+\mu \\
\hline
\end{array},$$

$$Q' = \begin{array}{|c|c|c|}
\hline
\omega_2 & \omega_2\mathbf{1}_{n-2} & \omega_3 \\
\hline
\mathbf{0}_{n-2}^\top & (1-(n-2s))I_{(n-2)\times(n-2)} & ((n-2s)-1)\mathbf{1}_{n-2}^\top \\
\hline
\mu & \mu\mathbf{1}_{n-2} & \omega_4 \\
\hline
\end{array},$$

$$R' = \begin{array}{|c|c|c|}
\hline
1 & \mathbf{1}_{n-2} & 1-(n-2s) \\
\hline
\mathbf{0}_{n-2}^\top & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^\top \\
\hline
0 & \mathbf{0}_{n-2} & 1 \\
\hline
\end{array},$$

$$S' = \begin{array}{|c|c|c|}
\hline
-1 & -\mathbf{1}_{n-2} & (n-2s)-1 \\
\hline
\mathbf{0}_{n-2}^\top & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^\top \\
\hline
0 & \mathbf{0}_{n-2} & 1 \\
\hline
\end{array},$$

$$T' = \begin{array}{|c|c|}
\hline
 & 1-\frac{\nu}{h}(1+\mu) \\
\cline{2-2}
O_{n\times(n-1)} & \mathbf{0}_{n-1}^\top \\
\hline
\end{array},$$

$$U' = \begin{array}{|c|c|c|} \hline 1-\frac{\nu}{h}\mu & (1-\frac{\nu}{h}\mu)\mathbf{1}_{n-2} & \omega_5 \\ \hline \mathbf{0}_{n-2}^{\top} & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^{\top} \\ \hline 0 & \mathbf{0}_{n-2} & 1 \\ \hline \end{array},$$

and $m = (n-s)(n-t)+st$, $\mu = \frac{2\theta+\phi+\xi-1}{2}$, $\nu = n-t+s$, $\omega_1 = 1-\frac{m}{h}[1+(1-\frac{\nu}{h})(1+\mu)]$, $\omega_2 = 1-\frac{m}{h}[1+(1-\frac{\nu}{h})\mu]$, $\omega_3 = 1-\frac{m}{h}(\omega_4+\omega_5)$, $\omega_4 = 1-(n-2s)+(2-(n-2s))\mu$, $\omega_5 = 2-(n-2s)-\frac{\nu}{h}\omega_4$. The first row of $E$ corresponds to the diagonal entry $\frac{2m\tilde{g}}{h\delta}$, the $n$-th row corresponds to the diagonal entry $h$, the $(n(n-1)+1)$-th row corresponds to the diagonal entry $\delta\tilde{g}$, the second to the $(n-1)$-th row and the $(1+in)$-th row, $1 \leqslant i \leqslant n-2$, correspond to the diagonal entries $2\tilde{g}$, the $(i+jn)$-th row, $2 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n-2$, corresponds to the diagonal entry 2, while the $(in)$-th row, $2 \leqslant i \leqslant n-1$, and the $((n-1)n+2)$-th to the last row correspond to the diagonal entries 1.

This matrix works as a front since the first, $n$-th and $(n(n-1)+1)$-th rows come from the row operations in the proof of part (c) of Theorem 4.2. As for the other rows, we can multiply to $N$ directly to check.

Now, if we multiply $(1, \underbrace{0,\ldots,0}_{n-2\ 0\text{'s}}, \frac{m}{h}, \underbrace{0,\ldots,0}_{n(n-1)\ 0\text{'s}}, \frac{m}{h}, \underbrace{0,\ldots,0}_{n-1\ 0\text{'s}})$ to the left of both sides of $EN = DF$, we will get $m\mathbf{1} = \frac{2m\tilde{g}}{h\delta}F_1 + \frac{m}{h}hF_n + \frac{m}{h}\delta\tilde{g}F_{n(n-1)+1}$, where $F_i$ denotes the $i$-th row in $F$. By dividing $m$ from both sides of the equation, we have $\mathbf{1} = \frac{2\tilde{g}}{h\delta}F_1 + F_n + \frac{\delta\tilde{g}}{h}F_{n(n-1)+1}$, which is the unique way to obtain $\mathbf{1}$ in the row space of $F$ over any field $\mathbb{Z}_p$ since $F$ is unimodular.

