

University of Florida
Dept. of Computer & Information Science & Engineering
COT 3100
Applications of Discrete Structures
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Slides for a Course Based on the Text
Discrete Mathematics & Its Applications
(5th Edition)
by Kenneth H. Rosen

Module #4:
Functions

Rosen 5th ed., §1.8
~44 slides, ~2 lectures

On to section 1.8... Functions

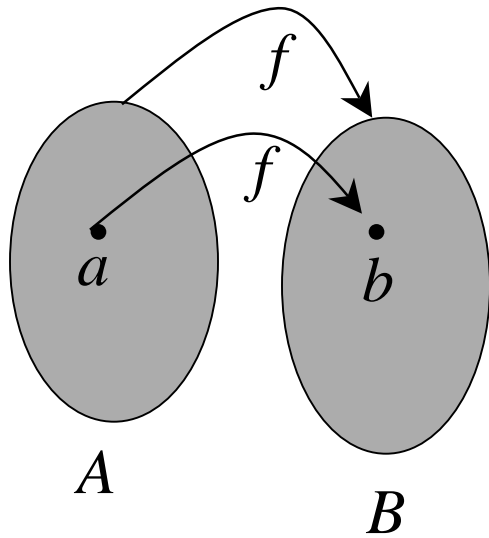
- From calculus, you are familiar with the concept of a real-valued function f , which assigns to each number $x \in \mathbf{R}$ a particular value $y = f(x)$, where $y \in \mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set.

Function: Formal Definition

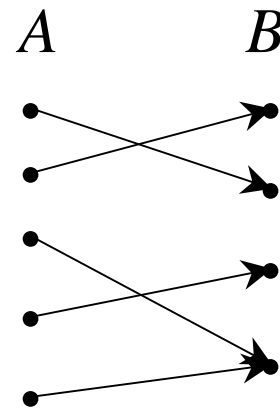
- For any sets A, B , we say that a *function* f from (or “mapping”) A to B ($f:A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Some further generalizations of this idea:
 - A *partial* (non-total) function f assigns *zero or one* elements of B to each element $x \in A$.
 - Functions of n arguments; relations (ch. 6).

Graphical Representations

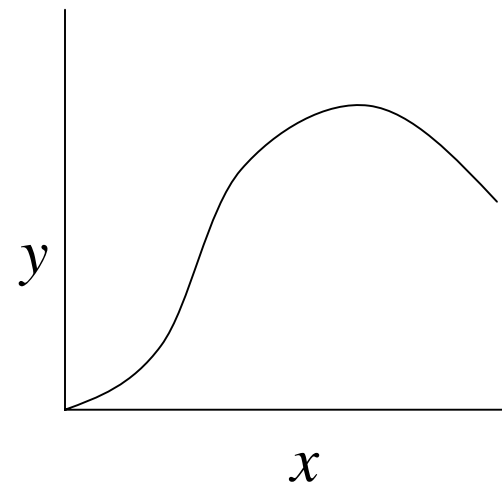
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from “situations” to truth values $\{\mathbf{T}, \mathbf{F}\}$
 - A logic system called *situation theory*.
 - p = “It is raining.”; s = our situation here, now
 - $p(s) \in \{\mathbf{T}, \mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: $\vee((\mathbf{F}, \mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T}, \mathbf{F})) = \mathbf{F}$.

More functions so far...

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values): $P \equiv$ “is 7 feet tall”;
 $P(\text{Mike}) =$ “Mike is 7 feet tall.” = **False**.
- A *bit string* B of length n can be viewed as a function from the numbers $\{1, \dots, n\}$ (bit *positions*) to the *bits* $\{0, 1\}$.
E.g., $B=101 \rightarrow B(3)=1$.

