On Properties and Aspects of Two Quasi-Cantoresque Sets

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Abstract
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Our paper, "On Properties and Aspects of Two Quasi-Cantoresque Sets," begins with a brief synopsis of the classic Cantor middle-third set, $\mathcal{C}$, and how the study of said led to our work on quasi-Cantoresque sets.

We introduce the quasi-Cantoresque sets that Krizan created, the Krizan Set, $\mathcal{K}$, and the Bubba Set, $\mathcal{B}$. We explore aspects of and we prove some nice results on each of the aforementioned quasi-Cantoresque sets; that they are compact, uncountable, perfect, and nowhere dense, totally disconnected subsets of $\mathbb{R}$ but do not necessarily have the same Hausdorff dimension as $\mathcal{C}$; moreover, they are non-self similar but symmetric ($\mathcal{K}$ and $\mathcal{B}$) so we opine as to a generalised Hausdorff dimension of $\mathcal{K}$ and we offer a limiting argument for said.

1. Introduction, Background, and Preliminary Definitions and Theorems

We note the debt owed to the great mathematician Georg Cantor, the founder of the theory of infinite sets, and comment that this work is but an extension of his work.

The primary author wrote his senior thesis under the direction of the second author [2] and then followed that work with more investigations of the Cantor Set and some sets that are somewhat like the Cantor set, Cantoresque sets and quasi-Cantoresque sets that led to papers that were presented at two undergraduate conferences, the Central Pennsylvania Regional Student Science & Mathematics Conference of Kappa Mu Epsilon at Bloomsburg University [3] and the Eastern Pennsylvania and Delaware (EPaDel) Section of the Mathematical Association of America Conference at Ursinus College [4].

Let U be a well defined universe.[1]
We assume the Zermelo - Fraenkel (with the axiom of choice) axioms of Set Theory; the field, order, and completeness axioms of $\mathbb{R}$; the Kolmogorov axioms of probability theory; the Peano axioms of $\mathbb{N}$; and, we assume the reader has a basic understanding of Real Analysis.
Definition 1.01: $\mathbb{N}^* = \{0, 1, 2, 3, \ldots, (k-1), k, \ldots\}$.
Definition 1.02: $\mathbb{N} = \{1, 2, 3, \ldots, (k-1), k, \ldots\}$.
Definition 1.03: $\mathbb{N}_k^* = \{0, 1, 2, 3, \ldots, (k-1), k\}$.
Definition 1.04: $\mathbb{N}_k = \{1, 2, 3, \ldots, (k-1), k\}$.
Definition 1.05: Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that $a < b$. Then a segment is $(a, b) = \{x | a < x < b\}$ and an interval is $[a, b] = \{x | a \leq x \leq b\}$. We note that $\ell([a, b]) = |b - a| = b - a = \ell((a, b))$ which is length (the usual metric on $\mathbb{R}$).
Theorem 1.01: (Archimedean Property of $\mathbb{N}$ in $\mathbb{R}$) The set $\mathbb{N}$ is unbounded above in $\mathbb{R}$.
Definition 1.06: A set S is finite if $S = \emptyset$ or if there exists $n \in \mathbb{N}$ and a bijection $f : \mathbb{N}_n \longrightarrow S$. We say the cardinality of S is 0 if $S = \emptyset$ and the cardinality of S is n if there exists $n \in \mathbb{N}$ and a bijection $f : \mathbb{N}_n \longrightarrow S$.
Definition 1.07: If S is not finite, it is infinite.
Definition 1.08: A set S is denumerable if there exists a bijection $f : \mathbb{N} \rightarrow S$. We say the cardinality of S is $\aleph_0$ when S is denumberable.
Notation 1.01: Let S be a set. denote the cardinality of S as $|S|$.
Definition 1.09: Let S and T be sets. The sets S and T are equipollent (or equinumerous or equipotent) if there exists a well defined bijection $f : S \rightarrow T$. We say the cardinality of S is equal to the cardinality of T or simply note either $S \cong T$ or $|S| = |T|$.
Theorem 1.02: Let S and T be sets. $|S| \leq |T|$ iff $\exists f : S \longrightarrow T$ such that f is a well defined injective function.
Theorem 1.03: Let S and T be sets. $|S| \geq |T|$ iff $\exists f : S \longrightarrow T$ such that f is a well defined surjective function.
Definition 1.10: A point x is a limit point of a set S if $\forall \varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap$

[1] For our paper all but a momentary side discussion has the property that we will be discussing set where $U = \mathbb{R}$ and it is linearly ordered by the relation less than or equal to ($\leq$).