*Case* 1: $p \mid h$.

The coefficient of $F_n$ in $\frac{2\tilde{g}}{h\delta}F_1 + F_n + \frac{\delta\tilde{g}}{h}F_{n(n-1)+1}$ is non-zero in $\mathbb{F}_p$. However, $F_n$ is not in $\text{row}_p(DF)$ since the $n$-th entry of $D$ is $h$ which is 0 in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is not in $\text{row}_p(N)$.

*Case* 2: $p \nmid h$ and $p \nmid \frac{2\tilde{g}}{\delta}$.

The coefficient of $F_1$ in $\frac{2\tilde{g}}{h\delta}F_1 + F_n + \frac{\delta\tilde{g}}{h}F_{n(n-1)+1}$ is non-zero in $\mathbb{F}_p$. However, $F_1$ is not in $\text{row}_p(DF)$ since the first entry of $D$ is $\frac{2m\tilde{g}}{h\tilde{g}}$ which is 0 in $\mathbb{F}_p$. Therefore, $\mathbf{1}$ is not in $\text{row}_p(N)$.

*Case* 3: $p \nmid h$ and $p \mid \frac{2\tilde{g}}{\delta}$.

The coefficient of $F_1$ in $\frac{2\tilde{g}}{h\delta}F_1 + F_n + \frac{\delta\tilde{g}}{h}F_{n(n-1)+1}$ is 0 in $\mathbb{F}_p$. As $p \nmid h$, $F_n$ is in $\text{row}_p(DF)$. If $p \mid \delta\tilde{g}$, then the coefficient of $F_{n(n-1)+1}$ is also 0 in $\mathbb{F}_p$; if $p \nmid \delta\tilde{g}$, then $F_{n(n-1)+1}$ is in $\text{row}_p(DF)$ since the $(n(n-1)+1)$-th entry of $D$ is $\delta\tilde{g}$. Therefore, $\mathbf{1}$ is in $\text{row}_p(N)$. $\square$

## 5. Zero-sum (mod 2) bipartite Ramsey numbers

Let $G$ be a simple nonempty bipartite graph with $m$ edges. A $p$-coloring on the edges of $G$ is a function $c : E(G) \to \mathbb{Z}_p$. If $\sum_{e \in E(G)} c(e) = 0$ over $\mathbb{Z}_p$, then we say that $G$ is a *zero-sum* (mod $p$) graph with respect to $c$. If $p \mid m$, then the *zero-sum bipartite Ramsey number* $B_p(G)$ is the smallest integer $n$ such that, for every $p$-coloring on the edges of $K_{n,n}$, there exists a zero-sum (mod $p$) copy of $G$ in $K_{n,n}$.

Zero-sum Ramsey problems were first studied by Bialostocki and Dierker [2] and Alon and Caro [1], and the zero-sum (mod 2) bipartite Ramsey numbers are fully characterized by Caro and Yuster [3] in the following theorem. We are going to provide our own proof here.