Still More Functions

- A *set* S over universe U can be viewed as a function from the elements of U to $\{\mathbf{T}, \mathbf{F}\}$, saying for each element of U whether it is in S . $S = \{3\}$; $S(0) = \mathbf{F}$, $S(3) = \mathbf{T}$.
- A *set operator* such as $\cap, \cup, \bar{}$ can be viewed as a function from pairs of sets to sets.
 - Example: $\cap((\{1,3\}, \{3,4\})) = \{3\}$

A Neat Trick

- Sometimes we write Y^X to denote the set F of *all* possible functions $f: X \rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y , $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{T} \equiv \mathbf{1}$, $\mathbf{2} \equiv \{\mathbf{0}, \mathbf{1}\} = \{\mathbf{F}, \mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If $f:A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:
 - A is the *domain* of f .
 - B is the *codomain* of f .
 - b is the *image* of a under f .
 - a is a *pre-image* of b under f .
 - In general, b may have more than 1 pre-image.
 - The *range* $R \subseteq B$ of f is $\{b \mid \exists a f(a)=b\}$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: “ f is a function mapping students in this class to the set of grades $\{A,B,C,D,E\}$.”
- At this point, you know f 's codomain is: $\{A,B,C,D,E\}$, and its range is unknown!
- Suppose the grades turn out all As and Bs.
- Then the range of f is $\{A,B\}$, but its codomain is still $\{A,B,C,D,E\}$!.

Operators (general definition)

- An n -ary operator over the set S is any function from the set of ordered n -tuples of elements of S , to S itself.
- *E.g.*, if $S = \{\mathbf{T}, \mathbf{F}\}$, \neg can be seen as a unary operator, and \wedge, \vee are binary operators on S .
- Another example: \cup and \cap are binary operators on the set of all sets.

Constructing Function Operators

- If \bullet (“dot”) is any operator over B , then we can extend \bullet to also denote an operator over functions $f:A\rightarrow B$.
- *E.g.:* Given any binary operator $\bullet:B\times B\rightarrow B$, and functions $f,g:A\rightarrow B$, we define $(f\bullet g):A\rightarrow B$ to be the function defined by:
 $\forall a\in A, (f\bullet g)(a) = f(a)\bullet g(a)$.

Function Operator Example

- $+, \times$ (“plus”, “times”) are binary operators over \mathbf{R} . (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f, g: \mathbf{R} \rightarrow \mathbf{R}$:
 - $(f + g): \mathbf{R} \rightarrow \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
 - $(f \times g): \mathbf{R} \rightarrow \mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

- For functions $g:A\rightarrow B$ and $f:B\rightarrow C$, there is a special operator called *compose* (“ \circ ”).
 - It composes (creates) a new function out of f,g by applying f to the result of g .
 - $(f\circ g):A\rightarrow C$, where $(f\circ g)(a) = f(g(a))$.
 - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
 - Note that \circ (like Cartesian \times , but unlike $+, \wedge, \cup$) is non-commuting. (Generally, $f\circ g \neq g\circ f$.)

Images of Sets under Functions

- Given $f:A \rightarrow B$, and $S \subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S .
$$f(S) ::= \{f(s) \mid s \in S\}$$
$$::= \{b \mid \exists s \in S: f(s)=b\}.$$
- Note the range of f can be defined as simply the image (under f) of f 's domain!

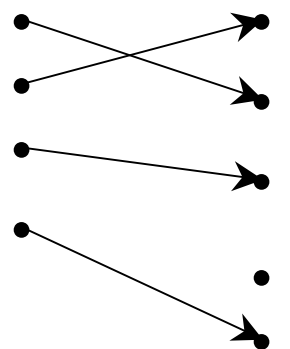
One-to-One Functions

- A function is *one-to-one* (1-1), or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
 - Formally: given $f:A \rightarrow B$,
“ x is injective” $:\equiv (\neg \exists x, y: x \neq y \wedge f(x) = f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
 - Domain & range have same cardinality. What about codomain?
- Each element of the domain is injected into a different element of the range.
 - Compare “each dose of vaccine is injected into a different patient.”

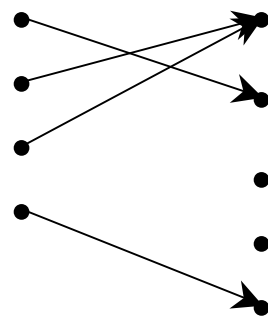
May Be
Larger

One-to-One Illustration

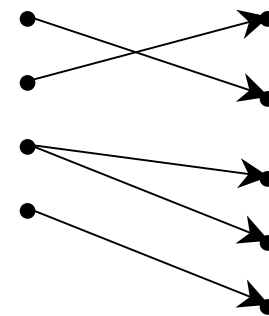
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Sufficient Conditions for 1-1ness

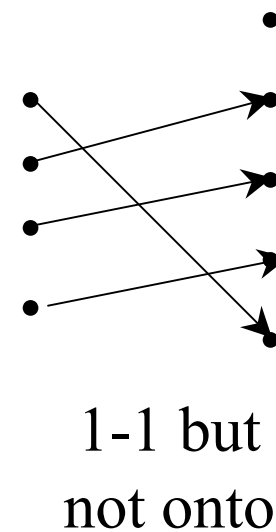
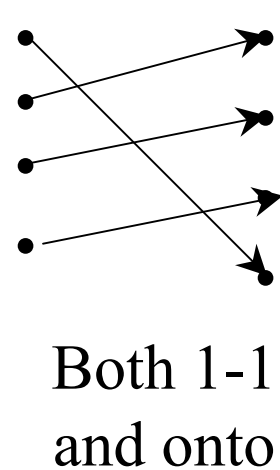
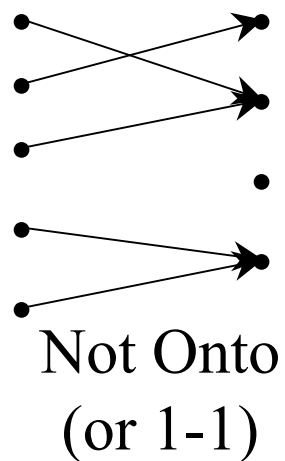
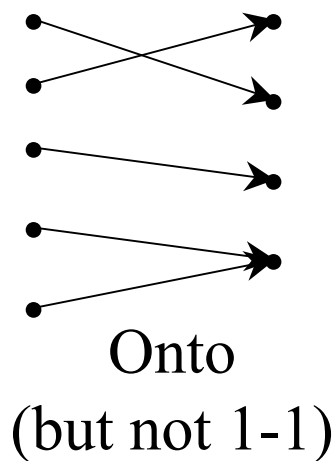
- For functions f over numbers,
 - f is *strictly* (or *monotonically*) *increasing* iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
 - f is *strictly* (or *monotonically*) *decreasing* iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
 - *Converse is not necessarily true. E.g.* $1/x$

Onto (Surjective) Functions

- A function $f:A\rightarrow B$ is *onto* or *surjective* or a *surjection* iff its range is equal to its codomain ($\forall b\in B, \exists a\in A: f(a)=b$).
- An *onto* function maps the set A onto (over, covering) the *entirety* of the set B , not just over a piece of it.
- *E.g.*, for domain & codomain \mathbf{R} , x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Bijections

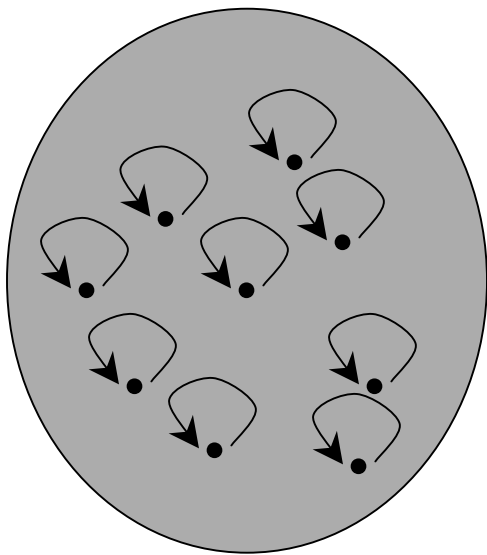
- A function f is a *one-to-one correspondence*, or a *bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f:A\rightarrow B$, there exists an *inverse of f* , written $f^{-1}:B\rightarrow A$, which is the unique function such that $f^{-1} \circ f = I$ (the identity function)

The Identity Function

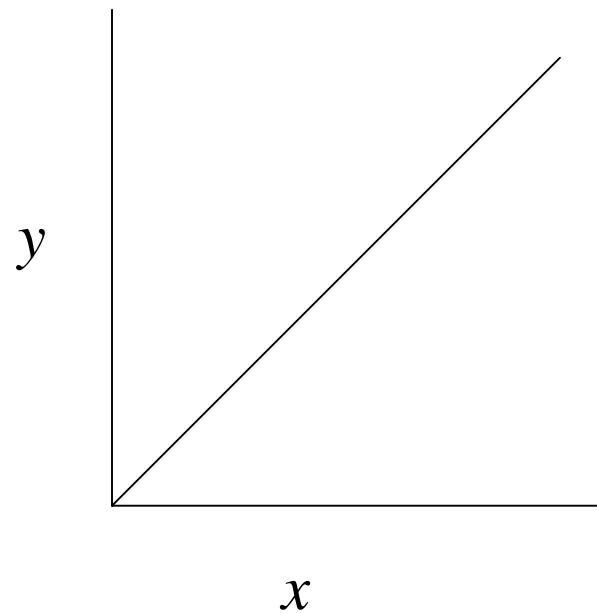
- For any domain A , the *identity function* $I:A\rightarrow A$ (variously written, I_A , $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a\in A: I(a)=a$.
- Some identity functions you've seen:
 - +ing 0, \cdot ing by 1, \wedge ing with \mathbf{T} , \vee ing with \mathbf{F} ,
 \cup ing with \emptyset , \cap ing with U .
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



A Fun Application

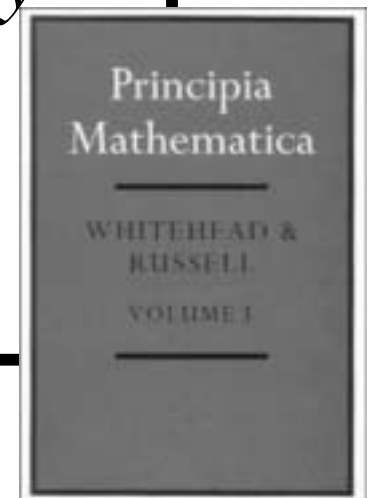
- In a computer, the function mapping *state at clock cycle #t* to *state at clock cycle #t+1* is called the computer's *transition function*.
- If the transition function is reversible (a bijection), then the computer's operation in theory requires *no energy expenditure*.
- The study of low-power *reversible computing* techniques based on this idea is my primary research area.

Graphs of Functions

- We can represent a function $f:A\rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a\in A\}$.
- Note that $\forall a$, there is only 1 pair $(a,f(a))$.
 - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x .

Comment About Representations

- You can represent any type of discrete structure (propositions, bit-strings, numbers, sets, ordered pairs, functions) in terms of virtually any of the other structures (or some combination thereof).
- Probably none of these structures is *truly* more fundamental than the others (whatever that would mean). However, strings, logic, and sets are often used as the foundation for all else. *E.g.* in →



A Couple of Key Functions

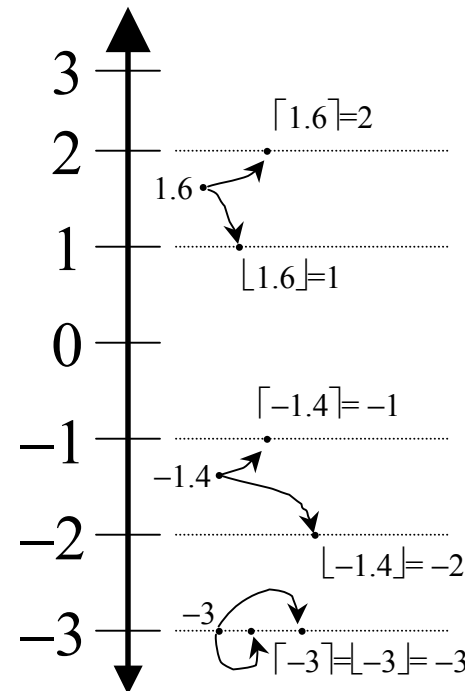
- In discrete math, we will frequently use the following functions over real numbers:
 - $\lfloor x \rfloor$ (“floor of x ”) is the largest (most positive) integer $\leq x$.
 - $\lceil x \rceil$ (“ceiling of x ”) is the smallest (most negative) integer $\geq x$.

Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”

- Note that if $x \notin \mathbf{Z}$,
 $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ &
 $\lceil -x \rceil \neq -\lceil x \rceil$

- Note that if $x \in \mathbf{Z}$,
 $\lfloor x \rfloor = \lceil x \rceil = x$.