$(S \backslash \{x\}) \neq \emptyset$. The set of all limit points of S is denoted S'.
Definition 1.11: The closure of S, denoted $\overline{S}$, is defined by $\overline{S} = S \cup S'$.
Definition 1.12: The boundary of S, denoted $bd(S)$, is defined by $bd(S) = \overline{S} \cup \overline{S^C}$.
Definition 1.13: A set S is closed iff $S = \overline{S}$.
Definition 1.14: A set S is bounded below if $\exists m \in \mathbb{R} \ni m \leq s \forall s \in S$.
Definition 1.15: A set S is not bounded below if $\forall m \in \mathbb{R} \exists s \in S \ni s < m$.
Definition 1.16: A set S is bounded above if $\exists p \in \mathbb{R} \ni p \geq s \forall s \in S$.
Definition 1.17: A set S is not bounded above if $\forall p \in \mathbb{R} \exists s \in S \ni s > p$.
Definition 1.18: A set S is bounded if is bounded above and bounded below.
Definition 1.19: A set S is compact iff every open cover of that set contains a finite subcover.
Theorem 1.04: Let $\Omega = \{A_i\}$ be a collection of closed subsets of $\mathbb{R}$. It is the case that $\bigcap \Omega$ is closed.
Theorem 1.05 (Heine-Borel): A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.
Definition 1.20: If a point $x \in S$ and $x \notin S'$, then x is called an isolated point.
Definition 1.21: A set S is perfect iff A is closed and A contains no isolated points.
Definition 1.22: A set S is disconnected iff there exists an open set U and an open set V such that $S \subseteq U \cup V$, $U \cap V = \emptyset$, $S \cap U \neq \emptyset$, and $S \cap V \neq \emptyset$.
Definition 1.23: If a set S is not disconnected, it is connected.
Definition 1.24: A set S is totally disconnected iff for every $x \in S$ and $y \in S$ such that $x \neq y$ there exists an open set V such that $U \cap V = \emptyset$, $x \in U$, $y \in V$, and $S \subseteq U \cup V$.
Definition 1.25: A set S is nowhere dense in $\mathbb{R}$ iff S^C is dense in $\mathbb{R}$.
Definition (Alternate 1.25): A set $S \subseteq T$ is nowhere dense in T iff $\overline{S}$ contains no segments that are subsets of T.
Definition 1.26: A set S is countable iff it is finite or denumerable. We say the cardinality of S, $|S| \leq \aleph_0$ if S is countable.
Definition 1.27: A set S is uncountable iff it is not countable. We say the cardinality of S, $|S| \geq \aleph_0$ if S is uncountable.
Definition 1.28: A set S has measure zero if $\forall \epsilon > 0 \quad \exists$ a collection of segments, $\Omega = \{S_i\}_{i \in I} \quad \ni |I| \leq \aleph_0$ such that Ω covers the set $S \ni \sum_{k=1}^{\infty} \ell(S_i) < \varepsilon$.

The classic middle 3^{rd} Cantor set is constructed by considering the universe to be $\mathbb{R}$. Then starting with the interval $C_0 = [0, 1]$ we remove the middle 3^{rd} segment $\left(\frac{1}{3}, \frac{2}{3}\right)$. What is left is $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Next we remove the middle 3^{rd} segments from these intervals; thus, $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ is left; so, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Next we remove the middle 3^{rd} segments from these intervals; thus, $C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$. We continue to remove these middle 3^{rd} segments from the remaining intervals on each C_j level $\forall i \in \mathbb{N}$. Thus, $\mathcal{C}$ (the classic middle 3^{rd} Cantor set) is inductively understood to be $\mathcal{C} = \bigcap_{j=1}^{\infty} C_j$.

This set cannot be called *the* Cantor set in general because there are other sets that are homomorphic to the classic middle 3^{rd} Cantor set (a generalised Cantor set, a middle 4^{th} Cantor set, middle 5^{th} Cantor set, etc.). Therefore, these other sets can also be called Cantor sets. All these sets have the same properties and characteristics.

