Example 4

Evaluate $\int \frac{x-2}{x^2(x-1)^2}\,dx$.

$$\frac{x-2}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2},$$

$$\begin{aligned} x-2 &= Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B)x + B; \end{aligned}$$

$$\begin{aligned} A + C &= 0, \\ -2A + B - C + D &= 0, \\ A - 2B &= 1, \\ B &= -2. \end{aligned}$$

Solving simultaneously, we have

$$A = -3, \qquad B = -2, \qquad C = 3, \qquad D = -1.$$

Thus,

$$\begin{aligned} \int \frac{x-2}{x^2(x-1)^2}\,dx &= \int \left(-\frac{3}{x} - \frac{2}{x^2} + \frac{3}{x-1} - \frac{1}{(x-1)^2} \right) dx \\ &= -3 \ln |x| + \frac{2}{x} + 3 \ln |x-1| + \frac{1}{x-1} + K \\ &= 3 \ln \left| \frac{x-1}{x} \right| + \frac{3x-2}{x(x-1)} + K. \end{aligned}$$

Problems

Evaluate the following integrals.

1. $\int \frac{dx}{x(x+1)}$.
2. $\int \frac{dx}{x(x-2)}$.
3. $\int \frac{x\,dx}{(x+1)(x+2)}$.
4. $\int \frac{dx}{x^2-4x-5}$.
5. $\int \frac{x^2+2}{x^2+2x}\,dx$.
6. $\int \frac{x^3-1}{x^2-3x}\,dx$.
7. $\int \frac{dx}{x^3-x}$.
8. $\int \frac{dx}{(x+1)(x+2)(x+3)}$.
9. $\int \frac{(x^3+2)\,dx}{x^3-3x^2+2x}$.
10. $\int \frac{x^3+4}{9x^3-4x}\,dx$.
11. $\int \frac{x\,dx}{(x-4)^2}$.
12. $\int \frac{x\,dx}{(x+1)^2}$.
13. $\int \frac{x^2\,dx}{(x+1)^2}$.
14. $\int \frac{x-2}{x(x-1)^2}\,dx$.
15. $\int \frac{dx}{x^3-x^2}$.
16. $\int \frac{dx}{x^2(x-3)^2}$.