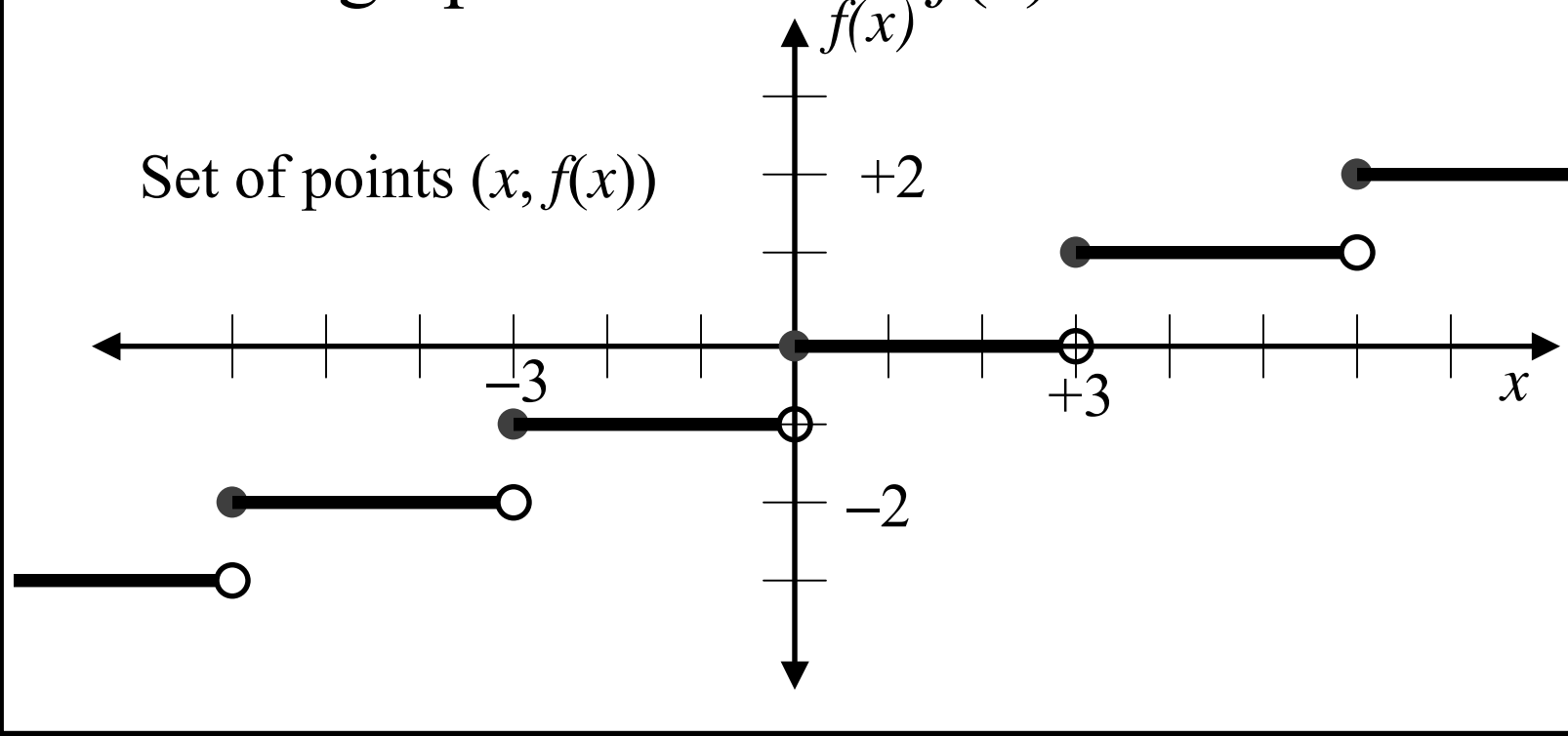


Plots with floor/ceiling

Note that for $f(x) = \lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a < 1$, but not for $a = 1$. We say that the set of points $(a, 0)$ that is in f does not include its *limit* or *boundary* point $(a, 1)$. Sets that do not include all of their limit points are called *open sets*. In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of §1.8 (Functions)

- Function variables f, g, h, \dots
- Notations: $f:A \rightarrow B, f(a), f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1} , binary operators $+, -, \text{etc.}$, and \circ .
- The $\mathbf{R} \rightarrow \mathbf{Z}$ functions $\lfloor x \rfloor$ and $\lceil x \rceil$.

Module #4, Topic # ∞ :
Cardinality & Infinite Sets

Rosen 5th ed., §3.2
~1 lecture

Infinite Cardinalities (from §3.2)

- Using what we learned about *functions* in §1.8, it's possible to formally define cardinality for infinite sets.
- We show that infinite sets come in different *sizes* of infinite!
- This gives us some interesting proof examples, in anticipation of chapter 3.