More rigorously let us note that when we let $P = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{2^{i-1}} S_{i,k}$ where $S_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$
$S_{2,1} = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $S_{2,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$
$S_{3,1} = \left(\frac{1}{27}, \frac{2}{27}\right)$, $S_{3,2} = \left(\frac{7}{27}, \frac{8}{27}\right)$, $S_{3,3} = \left(\frac{19}{27}, \frac{20}{27}\right)$, and $S_{3,4} = \left(\frac{25}{27}, \frac{26}{27}\right)$
It is the case that $C_0 = [0, 1]$
$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$
$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$
$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \ . \ . \ . \cup \left[\frac{27}{27}, 1\right]$
So, for each $i \in \mathbb{N}$, C_i is derived by deleting the middle 2^{i-1} segments of length 3^{-i+1} from the intervals of the $(i-1)^{th}$ level to construct that which remains which are 2^i intervals of length 3^{-i}.
Therefore,

$$\mathcal{C} = [0, 1] - P = C_0 - \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{2^{i-1}} S_{i,k} = \bigcap_{j=1}^{\infty} C_j$$

Krizan[2] proved in [2] and [3] that $\mathcal{C}$ is a perfect, compact, and uncountable set such that it is nowhere dense in $[0, 1]$, is a totally disconnected subset of $\mathbb{R}$, with measure zero, and with Hausdorff dimension $\frac{\ln(2)}{\ln(3)}$.
Definition 1.29: A set S is a Cantor middle m- set iff the length of the middle segment removed from each interval are of the form $m^{-n} \quad \forall n \in \mathbb{N} \ni m \in \mathbb{N} \setminus \{1\}$.
Definition 1.30: A set S is a generalised Cantor set iff $\exists \alpha \in (0, 1)$ such that the length of the middle segment removed from each interval are of the form $\alpha \cdot 3^{-n} \quad \forall n \in \mathbb{N}$.
Definition 1.31: A set S is a Cantoresque middle $\frac{k}{m}$ set iff $\exists k \in \mathbb{N}, m \in \mathbb{N} \setminus \mathbb{N}_2, k \leq (m-2) \wedge \exists p \in \mathbb{N} \ni m = (2p-1)$ where the construction of the set inductively defined so that the middle segments removed are of length $m^{-n} \quad \forall n \in \mathbb{N}$ and the number of segments removed is k.
Definition 1.32: A set S is a generalised Cantoresque middle $\frac{k}{m}$ set iff $\exists \beta \in (0, 1), \exists k \in \mathbb{N}, m \in \mathbb{N} \setminus \mathbb{N}_3, k \leq (m-2) \wedge \exists p \in \mathbb{N} \ni m = (2p-1)$ where the construction of the set inductively defined so that the middle segments removed are of length $\beta \cdot m^{-n} \quad \forall n \in \mathbb{N}$ and the number of segments removed is k.
Definition 1.33: A set S is a β - Cantoresque set if $\exists \beta \in (1, \infty)$ such that the construction of the set inductively defined so that the middle segments removed are of length $\beta^{-n} \quad \forall n \in \mathbb{N}$ and the number of segments removed is $m \in \mathbb{N}^* \ni m < n$.
Definition 1.34: A set S is a quasi-Cantoresque set if it is the case that in the construction of the set inductively defined so that the middle segments removed are not necessarily of the same ratio in the sequence of construction levels to construction levels (e.g.: $\beta_n^{-n} \quad \forall n \in \mathbb{N}$ and the number of segments removed is $m \in \mathbb{N} \ni m < n$) or it is the case that in the construction of the set inductively defined so that there are middle segments removed then intervals added, then segments removed then intervals added, etc. such that the total length of the intervals added in a step does not exceed the total length of the segments removed previous to the addition of intervals in the construction of the set.
Example 1.01: Let $\alpha = \frac{1}{5}$. Consider the set S, a generalised Cantor set with $\alpha = \frac{1}{5}$ under definition 1.29.

[2]It has been proven in some books as well but Krizan did not use said; he was directed in his work under a modified Moore Method by the second author.

