4.5 GENERALIZED PERMUTATIONS AND COMBINATIONS

Some counting problems involve repetition.

SEQUENCES with UNRESTRICTED REPETITION

Prop 4.5.1. *The number of sequences of length r selected from a set of n objects is n^r.*

Proof: Rule of Product. ♢

Example 4.5.1: There are 26^4 four-letter alphabetic strings in the English alphabet: $AAAA, AAAB, \ldots, ZZZZ$.

Example 4.5.2: There are 10^3 (unsigned) base-ten numerals with three or fewer digits: 0 (means 000), 1 (means 001), . . ., 999.

Example 4.5.3: Most license plates in New Jersey are formed by an ordered pair of 3-strings whose characters are either letters or digits. There are 36^6 such pairs.

SEQUENCES with RESTRICTED REPETITION

Prop 4.5.2. *Let $S = \{a_1, a_2, \ldots, a_k\}$, and let $n = n_1 + n_2 + \cdots + n_k$. The number of length-$n$ sequences in S with n_j occurrences of object a_j, for $j = 1, \ldots, k$ is*

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Proof: Rule of Quotient. $\diamondsuit$

Example 4.5.4: How many different strings can be formed by rearranging the letters of the word BANANA?
Solution: 3A, 1B, 2N.

$$\frac{6!}{3!1!2!} = 60$$

Example 4.5.5: How many different strings can be formed by rearranging the letters of the word BOOKKEEPER?
Solution: 1B, 3E, 2K, 2oh, 1P, 1R.

$$\frac{10!}{1!3!2!2!1!1!}$$

COMBINATIONS with UNRESTRICTED REPETITION

Example 4.5.6: Three dice are rolled. Count the number of possible combinations of values.

NOTATION: Represent a dice outcome as a non-descending 3-sequence in the set $\{1, 2, \ldots, 6\}$.

Method 1. One at a time.

111	112	113	114	115	116				
122	123	124	125	126	133	134	135	136	
144	145	146	155	156	166				
222	223	224	225	226	233	234	235	236	
244	245	246	255	256	266				
333	334	335	336	344	345	346	355	356	366
444	445	446	455	456	466	555	556	566	666

$7 \cdot 6 \ + \ 6 \ + \ 8 = \mathbf{56}$

Method 2. By partition structure.

$$\begin{cases} \binom{6}{1} = 6 & \text{3-of-a-kind xxx} \\ \binom{6}{1}\binom{5}{1} = 30 & \text{one-pair xxy} \\ \binom{6}{3} = 20 & \text{all different xyz} \end{cases}$$

$6 + 30 + 20 = \mathbf{56}$

Method 3. By an ingenious bijection, which is a formal name for pigeonholing.

Domain D: nondescending triples in $\{1, ..., 6\}$.

Codomain R: 8-bit strings with exactly three 0's

Strategy

Domain D models the set to be counted.
Codomain R is easy to count.
Construct a bijection $f : D \to R$, and count R.

Fact 1. *The set of triples* $[t_1, t_2, t_3]$ *correspond bijectively to the set of 6-tuples* $(k_1, k_2, \ldots, k_6)$ *such that*

$$k_j = \#\textit{occurrences of } j \textit{ in } [t_1, t_2, t_3]$$

and $\sum_{j=1}^{6} k_j = 3$.

Proof: Each such 6-tuple is the image of a unique triple. $\diamondsuit$

Example 4.5.7: $[2, 2, 4] \mapsto (0, 2, 0, 1, 0, 0)$.

Fact 2. *The set of 6-tuples* $(k_1, k_2, \ldots, k_6)$ *as in Fact 1 correspond bijectively to the set of length-8 bitstrings with 5 ones and 3 zeroes.*

Proof: This algorithm is the bijection:

Algorithm 4.5.1: tuples to bitstrings

Input: a 6-tuple $(k_1, k_2, \ldots, k_6)$ as in Fact 1
Output: a bitstring of length $6 + 3 - 1$,
with 3 zeroes

Initialize output string $s := \lambda$

For $j := 1$ **to** 5
  For $i := 1$ **to** k_j $\{s := s \cdot 0\}$
  $s := s \cdot 1$
  continue with next j

For $i := 1$ **to** k_6 $\{s := s \cdot 0\}$

Example 4.5.8:
$[2, 2, 4] \mapsto (0, 2, 0, 1, 0, 0) \mapsto 10011011.$

Conclusion: The number of possible outcomes of a roll of three indistinguishable dice is $\binom{8}{3}$

Theorem 4.5.3. *The number of ways to choose r objects from a set of cardinality n, if unrestricted repetitions are allowed, is*

$$\binom{n+r-1}{r}$$

Proof: Generalize the example above. $\diamondsuit$

CLASSROOM EXERCISE

How many possible combinations of seven coins can be formed from U.S. coins presently in circulation? 1, 5, 10, 25, 50, 100

ANSWER: