3.2 SEQUENCES AND SUMMATIONS

DEF: A *sequence in a set* $A$ is a function $f$ from a subset of the integers (usually $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to $A$. The values of a sequence are also called *terms* or *entries*.

NOTATION: The value $f(n)$ is usually denoted $a_n$. A sequence is often written $a_0, a_1, a_2, \ldots$.

**Example 3.2.1:** Two sequences.

$$a_n = \frac{1}{n}, \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

$$b_n = (-1)^n, \quad 1, -1, 1, -1, \ldots$$

**Example 3.2.2:** Five ubiquitous sequences.

$$n^2 \quad 0, 1, 4, 9, 16, 25, 36, 49, \ldots$$

$$n^3 \quad 0, 1, 8, 27, 64, 125, 216, 343, \ldots$$

$$2^n \quad 1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

$$3^n \quad 1, 3, 9, 27, 81, 243, 729, 2187, \ldots$$

$$n! \quad 1, 1, 2, 6, 24, 120, 720, 5040, \ldots$$
STRINGS

DEF: A set of characters is called an alphabet.

Example 3.2.3: Some common alphabets:
{0, 1} the binary alphabet
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9} the decimal digits
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F} the hexadecimal digits
{A, B, C, D, . . . , X, Y, Z} English uppercase ASCII

DEF: A string is a sequence in an alphabet.

NOTATION: Usually a string is written without commas, so that consecutive characters are juxtaposed.

Example 3.2.4: If \( f(0) = M, f(1) = A, f(2) = T, \) and \( f(3) = H, \) then write “MATH”.
SPECIFYING a RULE

**Problem:** Given some initial terms \(a_0, a_1, \ldots, a_k\) of a sequence, try to construct a rule that is consistent with those initial terms.

**Approaches:** There are two standard kinds of rule for calculating a generic term \(a_n\).

**DEF:** A **recursion** for \(a_n\) is a function whose arguments are earlier terms in the sequence.

**DEF:** A **closed form** for \(a_n\) is a formula whose argument is the subscript \(n\).

**Example 3.2.5:** \(1, 3, 5, 7, 9, 11, \ldots\)

- **recursion:** \(a_0 = 1; \quad a_n = a_{n-1} + 2\) for \(n \geq 1\)
- **closed form:** \(a_n = 2n + 1\)

The differences between consecutive terms often suggest a recursion. Finding a recursion is usually easier than finding a closed formula.

**Example 3.2.6:** \(1, 3, 7, 13, 21, 31, 43, \ldots\)

- **recursion:** \(b_0 = 1; \quad b_n = b_{n-1} + 2n\) for \(n \geq 1\)
- **closed form:** \(b_n = n^2 + n + 1\)
Sometimes, it is significantly harder to construct a closed formula.

**Example 3.2.7:** 1, 1, 2, 3, 5, 8, 13, 21, 34, …

recursion: \( c_0 = 1, c_1 = 1; \)
\[ c_n = c_{n-1} + c_{n-2} \text{ for } n \geq 1 \]

closed form: \( c_n = \frac{1}{\sqrt{5}} [G^{m+1} - g^{m+1}] \)

where \( G = \frac{1 + \sqrt{5}}{2} \) and \( g = \frac{1 - \sqrt{5}}{2} \)

**INFERRING a RULE**

The ESSENCE of science is inferring rules from partial data.

**Example 3.2.8:** Sit under apple tree. Infer gravity.

**Example 3.2.9:** Watch starlight move 0.15 arc-seconds in total eclipse. Infer relativity.

**Example 3.2.10:** Observe biological species. Infer DNA.
Important life skill: Given a difficult general problem, start with special cases you can solve.

Example 3.2.11: Find a recursion and a closed form for the arithmetic progression:
\[ c, c + d, c + 2d, c + 3d, \ldots \]

recursion: \[ a_0 = c; \quad a_n = a_{n-1} + d \]
closed form: \[ a_n = c + nd. \]

Q: How would you decide that a given sequence is an arithmetic progression?

A: Calculate differences between consecutive terms.

DEF: The **difference sequence** for a sequence \( a_n \) is the sequence \( a'_n = a_n - a_{n-1} \) for \( n \geq 1 \).

Example 3.2.5 redux:

\[
\begin{array}{cccccc}
\quad a_n & : & 1 & 3 & 5 & 7 & 9 & 11 \\
\quad a'_n & : & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

Analysis: Since \( a'_n \) is constant, the sequence is specified by this recursion:

\[ a_0 = 1; a_n = a_{n-1} + 2 \text{ for } n \geq 1. \]

Moreover, it has this closed form:

\[
a_n = a_0 + a'_1 + a'_2 + \cdots + a'_n = a_0 + 2 + 2 + \cdots + 2 = 1 + 2n
\]
If you don’t get a constant sequence on the first difference, then try reiterating.