$S_0 = [0,1]$
$S_1 = \left[0, \frac{14}{30}\right] \cup \left[\frac{16}{30}, \frac{30}{30}\right]$
$S_2 = \left[0, \frac{20}{90}\right] \cup \left[\frac{22}{90}, \frac{42}{90}\right] \cup \left[\frac{48}{90}, \frac{68}{90}\right] \cup \left[\frac{70}{90}, \frac{90}{90}\right]$
$\vdots$

$$S = \bigcap_{n=0}^{\infty} S_n$$

$\ell(S_1) = \frac{14}{15}$, $\ell(S_2) = \frac{8}{9}$, $\ell(S_3) = \frac{116}{135}$, ... which inductively defines that

$$\lim_{n\to\infty} (\ell(S_n)) = \ell(S)$$

Note we derive a geometric series that sums to $\frac{1}{5}$ by considering the lengths of the complements of each $S_i \ni i \in \mathbb{N}$
$\ell(S_1^C) = \frac{1}{15}, \quad \ell(SC_2^C) = \left(\frac{1}{15} + \frac{2}{45}\right), \quad \ell(S_3^C) = \left(\frac{1}{15} + \frac{2}{45} + \frac{4}{135}\right), \quad \ldots$

$$\ell(S^C) = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{5} \cdot \frac{\frac{1}{3}}{1-\frac{2}{3}} = \frac{1}{5}$$

$$\implies \ell(S) = \frac{4}{5}$$

Hence, we clearly see that a Cantor set need not have measure zero.
Example 1.02: Consider the set $C_{\frac{2}{5}}$, a Cantoresque set constructed by removing middle two-fifths segments from each interval.
$C_0 = [0,1]$
$C_1 = \left[0, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{3}{5}\right] \cup \left[\frac{4}{5}, 1\right]$
$C_2 = \left[0, \frac{1}{25}\right] \cup \left[\frac{2}{25}, \frac{3}{25}\right] \cup \left[\frac{4}{25}, \frac{5}{25}\right] \cup \left[\frac{10}{25}, \frac{11}{25}\right] \cup \left[\frac{12}{25}, \frac{13}{25}\right] \cup \left[\frac{14}{25}, \frac{15}{25}\right] \cup \left[\frac{20}{25}, \frac{21}{25}\right] \cup \left[\frac{22}{25}, \frac{23}{25}\right] \cup \left[\frac{24}{25}, 1\right]$
$C_3 = \left[0, \frac{1}{125}\right] \cup \left[\frac{2}{125}, \frac{3}{125}\right] \cup \left[\frac{4}{125}, \frac{5}{125}\right] \cup \ldots \cup \left[\frac{120}{125}, \frac{121}{125}\right] \cup \left[\frac{122}{125}, \frac{123}{125}\right] \cup \left[\frac{124}{125}, 1\right]$
$\vdots$

$$C_{\frac{2}{5}} = \bigcap_{n=0}^{\infty} C_n$$

$\ell(C_1) = \frac{3}{5}$, $\ell(C_2) = \frac{9}{25}$, $\ell(C_3) = \frac{27}{125}$, ... which inductively defines that $\lim_{n\to\infty} (\ell(C_n)) = \ell(C_{\frac{2}{5}})$
Note we derive a geometric series that sums to one by considering the lengths of the complements of each $C_i \ni i \in \mathbb{N}$
$\ell(C_1^c) = \frac{2}{5}, \quad \ell(C_2^c) = \left(\frac{2}{5} + \frac{6}{25}\right), \quad \ell(C_3^c) = \left(\frac{2}{5} + \frac{6}{25} + \frac{18}{125}\right), \quad \ldots$

$$\ell(C_{\frac{2}{5}}^C) = \sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{3}{5}\right)^{n-1} = 1$$

Therefore, the length of $C_{\frac{2}{5}}$ is zero which, of course, implies the measure of $C_{\frac{2}{5}}$ is zero. Indeed $C_{\frac{2}{5}}$ is interesting, Krizan in [4] proved that $C_{\frac{2}{5}}$ is compact, perfect, uncountable, totally disconnected, measure zero, and more. Of special note is that one can construct an example of a generalised Cantor set or a generalised Cantoresque set that does not have measure zero and one can construct an example of a generalised Cantor set or a generalised Cantoresque set that does have measure zero.

We now review the concept of the dimension of a point-set.

