Section 4.1
The Basics of Counting

THE RULE OF SUM

If A and B are disjoint sets then $|A \cup B| = |A| + |B|

Example:

Suppose statement labels in a programming language must be a single letter or a single decimal digit.

Since a label cannot be both at the same time,

the number of labels

= the number of letters + the number of decimal digits

= 26 + 10 = 36.

THE RULE OF PRODUCT

$|A \times B| = |A| \times |B|$
Example:

- Statement labels in Basic can be either
  - a single letter or
  - a letter followed by a digit.

Find the number of possible labels.

We can partition the set of all labels $L$ into the disjoint subsets consisting of

- the set of single letter labels $S$

and

- the set of single letters followed by a digit $D$

and

- $L = S \cup D$.

Use the rule of sum to compute the cardinality of $L$ if we can compute the cardinality of $D$.

- The elements of $D$ are ordered pairs of the form $[a, d]$ where $a$ is an alphabetic character and $d$ is a digit.

- By the rule of product the cardinality of $D$ is the product of the cardinality of the two sets:

$$= (26)(10)$$
= 260.

The cardinality of \( L \) is \( 26 + 260 = 286 \).

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**THE PRINCIPLE OF INCLUSION-EXCLUSION**

If \( A \) and \( B \) are not disjoint:

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

Don't count objects in the intersection of two sets more than once!

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Example:

Count the number of bit strings of length 4 which begin with a 1 or end with a 00.

The set can be expressed as the union of

- the subset \( S \) of strings which begin with 1

and

- the subset \( O \) that end in 00.

Unfortunately the two subsets overlap.

- The cardinality of \( S \) is 8 (why?)

- The cardinality of \( O \) is 4 (why?).
Hence, by the exclusion-inclusion principle, the cardinality of the union is 12 minus the cardinality of the intersection.

How many strings are in the intersection?

Those strings that begin with 1 and end in 00 or 2 such strings.

The total number is $10 = 8 + 4 - 2$.

Check:

- Strings in S that begin with 1:
  1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111
- Strings O that end with 00:
  0000, 0100, 1000, 1100
- 1000 and 1100 appear in both sets.

Count them once.

More Counting Examples:

Find the number of three-letter initials where none of the letters is repeated.

Apply the rule of product remembering that a letter cannot appear twice to get

$$(26)(25)(24)$$. 
Count the number of bit strings of length 4.

Apply the rule of product to get $2^4$.

Count the number of bit strings of length 4 or less.

Apply the rule of sum to get the disjoint subsets of length 1, 2, 3 and 4.

Then apply the rule of product to count each subset to get

$$2 + 4 + 8 + 16 = 2^1 + 2^2 + 2^3 + 2^4.$$ 

Count the set $S$ of 3 digit numbers which begin or end with an even digit.

Assume that 0 is even but a number cannot begin with 0.

The set is the union of the two subsets:

- The set $B$ of three digit numbers that begin with 2, 4, 6 or 8.

This set has cardinality

$$(4)(10)(10).$$ 

(why?)
• The set \( C \) of three digit numbers that end with 0, 2, 4, 6, or 8 and do not begin with 0.

This set has cardinality

\[ (5)(9)(10). \]

(why?)

• Now we use the inclusion-exclusion principle to eliminate the overlap of sets \( B \) and \( C \).

Their intersection:

The 3 digit numbers that begin with 2, 4, 6, or 8 and end with 0, 2, 4, 6, or 8.

The intersection has the cardinality

\[ (4)(10)(5) \]

Hence the cardinality is

\[ 400 + 450 - 200 = 650. \]