6.2 SOLVING EASY RECURRENCES

We identify a type of recurrence that can be solved by special methods.

DEF: A recurrence relation

$$a_n = f(a_0, \ldots, a_{n-1})$$

has ***degree*** k if the function f depends on the term a_{n-k} and if it depends on no terms of lower index. It is ***linear of degree*** k if it has the form

$$a_n \quad = \quad c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + g(n)$$

where each c_k is a real function and $c_k \neq 0$. It is ***homogenous*** if $g(n) = 0$.

Example 6.2.1: The recurrence system with initial condition

$$a_0 = 0$$

and recurrence relation

$$a_n \quad = \quad a_{n-1} + 2n - 1$$

is linear of degree one and non-homogeneous.

Remark: Similarly, the interest recursion and the Tower of Hanoi recursion are linear of degree one and non-homogeneous.

Example 6.2.2: Fibonacci Numbers
$f_0 = 1 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$
The Fibonacci recurrence is linear of degree two and homogeneous.

Example 6.2.3: Catalan Recursion
$c_0 = 1$
$c_n = c_0c_{n-1} + c_1c_{n-2} + \cdots + c_{n-1}c_0$ for $n \geq 1$.
The Catalan recusion is quadratic, homogeneous, and not of fixed degree.

Remark: Solving the Catalan recursion is well beyond the level of this course.

SOLVING HOMOG LINEAR RR's with CONST COEFF'S

DEF: The ***special method*** for solving an homogeneous linear RR with constant coeff's

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is as follows:

1. Assume there exists a solution of the form $a_n = r^n$ and substitute it into the recurrence:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Cancelling the excess powers of r and normalizing yields what is called the ***characteristic equation:***

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

2. Find the roots $r_1, r_2, \ldots, r_k$ of the char eq, which are called the ***characteristic roots.***

3. Form the ***general solution***

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

4. Use initial conditions to form k simultaneous linear equations in $\alpha_1, \ldots, \alpha_k$ and solve for them.

DEGREE ONE, LINEAR HOMOGENEOUS

Example 6.2.4: General RR of Degree 1
$a_0 = d$ initial condition
$a_n = ca^{n-1}$ recurrence
char eq: $r - c = 0$ has root $r = c$
general solution: $a_n = \alpha_1 c^n$
simultaneous linear equations: $d = \alpha_1 c^0 = \alpha_1$
solution to simult lin eq: $\alpha_1 = d$
problem solution: $a_n = dc^n$

Example 6.2.5: Compound Interest again
Deposit \$3 to be compounded annually at rate r.
$p_0 = 3 \quad p_n = (1+r)p_{n-1}$
Solution: $p_n = 3(1+r)^n$

DEGREE TWO, LINEAR HOMOGENEOUS

Example 6.2.6: Easy degree two recurrence.
$a_0 = 1 \quad a_1 = 4$ initial conditions
$a_n = 5a_{n-1} - 6a_{n-2}$ recurrence
char eq: $r^2 - 5r + 6 = 0$ has roots $r_1 = 3 \quad r_2 = 2$.
gen sol: $a_n = \alpha_1 3^n + \alpha_2 2^n$

simult lin eqns $$\begin{aligned} a_0 &= 1 = \alpha_1 + \alpha_2 \\ a_1 &= 4 = 3\alpha_1 + 2\alpha_2 \end{aligned}$$

have solution: $\alpha_1 = 2 \quad \alpha_2 = -1$.
$\Rightarrow$ problem solution: $a_n = 2 \cdot 3^n - 2^n$

Consider changing the initial conditions to $a_0 = 2 \quad a_1 = 5$. Then the

simult lin eqns $$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 \\ a_1 &= 5 = 3\alpha_1 + 2\alpha_2 \end{aligned}$$

have solution: $\alpha_1 = 1 \quad \alpha_2 = 1$.
$\Rightarrow$ problem solution: $a_n = 3^n + 2^n$

Example 6.2.7: Fibonacci Numbers again

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

char eq: $r^2 - r - 1 = 0$ has roots

$$\frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}$$

Etc. The complete solution is

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right] \\ &= \frac{1}{2^n}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2j+1} \end{aligned}$$

$$\begin{aligned} \text{e.g. } f_5 &= \frac{1}{16}\left[\binom{5}{1} + \binom{5}{3}5 + \binom{5}{5}5^2\right] \\ &= \frac{1}{16}[5+50+25] = \frac{80}{16} = 5 \end{aligned}$$

DEGREE THREE, LINEAR HOMOGENEOUS

Example 6.2.8: $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$
init conds: $a_0 = 2, a_1 = 5, a_2 = 15$

char eq:
$0 = r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$

char roots:
$r = 1, 2, 3$

gen sol:
$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$

simult lin eq:

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$
$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$$
$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$$

coeff solns:
$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2$

unique sol:
$a_n = 1 - 2^n + 2 \cdot 3^n$

NONHOMOGENEOUS LINEAR RECURRENCES

We split the solution into a homogeneous part and a particular part.

Example 6.2.9: Tower of Hanoi, again
$h_0 = 0$ initial condition
$h_n = 2h_{n-1} + 1$ recurrence

assoc homog relation $\hat{h}_n = 2\hat{h}_{n-1}$ has

homogeneous solution $\hat{h}_n = \alpha 2^n$

assoc partic relation $\dot{h}_n = 2\dot{h}_{n-1} + 1$ has

particular solution $\dot{h}_n = -1$

simult lin eqn:

$$h_0 = 0 = \hat{h}_0 + \dot{h}_0 = \alpha 2^0 - 1$$

has solution $\alpha = 1$

problem solution: $h_n = 2^n - 1$.

Remark: The form of the particular solution usuallly resembles the function of n. In this case

$$g(n) = 1$$

is a constant function. So we tried $\dot{h}_n = K$, and we solved the equation $K = 2K + 1$, and obtained $K = -1$.

Example 6.2.10: $a_n = 3a_{n-1} + 2n$
init cond: $a_1 = 3$

homog soln:
$\hat{a}_n = \alpha 3^n$

partic rec rel:
$\hat{a}_n = 3\hat{a}_{n-1} + 2n$

trial soln:
$\hat{a}_n = cn + d$

Then $cn + d = 3[c(n-1) + d] + 2n$,
i.e., $0 = n(2c+2) + (2d - 3c)$
$\Rightarrow c = -1, d = -3/2$

partic soln:
$\hat{a}_n = -n - 3/2$

general soln:
$a_n = \alpha 3^n - n - 3/2$

simult eq:
$a_1 = 3 = \alpha 3 - 1 - 3/2 = 3\alpha - 5/2$

coeff solns:
$\alpha = 11/6$

unique sol:

$$a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$$

REPEATED ROOTS

Example 6.2.11: A recurrence system
$a_0 = -2 \quad a_1 = 2$ initial conditions
$a_n = 4a_{n-1} - 4a_{n-2}$ recurrence
char eq: $r^2 - 4r + 4 = 0$ has roots 2, 2.
gen sol: $a_n = \alpha_1 2^n + \alpha_2 n 2^n$

simult lin eqns $$\begin{aligned} a_0 &= -2 = \alpha_1 \\ a_1 &= 2 = 2\alpha_1 + 2\alpha_2 \end{aligned}$$

have solution: $\alpha_1 = -2 \quad \alpha_2 = 3$.
problem solution: $a_n = (-2) \cdot 2^n + 3 \cdot n2^n$