Module #13 – Inductive Proofs

University of Florida
Dept. of Computer & Information Science & Engineering

COT 3100
Applications of Discrete Structures
Dr. Michael P. Frank

Slides for a Course Based on the Text
Discrete Mathematics & Its Applications
(5th Edition)
by Kenneth H. Rosen
Module #13 – Inductive Proofs

Module #13: Inductive Proofs

Rosen 5th ed., §3.3
~11 slides, ~1 lecture
§3.3: Mathematical Induction

• A powerful, rigorous technique for proving that a predicate \( P(n) \) is true for every natural number \( n \), no matter how large.

• Essentially a “domino effect” principle.

• Based on a predicate-logic inference rule:

\[
P(0)\]
\[
\forall n \geq 0 \ (P(n) \rightarrow P(n+1))
\]
\[
\therefore \forall n \geq 0 \ P(n)
\]

“The First Principle of Mathematical Induction”
Proof that $\forall k\geq 0 \ P(k)$ is a valid consequent:
Given any $k\geq 0$, $\forall n\geq 0 \ (P(n)\rightarrow P(n+1))$ (antecedent 2) trivially implies $\forall n\geq 0 \ (n<k)\rightarrow(P(n)\rightarrow P(n+1))$, or $(P(0)\rightarrow P(1)) \land (P(1)\rightarrow P(2)) \land \ldots \land (P(k-1)\rightarrow P(k))$. Repeatedly applying the hypothetical syllogism rule to adjacent implications $k-1$ times then gives $P(0)\rightarrow P(k)$; which with $P(0)$ (antecedent #1) and modus ponens gives $P(k)$. Thus $\forall k\geq 0 \ P(k)$. 

Module #13 – Inductive Proofs

The Well-Ordering Property

• The validity of the inductive inference rule can also be proved using the well-ordering property, which says:
  – Every non-empty set of non-negative integers has a minimum (smallest) element.
  – ∀ ∅ ⊂ S ⊆ N : ∃ m ∈ S : ∀ n ∈ S : m ≤ n
• Implies \{ n \mid \neg P(n) \} has a min. element m, but then P(m-1) → P((m-1)+1) contradicted.
Module #13 – Inductive Proofs

Outline of an Inductive Proof

- **Want to prove** $\forall n \ P(n)$...
- **Base case (or basis step):** Prove $P(0)$.
- **Inductive step:** Prove $\forall n \ P(n) \rightarrow P(n+1)$.
  - *E.g.* use a direct proof:
  - Let $n \in \mathbb{N}$, assume $P(n)$. *(inductive hypothesis)*
  - Under this assumption, prove $P(n+1)$.
- **Inductive inference rule** then gives $\forall n \ P(n)$. 
Generalizing Induction

- Can also be used to prove $\forall n \geq c \ P(n)$ for a given constant $c \in \mathbb{Z}$, where maybe $c \neq 0$.
  - In this circumstance, the base case is to prove $P(c)$ rather than $P(0)$, and the inductive step is to prove $\forall n \geq c \ (P(n) \rightarrow P(n+1))$.

- Induction can also be used to prove $\forall n \geq c \ P(a_n)$ for an arbitrary series $\{a_n\}$.

- Can reduce these to the form already shown.
Second Principle of Induction

- Characterized by another inference rule:

\[
P(0) \quad \text{P is true in all previous cases} \\
\forall n \geq 0: \left( \forall 0 \leq k \leq n \ P(k) \right) \rightarrow P(n+1) \\
\therefore \forall n \geq 0: P(n)
\]

- Difference with 1st principle is that the inductive step uses the fact that \( P(k) \) is true for all smaller \( k < n+1 \), not just for \( k=n \).
Induction Example (1st princ.)

• Prove that the sum of the first $n$ odd positive integers is $n^2$. That is, prove:
\[ \forall n \geq 1 : \sum_{i=1}^{n} (2i-1) = n^2 \]

• Proof by induction. $P(n)$
  – Base case: Let $n=1$. The sum of the first 1 odd positive integer is 1 which equals $1^2$.
  (Cont…)
Module #13 – Inductive Proofs

Example cont.

- Inductive step: Prove \( \forall n \geq 1: P(n) \rightarrow P(n+1) \).
  - Let \( n \geq 1 \), assume \( P(n) \), and prove \( P(n+1) \).
    \[
    \sum_{i=1}^{n+1} (2i - 1) = \left( \sum_{i=1}^{n} (2i - 1) \right) + (2(n + 1) - 1)
    \]
    \[= n^2 + 2n + 1 \]
    \[= (n + 1)^2 \]
    By inductive hypothesis \( P(n) \)
Another Induction Example

• Prove that \( \forall n>0, n<2^n \). Let \( P(n)=(n<2^n) \)
  
  – Base case: \( P(1)=(1<2^1)=(1<2)=T \).
  
  – Inductive step: For \( n>0 \), prove \( P(n) \rightarrow P(n+1) \).
    
    • Assuming \( n<2^n \), prove \( n+1 < 2^{n+1} \).
    
    • Note \( n + 1 < 2^n + 1 \) (by inductive hypothesis)
      
      \[ < 2^n + 2^n \text{ (because } 1<2=2\cdot2^0\leq2\cdot2^{n-1}=2^n) \]
      
      \[ = 2^{n+1} \]
    
    • So \( n + 1 < 2^{n+1} \), and we’re done.
Module #13 – Inductive Proofs

Example of Second Principle

• Show that every \( n > 1 \) can be written as a product \( p_1p_2 \cdots p_s \) of some series of \( s \) prime numbers. Let \( P(n) = \text{“} n \text{ has that property”} \)
• Base case: \( n = 2 \), let \( s = 1, p_1 = 2 \).
• Inductive step: Let \( n \geq 2 \). Assume \( \forall 2 \leq k \leq n: P(k) \). Consider \( n+1 \). If prime, let \( s = 1, p_1 = n+1 \).
Else \( n+1 = ab \), where \( 1 < a \leq n \) and \( 1 < b \leq n \). Then \( a = p_1p_2 \cdots p_t \) and \( b = q_1q_2 \cdots q_u \). Then \( n+1 = p_1p_2 \cdots p_t q_1q_2 \cdots q_u \), a product of \( s = t + u \) primes.
Another 2nd Principle Example

• Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
• Base case: 12=3(4), 13=2(4)+1(5), 14=1(4)+2(5), 15=3(5), so \( \forall 12 \leq n \leq 15, P(n) \).
• Inductive step: Let \( n \geq 15 \), assume \( \forall 12 \leq k \leq n \ P(k) \). Note \( 12 \leq n-3 \leq n \), so \( P(n-3) \), so add a 4-cent stamp to get postage for \( n+1 \).