Cardinality: Formal Definition

- For any two (possibly infinite) sets A and B , we say that A and B *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (bijective function) from A to B .
- When A and B are finite, it is easy to see that such a function exists iff A and B have the same number of elements $n \in \mathbf{N}$.

Countable versus Uncountable

- For any set S , if S is finite or $|S|=|\mathbf{N}|$, we say S is *countable*. Else, S is *uncountable*.
- Intuition behind “**countable**:” we can *enumerate* (generate in series) elements of S in such a way that *any* individual element of S will eventually be *counted* in the enumeration. Examples: \mathbf{N} , \mathbf{Z} .
- **Uncountable**: *No* series of elements of S (even an infinite series) can include all of S 's elements. Examples: \mathbf{R} , \mathbf{R}^2 , $P(\mathbf{N})$

Countable Sets: Examples

- **Theorem:** The set \mathbf{Z} is countable.
 - **Proof:** Consider $f:\mathbf{Z}\rightarrow\mathbf{N}$ where $f(i)=2i$ for $i\geq 0$ and $f(i) = -2i-1$ for $i<0$. Note f is bijective.
- **Theorem:** The set of all ordered pairs of natural numbers (n,m) is countable.
 - Consider listing the pairs in order by their sum $s=n+m$, then by n . Every pair appears once in this series; the generating function is bijective.

Uncountable Sets: Example

- **Theorem:** The open interval $[0,1) := \{r \in \mathbf{R} \mid 0 \leq r < 1\}$ is uncountable.
- **Proof by diagonalization:** (Cantor, 1891)
 - Assume there is a series $\{r_i\} = r_1, r_2, \dots$ containing *all* elements $r \in [0,1)$.
 - Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index: ... (*continued on next slide*)



Georg Cantor
1845-1918

Uncountability of Reals, cont'd

A postulated enumeration of the reals:

$$r_1 = 0.d_{1,1} d_{1,2} d_{1,3} d_{1,4} d_{1,5} d_{1,6} d_{1,7} d_{1,8} \dots$$

$$r_2 = 0.d_{2,1} d_{2,2} d_{2,3} d_{2,4} d_{2,5} d_{2,6} d_{2,7} d_{2,8} \dots$$

$$r_3 = 0.d_{3,1} d_{3,2} d_{3,3} d_{3,4} d_{3,5} d_{3,6} d_{3,7} d_{3,8} \dots$$

$$r_4 = 0.d_{4,1} d_{4,2} d_{4,3} d_{4,4} d_{4,5} d_{4,6} d_{4,7} d_{4,8} \dots$$

- Now, consider a real number generated by taking
- all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.

That real doesn't appear in the list!

Uncountability of Reals, fin.

- *E.g.*, a postulated enumeration of the reals:

$$r_1 = 0.301948571\dots$$

$$r_2 = 0.103918481\dots$$

$$r_3 = 0.039194193\dots$$

$$r_4 = 0.918237461\dots$$

- OK, now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103... can't be on the list anywhere!

Transfinite Numbers

- The cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of the natural numbers, $\aleph_0 := |\mathbf{N}|$, is the *first transfinite cardinal* number. (There are none smaller.)
- The *continuum hypothesis* claims that $|\mathbf{R}| = \aleph_1$, the *second transfinite cardinal*.

Proven impossible to prove or disprove!

Do Uncountable Sets Really Exist?

- The set of objects that can be defined using finite-length strings of symbols (“descriptions”) is only *countable*.
- Therefore, any uncountable set must consist primarily of elements which individually have *no* finite description.
- Löwenheim-Skolem theorem: No consistent theory can ever *force* an interpretation involving uncountables.
- The “constructivist school” asserts that only objects constructible from finite descriptions exist. (e.g. $\neg\exists\mathbf{R}$)
- Most mathematicians are happy to use uncountable sets anyway, because postulating their existence has not led to any demonstrated contradictions (so far).

Countable vs. Uncountable

- You should:
 - Know how to define “same cardinality” in the case of infinite sets.
 - Know the definitions of *countable* and *uncountable*.
 - Know how to prove (at least in easy cases) that sets are either countable or uncountable.