Definition 1.35: Let U be a space with topology T. So, (U,T) is a topological space. (U,T) has small inductive dimension (or Menger - Urysohn dimension or just call it topological dimension) less than or equal to -1 iff $U=\emptyset$. Let $n \in \mathbb{N}^*$ and suppose that (U,T) has dimension less than or equal to $k \quad \ni k \in \mathbb{N}^* \quad \forall k \leq (n-1)$. Then, (U,T) has dimension less than or equal to n if it has a base, $\mathbf{B}$, such that $\forall B \in \mathbf{B}, bd\,(B)$ has dimension less than or equal to $(n-1)$.

Definition 1.36: Let (U,T) be a topological space. (U,T) has small inductive dimension (or Menger - Urysohn dimension or just call it topological dimension) -1 if $U=\emptyset$. Let $n \in \mathbb{N}^*$ and suppose that U has dimension less than or equal to n but it is false that U has dimension less than or equal to $(n-1)$, then, U has dimension n and we write $dim\,(U)=n$.

Theorem 1.06: Let (U,T) be a topological space and $dim\,(U)=n.$. Let $A \subseteq U. \quad dim\,(A) \leq n$

Theorem 1.07: $\mathbb{R}$ has a Menger - Urysohn dimension of 1.

Theorem 1.08: $x \in \mathbb{R}, y \in \mathbb{R}, x<y, \quad \wedge \quad A=[x,y]$ has Menger - Urysohn dimension of 1.

Definition 1.37: Let U be a well defined universe and S a set. A metric space is the set S along with a real-valued function, $f(\{a,b\}\,S, b \in S$ so that

(1) $f(\{a,a\})=0 \quad \forall a \in S$
(2) $f(\{a,b\})>0 \quad \forall a \in S, b \in S \ni a \neq b$
(3) $f(\{a,b\})=f(\{b,a\}) \quad \forall a \in S, b \in S$
(4) $f(\{a,c\}) \leq f(\{a,b\})+f(\{b,c\}) \quad \forall a \in S, b \in S, c \in S$

Definition 1.38: Let U be a metric space with metric $f$and S a set. Then $diam(S)$ is defined as the real number d such that $d=sup\,\{f(\{a,b\}: a \in S, b \in S\}$.

Definition 1.39: Let $\mathcal{F}$ be a collection of subsets of the metric space U be a space with topology T. Then $m\,(\mathcal{F})=sup\,\{diam\,(F): F \in \mathcal{F}\}$ where $diam\,(F)$ is the diameter of F, $m\,(\mathcal{F})$ is called the mesh of the collection $\mathcal{F}$.

Definition 1.40: $A \subseteq U$ where U is a metric space and let $k \in \mathbb{R} \ni k \geq 0$. Let $\mathcal{F}=\{F_i : i \in \mathbb{N}\}$ be a countable open cover of A. Then $S_k(A,\mathcal{F})=\sum_{i=1}^{\infty}(diamF_i)^k$.

Definition 1.41: $A \subseteq U$ where U is a metric space and let $k \in \mathbb{R} \ni k \geq 0$. Let $\varepsilon>0$. We define $H_k(A,\varepsilon)=inf\,\{S_k(A,\mathcal{F}): m\,(\mathcal{F})<\varepsilon\}$.
Then $M_k(A)=sup\,\{H_k(A,\varepsilon)\}$ is the Hausdorff k-measure of A.

Definition 1.42: $A \subseteq U$ where U is a metric space. The Hausdorff dimension of A is the real number, $dim_H(A)$, where $dim_H(A)=sup\,\{k>0 : M_k(A)>0\}$.

Whilst the definitions are quite involved, the understanding of the concepts of dimension and computation of them is not as difficult as the definition.

Theorem 1.09: $\mathbb{R}$ has Hausdorff dimension of 1.

Theorem 1.10: $x \in \mathbb{R}, y \in \mathbb{R}, x<y, \quad \wedge \quad A=[x,y]$ has Hausdorff dimension of 1.

Let $U=\mathbb{R}$. Let p be a point in $[0,1]$. $\{p\}$ has a Hausdorff dimension equal to its topological dimension which is 0.

Let $U=\mathbb{R}$. Notice that the interval $[0,1]$ is self-similar. It can be separated into $4=4^1$ pieces. Each is $\frac{1}{4}$ the size of the original. Each looks exactly like the original figure when magnified by a factor of 4 (a scaling factor). Thus, it has a Hausdorff dimension equal to its topological dimension which is 1.