**Theorem 5.1.** *(See [3].) Let $G$ be a simple nonempty bipartite graph with an even number of edges. Let $n$ be the minimum number such that the vertices of $G$ can be divided into two partite sets, each of size not exceeding $n$. Then isolated vertices are added to $G$ if necessary to make each partite set of $G$ have size $n$. Let $B_2(G)$ denote the zero-sum bipartite Ramsey number of $G$ modulo 2. Then $B_2(G) = n + 1$ if and only if one of the following holds:*

(i) *$G$ is primitive with all degrees odd,*
(ii) *$G$ is primitive such that any representation of $G$, where each partite set has size $n$, has all degrees in one partite set odd and all degrees in the other partite set even,*
(iii) *$G = K_{n,n}$,*
(iv) *$G = K_{s,n} \sqcup \{n - s \text{ isolated vertices}\}$ with $s$ odd,*
(v) *$G = K_{s,t} \sqcup K_{n-s,n-t}$ with $n$ even and at least one of $s$ and $t$ is odd.*

*Otherwise, $B_2(G) = n$.*

**Proof.** Let $p = 2$. By Theorems 3.7 and 4.3, the vector of all 1's $\mathbf{1}$ is in $\mathrm{row}_2(N)$ if and only if one of these five cases hold. We will check the only unobvious case, which is Theorem 4.3(iii) implying Theorem 5.1(v). If $n$ is odd, then $\tilde{g}$ is also odd. As $p \nmid h$ and $p \mid \frac{2\tilde{g}}{\delta}$, $s$ and $t$ are of the same parity, which contradicts that $p$ divides $m = st + (n-s)(n-t)$. If $n$ is even, then $p$ always divides $m$. Since $p \nmid h$, at least one of $s$ and $t$ is odd. If $s$ and $t$ are of opposite parity, then $\delta$ is odd and $p \mid \frac{2\tilde{g}}{\delta}$. If both $s$ and $t$ are odd, then $\tilde{g}$ is even and again $p \mid \frac{2\tilde{g}}{\delta}$.

$B_2(G) > n$ if and only if there exists a 2-coloring on the edges of $K_{n,n}$ such that all isomorphic copies of $G$ in $K_{n,n}$ have color sum equal to 1 (mod 2). In other words, there exists a row vector $\mathbf{c}$ of length $n^2$, each entry recording the color of the corresponding edge in $K_{n,n}$, such that when it is multiplied to the left of the matrix $N$, we get a vector of all 1's over $\mathbb{Z}_2$. This is equivalent to the condition that $\mathbf{1}$ is in $\mathrm{row}_2(N)$, which happens if and only if one of these five cases hold. Note that when two more isolated vertices are added to $G$ so that $G$ is embedded in $K_{n+1,n+1}$, none of these five cases hold, so we always have $B_2(G) \leqslant n+1$. Combining these two directions, this theorem is proved. □

## 6. Signed bipartite graph designs

Let $G$ be a nonempty proper subgraph of the complete bipartite graph $K_{n,n}$ with $2n$ vertices, some of which are possibly isolated. Let $\mathcal{G}$ be the collection of subgraphs $G'$ of $K_{n,n}$ which are isomorphic to $G$. We say that there exists an $(n, G, \lambda)$-*signed bipartite graph design* if there exists $z : \mathcal{G} \to \mathbb{Z}$ such that, for each edge $e \in E(K_{n,n})$,

$$\sum_{\substack{G' \in \mathcal{G}: \\ E(G') \ni e}} z(G') = \lambda.$$

If $\lambda = 1$ and $z : \mathcal{G} \to \{0, 1\}$, then this problem is related to graph decompositions, which is studied by Wilson [7] and many others. Ushio [5] gives necessary and sufficient conditions for a complete bipartite graph to be decomposed into complete bipartite graphs. Here, the necessary and sufficient conditions for the existence of an $(n, G, \lambda)$-signed bipartite graph design are given.

