Section 5.2
Solving Recurrence Relations

If

\[ a_{g(n)} = f(a_{g(0)}, a_{g(1)}, \ldots, a_{g(n-1)}) \]

find a closed form or an expression for \( a_{g(n)} \).

Recall:

- **nth degree polynomials have n roots:**
  \[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

- **If the coefficients are real then the roots are real or occur in complex conjugate pairs.**

Recall the **quadratic formula**: If

\[ ax^2 + bx + c = 0 \text{ then} \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

We assume you remember how to solve linear systems
\( Ax = b. \)

where \( A \) is an \( n \times n \) matrix.

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Solving recurrence relations can be very difficult unless the recurrence equation has a special form:

- \( g(n) = n \) (single variable)
- the equation is linear:
  - sum of previous terms
  - no transcendental functions of the \( a_i's \)
  - no products of the \( a_i's \)
- constant coefficients: the coefficients in the sum of the \( a_i's \) are constants, independent of \( n \).
- degree \( k \): \( a_n \) is a function of only the previous \( k \) terms in the sequence
- homogeneous: If we put all the \( a_i's \) on one side of the equation and everything else on the right side, then the right side is 0.

Otherwise \textit{inhomogeneous} or \textit{nonhomogeneous}.

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Examples:

• \( a_n = (1.02)a_{n-1} \)
  linear
  constant coefficients
  homogeneous
  degree 1

• \( a_n = (1.02)a_{n-1} + 2^{n-1} \)
  linear
  constant coefficients
  nonhomogeneous
  degree 1

• \( a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3} \)
  linear
  constant coefficients
  nonhomogeneous
  degree 3

• \( a_n = c a_{n/m} + b \)
  g does not have the right form

• \( a_n = n a_{n-1} + n^2 a_{n-2} + a_{n-1} a_{n-2} \)
  nonlinear
  coefficients are not constants
  homogeneous
  degree 2

Solution Procedure
linear
constant coefficients
homogeneous
degree k

\[ a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_{n-k}a_{n-k} \]

1. Put all \( a_i \)'s on the left side of the equation, everything else on the right. If nonhomogeneous, stop (for now).

\[ a_n - c_1a_{n-1} - c_2a_{n-2} - \ldots - c_{n-k}a_{n-k} = 0 \]

2. Assume a solution of the form \( a_n = b^n \).

3. Substitute the solution into the equation, factor out the lowest power of \( b \) and eliminate it.

\[ b^n - c_1b^{n-1} - c_2b^{n-2} - \ldots - c_{n-k}b^{n-k} = 0 \]

\[ b^{n-k}[b^k - c_1b^{k-1} - \ldots - c_{n-k}] = 0 \]

4. The remaining polynomial of degree \( k \),

\[ b^k - c_1b^{k-1} - \ldots - c_{n-k} \]

is called the characteristic polynomial.

Find its \( k \) roots, \( r_1, r_2, \ldots, r_k \).

5. If the roots are distinct, the general solution is

\[ a_n = \alpha_1r_1^n + \alpha_2r_2^n + \ldots + \alpha_kr_k^n \]
6. The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are found by enforcing the initial conditions.

Solve the resulting linear system of equations:

\[
\begin{align*}
    a_0 &= \alpha_1 r_1^0 + \alpha_2 r_2^0 + \ldots + \alpha_k r_k^0 \\
    a_1 &= \alpha_1 r_1^1 + \alpha_2 r_2^1 + \ldots + \alpha_k r_k^1 \\
    &\vdots \\
    a_{k-1} &= \alpha_1 r_1^{k-1} + \alpha_2 r_2^{k-1} + \ldots + \alpha_k r_k^{k-1}
\end{align*}
\]

Example:

\[
a_{n+2} = 3a_{n+1}, \quad a_0 = 4
\]

- Bring subscripted variables to one side:

\[
a_{n+2} - 3a_{n+1} = 0.
\]

- Substitute $a_n = b^n$ and factor lowest power of $b$:

\[
b^{n+1}(b-3) = 0 \quad \text{or} \quad b-3 = 0
\]

- Find the root of the characteristic polynomial:

\[
r_1 = 3
\]

- Compute the general solution:

\[
a_n = c3^n
\]

- Find the constants based on the initial conditions:
\[ a_0 = c(3^0) \text{ or } c = 4 \]

- Produce the specific solution:
  \[ a_n = 4(3^n) \]

Example:

\[ a_n = 3a_{n-2}, \quad a_0 = a_1 = 1 \]

- \( a_n - 3a_{n-2} = 0 \)

Note: the \( a_{n-1} \) term has a coefficient of 0.

- \( b^{n-2}(b^2 - 3) = 0 \text{ or } b^2 - 3 = 0 \)

- \( r_1 = \sqrt{3}, \quad r_2 = -\sqrt{3} \)

- \( a_n = \alpha_1 \sqrt{3}^n + \alpha_2 (-\sqrt{3})^n \)

- Solve the linear system for \( \alpha_1, \alpha_2 \):
  
  \[ a_0 = 1 = \alpha_1 \sqrt{3}^0 + \alpha_2 (-\sqrt{3})^0 = \alpha_1 + \alpha_2 \]
  
  \[ a_1 = 1 = \alpha_1 \sqrt{3}^1 + \alpha_2 (-\sqrt{3})^1 = \alpha_1 \sqrt{3} - \alpha_2 \sqrt{3} \]