Let $U=\mathbb{R}^2$. Take the unit square, $[0,1]\times[0,1]$ and stretch each side by 4, the result is 16 copies of the original and $16=4^2$ so the Hausdorff dimension is 2.

Let $U = \mathbb{R}^3$. The unit cube, $[0,1] \times [0,1] \times [0,1]$ The cube can be seen to have Hausdorff dimension 3 by noting stretching each side by 4 creates 64 copies of the original cube and $64 = 4^3$.

Let $U = \mathbb{R}$. The Cantor Set has Hausdorff dimension $\frac{ln(2)}{ln(3)} = log_3(2)$, which can be explained by tripling the size $\mathcal{C}$ produces two Cantor sets (whereas if you triple the size of the interval $[0,1]$ produces three copies, hence the interval has dimension $\frac{ln(3)}{ln(3)} = 1$ or if you quintuple the size of the interval $[0,1]$ produces five copies, hence the interval has dimension $\frac{ln(5)}{ln(5)} = 1$.

Let $U = \mathbb{R}$. Krizan proved in [4] that $C_{\frac{2}{5}}$ has Hausdorff dimension $\frac{ln(3)}{ln(5)}$, or if one so desires to express it as such, $log_5(3)$. Krizan also considered other Cantoresque sets which laid the ground-work for the Krizan set.

Example 1.03: Consider the set $C_{\frac{3}{7}}$, a Cantoresque set, constructed by removing middle three-seventh segments from each interval is compact, perfect, uncountable, totally disconnected, measure zero, and has Hausdorff dimension $\frac{ln(4)}{ln(7)}$.

Example 1.04: Consider the set $C_{\frac{4}{9}}$, a Cantoresque set, constructed by removing middle four-ninth segments from each interval is compact, perfect, uncountable, totally disconnected, measure zero, and has Hausdorff dimension $\frac{ln(5)}{ln(9)}$.

We are only concerned with sets in the universe $U = \mathbb{R}$ with the usual metric, $x \in \mathbb{R}, y \in \mathbb{R} \rightarrow d(x,y) = |x - y|$; hence the definition of the Hausdorff dimension for subsets of the reals suffices for this paper.

2. The Krizan Set

$\mathcal{K}$, the Krizan set[3], is created, like the Cantor set, by defining a first level:

$$Z_0 = [0, 1]$$

Note $\ell(Z_0) = 1$. The construction is defined by deleting segments from each preceding level $(n \in \mathbb{N}^*)$ $\forall n \in \mathbb{N}$.
So, we remove the middle third.

$$Z_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]$$

$\ell(Z_1) = \frac{2}{3}$ or $\ell(Z_1^C) = \frac{1}{3}$
Now, to construct Z_2 we remove the *two middle fifths from each interval*

$$Z_2 = \left[\frac{0}{15}, \frac{1}{15}\right] \cup \left[\frac{2}{15}, \frac{3}{15}\right] \cup \left[\frac{4}{15}, \frac{5}{15}\right] \cup \left[\frac{10}{15}, \frac{11}{15}\right] \cup \left[\frac{12}{15}, \frac{13}{15}\right] \cup \left[\frac{14}{15}, \frac{15}{15}\right]$$

$$Z_2 = \left[0, \frac{1}{15}\right] \cup \left[\frac{2}{15}, \frac{3}{15}\right] \cup \left[\frac{4}{15}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{11}{15}\right] \cup \left[\frac{12}{15}, \frac{13}{15}\right] \cup \left[\frac{14}{15}, 1\right]$$

$\ell(Z_2) = \frac{6}{15}$ or $\ell(Z_2^C) = \frac{9}{15}$
Next we remove the three middle sevenths from each existing interval of Z_2 to construct Z_3.