**Revisit Example 1.7.6:** 1, 3, 7, 13, 21, 31, 43, …

\[
\begin{align*}
\begin{array}{c}
b_n : \quad 1 & 3 & 7 & 13 & 21 & 31 & 43 \\
b'_n : \quad 2 & 4 & 6 & 8 & 10 & 12 \\
b''_n : \quad 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\end{align*}
\]

**Analysis:** Since \(b''_n\) is constant, we have
\[
b'_n = 2 + 2n
\]

Therefore,
\[
b_n = b_0 + b'_1 + b'_2 + \cdots + b'_n
\]
\[
= b_0 + 2 \sum_{j=1}^{n} j = 1 + (n^2 + n) = n^2 + n + 1
\]

**Consolation Prize:** Without knowing about finite sums, you can still extend the sequence:

\[
\begin{align*}
\begin{array}{c}
b_n : \quad 1 & 3 & 7 & 13 & 21 & 31 & 43 & 57 \\
b'_n : \quad 2 & 4 & 6 & 8 & 10 & 12 & 14 \\
b''_n : \quad 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\end{align*}
\]
SUMMATIONS

DEF: Let $a_n$ be a sequence. Then the **big-sigma** notation

\[ \sum_{j=m}^{n} a_j \]

means the sum

\[ a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n \]

TERMINOLOGY: $j$ is the *index of summation*

TERMINOLOGY: $m$ is the *lower limit*

TERMINOLOGY: $n$ is the *upper limit*

TERMINOLOGY: $a_j$ is the *summand*

**Theorem 3.2.1.** These formulas for summing falling powers are provable by induction (see §3.3):

\[ \sum_{j=1}^{n} j^1 = \frac{1}{2} (n + 1)^2 \quad \sum_{j=1}^{n} j^2 = \frac{1}{3} (n + 1)^3 \]

\[ \sum_{j=1}^{n} j^3 = \frac{1}{4} (n + 1)^4 \quad \sum_{j=1}^{n} j^k = \frac{1}{k+1} (n + 1)^{k+1} \]
Example 3.2.12: True Love and Thm 3.2.1
On the $j^{th}$ day ... True Love gave me

$$j + (j - 1) + \cdots + 1 = \frac{(j + 1)^2}{2} \text{ gifts.}$$

\[
= \frac{1}{2} \sum_{j=2}^{13} j^2 = \frac{1}{2} \left[ 2^2 + \cdots + 13^2 \right]
\]

\[
= \frac{1}{2} \left[ 2 + 6 + \cdots + 78 \right] = 364 \quad \text{slow}
\]

\[
= \frac{1}{2} \cdot \frac{14^3}{3} = 364 \quad \text{fast}
\]

Corollary 3.2.2. High-powered look-ahead to formulas for summing $j^k : j = 0, 1, ..., n.$

\[
\sum_{j=1}^{n} j^2 = \sum_{j=1}^{n} (j^2 + j^1) = \frac{1}{3} (n + 1)^3 + \frac{1}{2} (n + 1)^2
\]

\[
\sum_{j=1}^{n} j^3 = \sum_{j=1}^{n} (j^3 + 3j^2 + j^1) = \cdots
\]
POTLATCH RULES for CARDINALITY

**DEF:** non-dominating cardinality: Let $A$ and $B$ be sets. Then $|A| \leq |B|$ means that $\exists$ one-to-one function $f : A \to B$.

**DEF:** Set $A$ and $B$ have equal cardinality (write $|A| = |B|$) if $\exists$ bijection $f : A \to B$, which obviously implies that $|A| \leq |B|$ and $|B| \leq |A|$.

**DEF:** strictly dominating cardinality: Let $A$ and $B$ be sets. Then $|A| < |B|$ means that $|A| \leq |B|$ and $|A| \neq |B|$.

**DEF:** The cardinality of a set $A$ is $n$ if $|A| = |\{1, 2, \ldots, n\}|$ and 0 if $A = \emptyset$. Such cardinalities are called finite. NOTATION: $|A| = n$.

**DEF:** The cardinality of $\mathbb{N}$ is $\omega$ ("omega"), or alternatively, $\aleph_0$ ("aleph null").

**DEF:** A set is countable if it is finite or $\omega$.

**Remark:** $\aleph_0$ is the smallest infinite cardinality. The real numbers have cardinality $\aleph_1$ ("aleph one"), which is larger than $\aleph_0$, for reasons to be given.
**INFINITE CARDINALITIES**

**Proposition 3.2.3.** There are as many even nonnegative numbers as non-negative numbers.

**Proof:** \( f(2n) = n \) is a bijection. \( \diamond \)

**Theorem 3.2.4.** There are as many positive integers as rational fractions.

Proof: \[ f \left( \frac{p}{q} \right) = \frac{(p + q - 1)(p + q - 2)}{2} + p \] \( \diamond \)

**Example 3.2.13:** \[ f \left( \frac{2}{3} \right) = \frac{(4)(3)}{2} + 2 = 8 \]

Theorem 3.2.5. (G. Cantor) There are more positive real numbers than positive integers.

Semi-proof: A putative bijection $f : \mathbb{Z}^+ \to \mathbb{R}^+$ would generate a sequence in which each real number appears somewhere as an infinite decimal fraction, like this:

- $f(1) = 0.8841752032669031\ldots$
- $f(2) = 0.1415926531424450\ldots$
- $f(3) = 0.3202313932614203\ldots$
- $f(4) = 0.1679888138381728\ldots$
- $f(5) = 0.0452998136712310\ldots$

... 

$\cdots$ 

$f(?) = 0.73988\ldots$

Let $f(n)_k$ be the $k$th digit of $f(n)$, and let $\pi$ be the permutation $0 \mapsto 9, 1 \mapsto 0, \ldots 9 \mapsto 8$. Then the infinite decimal fraction whose $k$th digit is $\pi(f(n)_k)$ is not in the sequence. Therefore, the function $f$ is not onto, and accordingly, not a bijection. ♦