

University of Florida  
Dept. of Computer & Information Science & Engineering  
**COT 3100**  
**Applications of Discrete Structures**  
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Slides for a Course Based on the Text  
*Discrete Mathematics & Its Applications*  
(5<sup>th</sup> Edition)  
by Kenneth H. Rosen

**Module #12:  
Summations**

Rosen 5<sup>th</sup> ed., §3.2  
~19 slides, ~1 lecture

## Summation Notation

- Given a series  $\{a_n\}$ , an integer *lower bound* (or *limit*)  $j \geq 0$ , and an integer *upper bound*  $k \geq j$ , then the *summation of  $\{a_n\}$  from  $j$  to  $k$*  is written and defined as follows:

$$\sum_{i=j}^k a_i \equiv a_j + a_{j+1} + \dots + a_k$$

- Here,  $i$  is called the *index of summation*.

# Generalized Summations

- For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i \equiv a_j + a_{j+1} + \dots$$

- To sum a function over all members of a set

$$X = \{x_1, x_2, \dots\}: \sum_{x \in X} f(x) \equiv f(x_1) + f(x_2) + \dots$$

- Or, if  $X = \{x | P(x)\}$ , we may just write:

$$\sum_{P(x)} f(x) \equiv f(x_1) + f(x_2) + \dots$$

## Simple Summation Example

$$\begin{aligned}\sum_{i=2}^4 i^2 + 1 &= (2^2 + 1) + (3^2 + 1) + (4^2 + 1) \\ &= (4 + 1) + (9 + 1) + (16 + 1) \\ &= 5 + 10 + 17 \\ &= 32\end{aligned}$$

## More Summation Examples

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \dots = 1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x < 10}} x^2 = 2^2 + 3^2 + 5^2 + 7^2 = 4 + 9 + 25 + 49 = 87$$

# Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c \sum_x f(x) \quad (\text{Distributive law.})$$

$$\sum_x f(x) + g(x) = \left( \sum_x f(x) \right) + \sum_x g(x) \quad (\text{Application of commutativity.})$$

$$\sum_{i=j}^k f(i) = \sum_{i=j+n}^{k+n} f(i-n) \quad (\text{Index shifting.})$$

# More Summation Manipulations

- Other identities that are sometimes useful:

$$\sum_{i=j}^k f(i) = \left( \sum_{i=j}^m f(i) \right) + \sum_{i=m+1}^k f(i) \quad \text{if } j \leq m < k$$

(Series splitting.)

$$\sum_{i=j}^k f(i) = \sum_{i=0}^{k-j} f(k-i)$$

(Order reversal.)

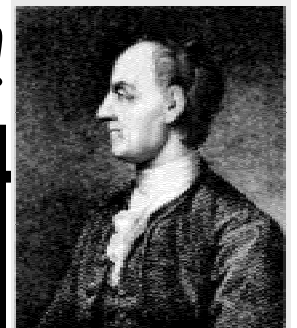
$$\sum_{i=0}^{2k} f(i) = \sum_{i=0}^k f(2i) + f(2i+1)$$

(Grouping.)

## Example: Impress Your Friends

- Boast, “I’m so smart; give me any 2-digit number  $n$ , and I’ll add all the numbers from 1 to  $n$  in my head in just a few seconds.”
- *I.e.*, Evaluate the summation: 
$$\sum_{i=1}^n i$$
- There is a simple closed-form formula for the result, discovered by Euler at age 12!

Leonhard  
Euler  
(1707-1783)





# Symbolic Derivation of Trick

$$\begin{aligned}
 \sum_{i=1}^n i &= \sum_{i=1}^{2k} i = \left( \sum_{i=1}^k i \right) + \sum_{i=k+1}^n i = \left( \sum_{i=1}^k i \right) + \sum_{i=0}^{n-(k+1)} (i + (k+1)) \\
 &= \left( \sum_{i=1}^k i \right) + \sum_{i=0}^{n-(k+1)} ((n-(k+1)) - i) + (k+1) \\
 &= \left( \sum_{i=1}^k i \right) + \sum_{i=0}^{n-(k+1)} (n-i) = \left( \sum_{i=1}^k i \right) + \sum_{i=1}^{n-k} (n-(i-1)) \\
 &= \left( \sum_{i=1}^k i \right) + \sum_{i=1}^{n-k} (n+1-i) = \left( \sum_{i=1}^k i \right) + \sum_{i=1}^k (n+1-i) = \dots
 \end{aligned}$$

# Concluding Euler's Derivation

$$\begin{aligned}
 \sum_{i=1}^n i &= \left( \sum_{i=1}^k i \right) + \sum_{i=1}^k (n+1-i) = \sum_{i=1}^k (i+n+1-i) \\
 &= \sum_{i=1}^k (n+1) = k(n+1) = \frac{n}{2}(n+1) \\
 &= n(n+1)/2
 \end{aligned}$$

The diagram illustrates the derivation of the sum formula. It shows the equation  $\sum_{i=1}^n i = \left( \sum_{i=1}^k i \right) + \sum_{i=1}^k (n+1-i) = \sum_{i=1}^k (i+n+1-i)$  with arrows pointing from the first sum to the second, and from the second sum to the third. The third sum is then simplified to  $\sum_{i=1}^k (n+1) = k(n+1) = \frac{n}{2}(n+1)$  and finally to  $= n(n+1)/2$ . The arrows indicate the flow of the derivation and the simplification of the terms.