**Theorem 6.1.** *Let $G$ be a nonempty proper subgraph of the complete bipartite graph $K_{n,n}$ with $2n$ vertices. If $G$ has only one connected component of size greater than 1, then there exists an $(n, G, \lambda)$-signed bipartite graph design if and only if all the following three conditions hold*:

(i) $h \mid \lambda n$,
(ii) $\tilde{g}c \mid \lambda n(\ell n + 2\ell' + \ell'')$,
(iii) $\frac{mg}{h} \mid \lambda n(n\mathrm{GCD}\{\frac{a_1+b_1}{h}, \frac{g}{\tilde{g}}\} + \alpha\frac{2m}{h} + \sigma\beta\frac{mg}{h\tilde{g}})$,

*where $\alpha$, $\beta$, $\sigma$, $\ell$, $\ell'$ and $\ell''$ are defined in Theorem 2.3.*

**Proof.** Note that there exists an $(n, G, \lambda)$-signed bipartite graph design if and only if there exists an integer solution $\mathbf{z}$ to $N(G)\mathbf{z} = \lambda\mathbf{1}^\top$. If $G$ is primitive, then by Theorem 3.2, $\tilde{E}NF = D$ for some unimodular matrix $F$, where $D$ contains the set of diagonal factors given in Theorem 3.2. Hence, $N\mathbf{z} = \lambda\mathbf{1}^\top$ has an integer solution $\mathbf{z}$ if and only if $\tilde{E}^{-1}DF^{-1}\mathbf{z} = \lambda\mathbf{1}^\top$ has an integer solution $\mathbf{z}$, and it is equivalent to $D\mathbf{z}' = \lambda\tilde{E}\mathbf{1}^\top$ has an integer solution $\mathbf{z}'$, which exists if and only if $d_i \mid \lambda E_i U\mathbf{1}^\top$ for $i = 1, 2, \ldots, 2n-1$, where $E_i$ is the $i$-th row of the front given in Theorem 2.3.

By definition, $U\mathbf{1}^\top = (n^2, n, \ldots, n)^\top$, so $\lambda E_i U\mathbf{1}^\top = 0$ which is divisible by $d_i$ for $i \geqslant 4$. When $i = 3$, 2 and 1, they correspond to the conditions (i), (ii) and (iii) respectively.

If $G$ is non-primitive, then $G = K_{s,n} \sqcup \{n - s \text{ isolated vertices}\}$. At this moment, condition (i) becomes trivial, since $h = \mathrm{GCD}\{n, s\}$. Condition (ii) is also trivial, since $g = \mathrm{GCD}\{n - 0, s - s\} = n$, $\tilde{g} = \mathrm{GCD}\{s - 0, g\} = h$, and $c = \mathrm{GCD}\{\frac{ns}{h}, \frac{s}{h}, \frac{n}{h}\} = 1$.

Finally, condition (iii) becomes $\frac{n^2 s}{h} \mid \lambda n(n\mathrm{GCD}\{\frac{s}{h}, \frac{n}{h}\} + \alpha\frac{2ns}{h} + \sigma\beta\frac{n^2 s}{h^2})$, which can be simplified into $\frac{s}{h} \mid \lambda(1 + \alpha\frac{2s}{h} + \sigma\beta\frac{ns}{h^2})$, or simply $\frac{s}{h} \mid \lambda$. This is equivalent to $s \mid \lambda n$ since $h = \mathrm{GCD}\{n, s\}$. It remains to verify that $s \mid \lambda n$ is the necessary and sufficient condition for the existence of an $(n, G, \lambda)$-signed bipartite graph design when $G = K_{s,n} \sqcup \{n - s$ isolated vertices$\}$.

Let $E$ be the front of $N(G)$ given in the proof of Theorem 4.3, and let $EN = DF$. Then the equation $N\mathbf{z} = \lambda\mathbf{1}^\top$ has an integer solution $\mathbf{z}$ if and only if $DF\mathbf{z} = \lambda E\mathbf{1}^\top$ has an integer solution $\mathbf{z}$, which in turn is equivalent to $D\mathbf{z}' = \lambda E\mathbf{1}^\top$ has an integer solution $\mathbf{z}'$. The column vector $\lambda E\mathbf{1}^\top$ has $\lambda n$ in the first and the $n$-th entries, $\lambda$ in the $(in)$-th entry, $2 \leqslant i \leqslant n$, and 0's in the rest. Recall from the proof of Theorem 4.3 that the first diagonal entry of $D$ is $s$, the $n$-th diagonal entry is $h$, the second to the $(n-1)$-th entries and the $(in)$-th entry, $2 \leqslant i \leqslant n$, are 1's, and the rest are 0's. Hence, $D\mathbf{z}' = \lambda E\mathbf{1}^\top$ has an integer solution if and only if $s \mid \lambda n$. □

