
6.1 RECURRENCE RELATIONS

DEF: A *recurrence system* is a finite set of *initial conditions*

$$a_0 = c_0, \quad a_1 = c_1, \quad \dots, \quad a_d = c_d$$

and a formula (called a *recurrence relation*)

$$a_n = f(a_0, \dots, a_{n-1})$$

that expresses a subscripted variable as a function of lower-indexed values. A sequence

$$\langle a_n \rangle = a_0, a_1, a_2, \dots$$

satisfying the initial conditions and the recurrence relation is called a *solution*.

Example 6.1.1: The recurrence system with initial condition

$$a_0 = 0$$

and recurrence relation

$$a_n = a_{n-1} + 2n - 1$$

has the sequence of squares as its solution:

$$\langle a_n \rangle = 0, 1, 4, 9, 16, 25, \dots$$

NÄIVE METHOD OF SOLUTION

Step 1. Use the recurrence to calculate a few more values beyond the given initial values.

Step 2. Spot a pattern and guess the right answer.

Step 3. Prove your answer is correct (by induction).

Example 6.1.1, continued:

Step 1. Starting from $a_0 = 0$, we calculate

$$a_1 = a_0 + 2 \cdot 1 - 1 = 0 + 1 = 1$$

$$a_2 = a_1 + 2 \cdot 2 - 1 = 1 + 3 = 4$$

$$a_3 = a_2 + 2 \cdot 3 - 1 = 4 + 5 = 9$$

$$a_4 = a_3 + 2 \cdot 4 - 1 = 9 + 7 = 16$$

Step 2. Looks like $f(n) = n^2$.

Step 3. BASIS: $a_0 = 0 = 0^2 = f(0)$.

IND HYP: Assume that $a_{n-1} = (n-1)^2$.

IND STEP: Then

$$\begin{aligned} a_n &= a_{n-1} + 2n - 1 && \text{from the recursion} \\ &= (n-1)^2 + 2n - 1 && \text{by IND HYP} \\ &= (n^2 - 2n + 1) + 2n - 1 && = n^2 \quad \diamond \end{aligned}$$

APPLICATIONS

Example 6.1.2: Compound Interest

Deposit \$1 to compound at annual rate r .

$$p_0 = 1 \quad p_n = (1 + r)p_{n-1}$$

EARLY TERMS: $1, 1 + r, (1 + r)^2, (1 + r)^3, \dots$

APPARENT PATTERN: $p_n = (1 + r)^n$

BASIS: True for $n = 0$.

IND HYP: Assume that $p_{n-1} = (1 + r)^{n-1}$

IND STEP: Then

$$\begin{aligned} p_n &= (1 + r)p_{n-1} \quad \text{by the recursion} \\ &= (1 + r)(1 + r)^{n-1} \quad \text{by IND HYP} \\ &= (1 + r)^n \quad \text{by arithmetic} \quad \diamond \end{aligned}$$

Example 6.1.3: Tower of Hanoi

RECURRENCE SYSTEM

$$h_0 = 0$$

$$h_n = 2h_{n-1} + 1$$

SMALL CASES: 0, 1, 3, 7, 15, 31, ...

APPARENT PATTERN: $h_n = 2^n - 1$ BASIS: $h_0 = 0 = 2^0 - 1$ IND HYP: Assume that $h_{n-1} = 2^{n-1} - 1$

IND STEP: Then

$$\begin{aligned} h_n &= 2h_{n-1} + 1 && \text{by the recursion} \\ &= 2(2^{n-1} - 1) + 1 && \text{by IND HYP} \\ &= 2^n - 1 && \text{by arithmetic } \diamond \end{aligned}$$

However, the naïve method has limitations:

- It can be non-trivial to spot the pattern.
- It can be non-trivial to prove that the apparent pattern is correct.

Example 6.1.4: Fibonacci Numbers

$$f_0 = 0 \quad f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

Fibo seq: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55,

APPARENT PATTERN (ha ha)

$$f_n = \frac{1}{2^n \sqrt{5}} \left[(1 + \sqrt{5})^n - (1 - \sqrt{5})^n \right]$$

It is possible, but not uncomplicated, to simplify this with the binomial expansion and to then use induction.

Sometimes there is no fixed limit on the number of previous terms used by a recursion.

Example 6.1.5: Catalan Recursion

$$c_0 = 1$$

$$c_n = c_0c_{n-1} + c_1c_{n-2} + \cdots + c_{n-1}c_0 \text{ for } n \geq 1.$$

SMALL CASES

$$c_1 = c_0c_0 = 1 \cdot 1 = 1$$

$$c_2 = c_0c_1 + c_1c_0 = 1 \cdot 1 + 1 \cdot 1 = 2$$

$$c_3 = c_0c_2 + c_1c_1 + c_2c_0 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$$

$$c_4 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$$

$$c_5 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$$

Catalan seq: 1, 1, 2, 5, 14, 42,

SOLUTION:
$$c_n = \frac{1}{n+1} \binom{2n}{n}$$

The Catalan recursion counts binary trees and other objects in computer science.

ADMONITION

- Most recurrence relations have no solution.
- Most sequences have no representation as a recurrence relation. (they are random)