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd  $n$  (prove this at home).

## Example: Geometric Progression

- A *geometric progression* is a series of the form  $a, ar, ar^2, ar^3, \dots, ar^k$ , where  $a, r \in \mathbf{R}$ .
- The sum of such a series is given by:

$$S = \sum_{i=0}^k ar^i$$

- We can reduce this to *closed form* via clever manipulation of summations...

# Geometric Sum Derivation

- Here we go...

$$\begin{aligned}
 S &= \sum_{i=0}^n ar^i \\
 rS &= r \sum_{i=0}^n ar^i = \sum_{i=0}^n rar^i = \sum_{i=0}^n arr^i = \sum_{i=0}^n ar^1 r^i \\
 &= \sum_{i=0}^n ar^{1+i} = \sum_{i=1}^{n+1} ar^{1+(i-1)} = \sum_{i=1}^{n+1} ar^i \\
 &= \left( \sum_{i=1}^n ar^i \right) + \sum_{i=n+1}^{n+1} ar^i = \left( \sum_{i=1}^n ar^i \right) + ar^{n+1} = \dots
 \end{aligned}$$

# Derivation example cont...

$$\begin{aligned}
 rS &= \left( \sum_{i=1}^n ar^i \right) + ar^{n+1} = (ar^0 - ar^0) + \left( \sum_{i=1}^n ar^i \right) + ar^{n+1} \\
 &= ar^0 + \left( \sum_{i=1}^n ar^i \right) + ar^{n+1} - ar^0 \\
 &= \left( \sum_{i=0}^0 ar^i \right) + \left( \sum_{i=1}^n ar^i \right) + ar^{n+1} - a \\
 &= \left( \sum_{i=0}^n ar^i \right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)
 \end{aligned}$$

## Concluding long derivation...

$$rS = S + a(r^{n+1} - 1)$$

$$rS - S = a(r^{n+1} - 1)$$

$$S(r - 1) = a(r^{n+1} - 1)$$

$$S = a \left( \frac{r^{n+1} - 1}{r - 1} \right) \quad \text{when } r \neq 1$$

$$\text{When } r = 1, S = \sum_{i=0}^n ar^i = \sum_{i=0}^n a1^i = \sum_{i=0}^n a \cdot 1 = (n + 1)a$$

## Nested Summations

- These have the meaning you'd expect.

$$\begin{aligned} \sum_{i=1}^4 \sum_{j=1}^3 ij &= \sum_{i=1}^4 \left( \sum_{j=1}^3 ij \right) = \sum_{i=1}^4 i \left( \sum_{j=1}^3 j \right) = \sum_{i=1}^4 i(1+2+3) \\ &= \sum_{i=1}^4 6i = 6 \sum_{i=1}^4 i = 6(1+2+3+4) \\ &= 6 \cdot 10 = 60 \end{aligned}$$

- Note issues of free vs. bound variables, just like in quantified expressions, integrals, etc.

## Some Shortcut Expressions

$$\sum_{k=0}^n ar^k = a(r^{n+1} - 1)/(r - 1), r \neq 1 \quad \text{Geometric series.}$$

$$\sum_{k=1}^n k = n(n + 1) / 2 \quad \text{Euler's trick.}$$

$$\sum_{k=1}^n k^2 = n(n + 1)(2n + 1) / 6 \quad \text{Quadratic series.}$$

$$\sum_{k=1}^n k^3 = n^2(n + 1)^2 / 4 \quad \text{Cubic series.}$$

# Using the Shortcuts

• Example: Evaluate

$$\sum_{k=50}^{100} k^2 .$$

– Use series splitting.

– Solve for desired summation.

– Apply quadratic series rule.

– Evaluate.

$$\sum_{k=1}^{100} k^2 = \left( \sum_{k=1}^{49} k^2 \right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left( \sum_{k=1}^{100} k^2 \right) - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338,350 - 40,425$$

$$= 297,925.$$

# Summations: Conclusion

- You need to know:

- How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^k a_i \quad \sum_{i=j}^{\infty} a_i \quad \sum_{x \in X} f(x) \quad \sum_{P(x)} f(x)$$

- Summation manipulation laws we covered.
- Shortcut closed-form formulas, & how to use them.