**Corollary 6.2.** *If $G = K_{s,t}$, $1 \leqslant s \leqslant t \leqslant n$, then there exists an $(n, G, \lambda)$-signed bipartite graph design if and only if $st \mid \lambda n^2$.*

**Proof.** If $s = t = n$, then an $(n, G, \lambda)$-signed bipartite graph design always exists, and the condition $st \mid \lambda n^2$ becomes $n^2 \mid \lambda n^2$ which is trivial. If $t = n$ and $s < t$, then the result follows from the proof of Theorem 6.1 for $G = K_{s,n} \sqcup \{n - s$ isolated vertices$\}$.

If $t < n$, then $h = \mathrm{GCD}\{s, t\}$, $g = \tilde{g} = \mathrm{GCD}\{s, t\} = h$, $m = st$, and $c = \mathrm{GCD}\{\frac{st}{h}, 0, \frac{h}{h}\} = 1$. Condition (ii) in Theorem 6.1 becomes $h \mid \lambda n(\ell n + 2\ell' + \ell'')$, which is redundant since condition (i) gives $h \mid \lambda n$. Condition (iii) becomes $\frac{sth}{h} \mid \lambda n(n\mathrm{GCD}\{0, \frac{h}{h}\} + \alpha\frac{2st}{h} + \sigma\beta\frac{sth}{h^2})$, which can be simplified into $st \mid \lambda n^2 + (2\alpha + \sigma\beta)\frac{\lambda n}{h}st$, and can be further simplified into $st \mid \lambda n^2$ since $h \mid \lambda n$ by condition (i). Lastly, notice that $st \mid \lambda n^2$ implies $h^2 \mid \lambda n^2$, which in turn implies $h \mid \lambda n$. □

## Acknowledgments

I acknowledge Lawrence Chung for his help in proving Lemma 2.2, Richard Wilson for his help in proving Proposition 3.1, Peter Sin for pointing out that the permutation group on the vertices of $K_{n,n}$ is the wreath product, and the reviewers of this paper for all the constructive comments.

## References

[1] N. Alon, Y. Caro, On three zero-sum Ramsey-type problems, J. Graph Theory 17 (1993) 177–192.
[2] A. Bialostocki, P. Dierker, Zero-sum Ramsey theorems, Congr. Numer. 70 (1990) 119–130.
[3] Y. Caro, R. Yuster, The characterization of zero-sum (mod 2) bipartite Ramsey numbers, J. Graph Theory 29 (1998) 151–166.
[4] A. Schrijver, Theory of Linear and Integer Programming, Wiley, 1986.
[5] K. Ushio, Bipartite decomposition of complete multipartite graphs, Hiroshima Math. J. 11 (1981) 321–345.
[6] B.L. van der Waerden, Algebra, vol. 2, Springer, 2003.

[7] R.M. Wilson, Decompositions of complete graphs into subgraphs isomorphic to a given graph, in: Proc. 5th Brit. Combinatorial Conf., 1975, pp. 647–659.

[8] R.M. Wilson, Signed hypergraph designs and diagonal forms for some incidence matrices, Des. Codes Cryptogr. 17 (1999) 289–297.

[9] R.M. Wilson, T.W.H. Wong, Diagonal forms of incidence matrices associated with $t$-uniform hypergraphs, European J. Combin. 35 (2014) 490–508.

[10] T.W.H. Wong, Diagonal forms, linear algebraic methods and Ramsey-type problems, Dissertation (Ph.D.), California Institute of Technology, 2013.