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Diagonal forms for incidence matrices and zero-sum Ramsey theory

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Abstract

We consider integer matrices N_t whose rows are indexed by the t-subsets of an n-set and whose columns are all distinct images of a particular column under the symmetric group S_n. Examples include matrices in the association algebras of the Johnson schemes. Three related problems are addressed. What is the Smith normal form (or a diagonal form) for N_t and the rank of N_t over a field of characteristic p? When does the equation $N_t\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$ in integers? When is the vector of all ones in the row space of N_t over the field of characteristic p? Previous work provides answers to these questions when the columns of N_t have at least t "isolated vertices", but interesting problems arise when this is not the case.

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1 Introduction

By a *t-vector* based on a set X, we mean a vector $\mathbf{h}$ whose coordinates are indexed by the t-subsets of the set X. If $\mathbf{h}$ is a t-vector and T a t-subset of X, then let $\mathbf{h}(T)$ denote the entry of $\mathbf{h}$ in coordinate position T.

Let H be a given t-uniform hypergraph on a v-set X. The *characteristic t-vector* of H is defined as the t-vector $\mathbf{h}$ based on X such that $\mathbf{h}(T) = 1$ when T is a hyperedge of H and 0 otherwise.

For an integer t-vector $\mathbf{h}$ based on a v-set X, we consider the matrix $N_t(\mathbf{h})$, or simply N_t, whose columns are all images of $\mathbf{h}$ under the symmetric group S_v. If $\mathbf{h}$ is the characteristic t-vector of H, then we may also denote the matrix by $N_t(H)$. If H is composed of a clique of size k and $v-k$ isolated vertices, then $N_t(H)$ is the incidence matrix of t-subsets against k-subsets of the v-set X, denoted by W^v_{tk}, or W_{tk} if the v-set is understood.

Let H have e edges. The zero-sum Ramsey theorem asserts that for any prime p dividing e, there exists a smallest integer $\mathrm{ZR}(H,p)$ such that for all $n \geq \mathrm{ZR}(H,p)$, for any coloring of the t-subsets of an n-set S by $\{0,1,\ldots,p-1\}$, there exists an isomorphic copy of H in $\binom{S}{t}$ such that the sum of the colors on its edges is 0 in $\mathbb{Z}_p$. In other words, there exists a smallest integer $\mathrm{ZR}(H,p)$ such that after extending H by adding $\mathrm{ZR}(H,p) - v$ isolated vertices, every vector in the row module of $N_t(H)$ over $\mathbb{Z}_p$ has at least one 0 entry.

2 Diagonal forms

If the rows of an integer matrix M are linearly independent over any field, we say M is *row-unimodular*. The term *unimodular matrix* is used for a square row-unimodular matrix. Every row-unimodular matrix M has *unimodular extensions*, i.e. unimodular matrices F whose row set includes the rows of M.

It is also convenient to consider Smith form and diagonal form of r by m matrix A as a square matrix of order r for any $m \geq r$. We say a square diagonal matrix D is a *diagonal form* for A when there is a unimodular matrix E of order r and an r by m row-unimodular matrix U so that

$$\boxed{E}\ \boxed{\quad A \quad} = \boxed{D}\ \boxed{\quad U \quad}.$$

The *Smith normal form* of an integer matrix A is the unique diagonal form so that the diagonal entries $d_1, d_2, \ldots, d_r$ are nonnegative and d_i divides d_{i+1} for $i = 1, 2, \ldots, r-1$.

The diagonal forms of a given matrix A is important in studying the row module over $\mathbb{Z}$. In fact, we have the property that

$$\mathbb{Z}^m/\mathrm{row}_{\mathbb{Z}}(A) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r} \oplus \mathbb{Z}^{m-r}.$$

Here, $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_1 = \{0\}$. Hence, by computing the diagonal form of A, we can investigate the row module over $\mathbb{Z}$ or row space over $\mathbb{Z}_p$.