Solve the first equation for the first variable and substitute in the second equation:

\[ \alpha_1 = 1 - \alpha_2 \]

\[ 1 = (1 - \alpha_2)\sqrt{3} - \alpha_2 \sqrt{3} = \sqrt{3} - \alpha_2 2\sqrt{3} \]
\[ \alpha_2 = (\sqrt{3} - 1) / 2\sqrt{3} \]
\[ \alpha_1 = 1 - (\sqrt{3} - 1) / 2\sqrt{3} = (\sqrt{3} + 1) / 2\sqrt{3} \]

If a root \( r_1 \) has multiplicity \( p \), then the solution is
\[ a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \ldots + \alpha_p n^{p-1} r_1^n + \ldots \]

Example:
\[ a_n = 6a_{n-1} - 9a_{n-2}, a_0 = a_1 = 1 \]
- Recurrence system:
\[ a_n - 6a_{n-1} + 9a_{n-2} = 0 \]
- Find roots of characteristic polynomial
\[ \begin{cases} b^2 - 6b + 9 = 0 \\ (b - 3)^2 = 0 \end{cases} \]
- Roots are equal:
\[ b_1 = b_2 = 3 \]
- General solutions is
\[ a_n = \alpha_1 3^n + \alpha_2 n 3^n \]
• Solve for coefficients:

\[
\begin{align*}
  a_0 &= 1 = \alpha_1 + 0 \\
  a_1 &= 1 = 1(3^1) + \alpha_2 (1)(3^1) \\
  \alpha_2 &= -\frac{2}{3}
\end{align*}
\]

You finish.

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**Nonhomogeneous Recurrence Relations**

- linear
- constant coefficients
- degree k

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{n-k} a_{n-k} + f(n)
\]

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_{n-k} a_{n-k}
\]

is the associated *homogeneous* recurrence equation

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**TELESCOPING**

Note: we introduce the technique here because it will be useful to solve recurrence systems associated with divide and conquer algorithms later.
For recurrences which are

- first degree

\[ a_n = \alpha a_{n-1} + f(n) \]

Method:

- back substitute
  
  - force the coefficient of \( a_{n-k} \) on the left side to agree with the coefficient of \( a_{n-k} \) in the previous equation

- stop when we get to the initial condition on the right side

- add the left sides of the equations and the right sides of the equations and cancel like terms

- add the remaining terms together to get a formula for \( a_n \).

Example:

- \( a_n = 2a_{n-1} + 1, a_0 = 3 \)

- Write down the equation:

\[ a_n = 2a_{n-1} + 1 \]
• Write the equation for \( a_{n-1} \):

\[
a_{n-1} = 2a_{n-2} + 1
\]

• Multiply by the constant which appears as a coefficient of \( a_{n-1} \) in the previous equation so the two will cancel when we add both sides:

\[
2a_{n-1} = 2^2 a_{n-2} + 2
\]

• Write down the equation for \( a_{n-2} \) and multiply both sides by the coefficient of \( a_{n-2} \) in the previous equation:

\[
a_{n-2} = 2a_{n-3} + 1
\]

becomes

\[
2^2 a_{n-2} = 2^3 a_{n-3} + 2^2
\]

• Continue until the initial condition appears on the right hand side:

\[
a_1 = 2a_0 + 1
\]

becomes

\[
2^{n-1} a_1 = 2^n a_0 + 2^{n-1}
\]

• Add both sides of the equations and cancel identical terms:

\[
a_n = (2a_{n-1}) + 1
\]

\[
(2a_{n-1}) = [2^2 a_{n-2}] + 2
\]
\[ [2^2 a_{n-2}] = 2^3 a_{n-3} + 2^2 \]

\[ \cdot \]

\[ \cdot \]

\[ 2^{n-1} a_1 = 2^n a_0 + 2^{n-1} \]

\[
a_n = 2^n a_0 + \sum_{i=0}^{n-1} 2^i
\]

- Substitute \( a_0 \) and simplify \( \sum_{i=0}^{n-1} 2^i \) to get the solution:

\[ a_n = 3(2^n) + 2^n - 1 = 2^{n+2} - 1 \]

Note: solution to nonhomogeneous case is sum of solution to associated homogeneous recurrence system and a particular solution to the nonhomogeneous case.

**Theorem:**

Let \( \{a_n^P\} \) be a particular solution to the nonhomogeneous equation and let \( \{a_n^H\} \) be the solution to the associated homogeneous recurrence system. Then every solution to the nonhomogeneous equation is of the form

\[ \{a_n^H + a_n^P\} \]
**Theorem:**

Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part $f(n)$ of the form

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

If $s$ is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) s^n$$

If $s$ is a root of multiplicity $m$, a particular solutions is of the form

$$n^m (c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) s^n$$

Example:

From the previous example the associated homogeneous recurrence equation is

$$a_n - 2a_{n-1} = 0$$

and

$$f(n) = 1$$

The root of the characteristic polynomial is 2 so the solution to the homogeneous part is
and a particular solution to the nonhomogeneous equation is 
\[ \{a_n^p\} = c_0. \]

Substituting \( c_0 \) into the nonhomogeneous equation we get 
\[ c_0 - 2c_0 = 1 \]

or 
\[ c_0 = -1 \]

Therefore the general solution is 
\[ \alpha 2^n - 1 \]

Using the initial condition we have 
\[ \alpha 2^0 - 1 = 3 \text{ or } \alpha = 4 = 2^2 \]

Hence, the solution is 
\[ a_n = 2^{n+2} - 1 \]

which is the same solution we obtained by telescoping.