3 Primitive and non-primitive graphs

A t-vector $\mathbf{h}$ is *primitive* when the GCD of $\langle \mathbf{f}, \mathbf{h} \rangle$ over all integer t-vectors $\mathbf{f}$ in the null space of $W_{t-1,t}$ is equal to 1. A t-uniform hypergraph is said to be *primitive* when its characteristic t-vector $\mathbf{h}$ is primitive. This concept of primitivity of hypergraphs appears implicitly in earlier work, e.g. [5].

Theorem 3.1 *A simple graph G with at least four vertices is primitive unless G is isomorphic to a complete graph, an edgeless graph, a complete bipartite graph, or a disjoint union of two complete graphs.*

Theorem 3.2 *Let G be a simple primitive graph with n vertices, m edges and degrees $\delta_1, \ldots, \delta_n$. Let $h = \gcd(\delta_1, \ldots, \delta_n, m)$. Let g denote the* gcd *of all differences $\delta_i - \delta_j$. Then a diagonal form of $N_2(G)$ is*

$$(1)^{\binom{n}{2}-n}, \quad (h)^1, \quad (g)^{n-2}, \quad (mg/h)^1.$$

Theorem 3.3 *Let G be a simple non-primitive graph with at least four vertices. Then diagonal forms of G can be given by the following.*

	G	*a diagonal form of* $N_2(G)$
(a)	K_n	$(1)^1,\ (0)^{\binom{n}{2}-1}$
(b)	*edgeless*	$(0)^{\binom{n}{2}}$
(c)	$K_{1,n-1}$	$(2)^1,\ (1)^{n-1},\ (0)^{\binom{n}{2}-n}$
(d)	$K_{r,n-r}$ $2 \le r \le \frac{n}{2}$	$\left(\frac{mg}{h}\right)^1,\ (h)^1,\ (2g)^{n-2},\ (2)^{\binom{n}{2}-(2n-2)},\ (1)^{n-2}$ $m = r(n-r),\ g = n-2r,\ h = \gcd(r,n)$
(e)	$K_1 \sqcup K_{n-1}$	$(n-2)^1,\ (1)^{n-1},\ (0)^{\binom{n}{2}-n}$
(f)	$K_r \sqcup K_{n-r}$ $2 \le r \le \frac{n}{2}$	*n odd:* $\left(\frac{2mg}{h}\right)^1,\ (h)^1,\ (2g)^{n-2},\ (2)^{\binom{n}{2}-(2n-1)},\ (1)^{n-1}$ *n even:* $\left(\frac{mg}{h}\right)^1,\ (2h)^1,\ (2g)^{n-2},\ (2)^{\binom{n}{2}-(2n-1)},\ (1)^{n-1}$ $m = \binom{r}{2} + \binom{n-r}{2},\ g = n-2r,\ h = \gcd(r-1,g,m)$

4 Relations with zero-sum Ramsey theory

Theorem 4.1 *Given a simple graph G with at least four vertices and a prime p such that $p|m$, the vector of all ones $\mathbf{j}$ is in the row space of $N_2(G)$ over $\mathbb{Z}_p$ if and only if either of the following holds:*
(i) G is primitive with $p|g$ but $p \nmid h$,
(ii) $G = K_n$,
(iii) $G = K_{1,n-1}$ with $p > 2$,
(iv) $G = K_1 \sqcup K_{n-1}$ with $p \nmid n-2$ or $p = 2$,
(v) $G = K_r \sqcup K_{n-r}$, $2 \leq r \leq \frac{n}{2}$, with $(p|g$ but $p \nmid h)$ or $p = 2$.

In particular, when $p = 2$, we have $\mathrm{ZR}(G, 2) = n$ unless G has all degrees odd, $G = K_n$ or $G = K_r \sqcup K_{n-r}$ for some $1 \leq r \leq \frac{n}{2}$, which agrees with the results in [1].

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