$$Z_3 = \left[0, \frac{1}{105}\right] \cup \left[\frac{2}{105}, \frac{3}{105}\right] \cup \left[\frac{4}{105}, \frac{5}{105}\right] \cup \left[\frac{6}{105}, \frac{7}{105}\right] \cup \left[\frac{14}{105}, \frac{15}{105}\right] \cup \left[\frac{16}{105}, \frac{17}{105}\right]$$
$$\cup \left[\frac{18}{105}, \frac{19}{105}\right] \cup \left[\frac{20}{105}, \frac{21}{105}\right] \cup \left[\frac{28}{105}, \frac{29}{105}\right] \cup \left[\frac{30}{105}, \frac{31}{105}\right] \cup \left[\frac{32}{105}, \frac{33}{105}\right] \cup \left[\frac{34}{105}, \frac{35}{105}\right]$$
$$\cup \left[\frac{70}{105}, \frac{71}{105}\right] \cup \left[\frac{72}{105}, \frac{73}{105}\right] \cup \left[\frac{74}{105}, \frac{75}{105}\right] \cup \left[\frac{76}{105}, \frac{77}{105}\right] \cup \left[\frac{84}{105}, \frac{85}{105}\right] \cup \left[\frac{86}{105}, \frac{87}{105}\right]$$
$$\cup \left[\frac{88}{105}, \frac{89}{105}\right] \cup \left[\frac{90}{105}, \frac{91}{105}\right] \cup \left[\frac{98}{105}, \frac{99}{105}\right] \cup \left[\frac{100}{105}, \frac{101}{105}\right] \cup \left[\frac{102}{105}, \frac{103}{105}\right] \cup \left[\frac{104}{105}, 1\right]$$

We continue with removing the four middle ninths from each existing interval of Z_3 to construct Z_4:

$$Z_4 = \left[0, \frac{1}{945}\right] \cup \left[\frac{2}{945}, \frac{3}{945}\right] \cup \left[\frac{4}{945}, \frac{5}{945}\right] \cup \left[\frac{6}{945}, \frac{7}{945}\right] \cup \left[\frac{8}{945}, \frac{9}{945}\right] \cup \left[\frac{126}{945}, \frac{127}{945}\right] \cup \ldots$$
$$\ldots \left[\frac{926}{945}, \frac{927}{945}\right] \cup \left[\frac{936}{945}, \frac{937}{945}\right] \cup \left[\frac{938}{945}, \frac{939}{945}\right] \cup \left[\frac{940}{945}, \frac{941}{945}\right] \cup \left[\frac{942}{945}, \frac{943}{945}\right] \cup \left[\frac{944}{945}, \frac{945}{945}\right]$$

Then we remove the five middle elevenths from each existing interval of Z_4 to construct Z_5, etc.
and therefore the Krizan set is:

$$\mathcal{K} = \bigcap_{n=0}^{\infty} Z_n$$

---

[3]McLoughlin insisted that it be called the Krizan set since Krizan created it. Hence, no assumption of any arrogance on the part of Krizan should be presumed.

3. The Bubba Set

$\mathcal{B}$, the Bubba set[4] is also begun like the Cantor set, by defining a first level; but it soon goes awry by not only deleting segments but by 'adding' intervals.

To start our construction we, like the Cantor set, define a first level: $B_0 = [0, 1]$. We delete the middle third segment, $\left(\frac{1}{3}, \frac{2}{3}\right)$, as with the Cantor set, so we have:

$$B_{(1,D)} = \left[\frac{0}{3}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right],$$

so we have two intevals of length $\frac{1}{3} \Longrightarrow \ell(B_{1,D}) = \frac{2}{3}$.

However, now we *'add'* two 'small' intervals within the removed segment, $\left(\frac{1}{3}, \frac{2}{3}\right)$, which are the 'largest' intervals (stage-intervals) to create a Cantor set through the deletion of middle-third segments such that (ignoring endpoints) they 'fit' within the removed segment,[5] $\left[\frac{3}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{6}{9}\right]$ to create

$$B_{(1,A)} = \left[\frac{0}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{9}{9}\right]$$

It is more illustrative and 'natural' to have it remain as unions of intervals (some of which intersect).

$$B_{(1,A)} = \left[\frac{0}{9}, \frac{3}{9}\right] \cup \left[\frac{3}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{6}{9}\right] \cup \left[\frac{6}{9}, \frac{9}{9}\right]$$

For $B_{(1,A)}$ so we have two intervals of length $\frac{1}{3}$ and two intevals of length $\frac{1}{9}$ each so

$\ell(B_{(1,A)}) = \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$

Now we delete middle third segment from each interval to yield $B_{(2,D)} =$

$$\left[\frac{0}{27}, \frac{3}{27}\right] \cup \left[\frac{6}{27}, \frac{9}{27}\right] \cup \left[\frac{9}{27}, \frac{10}{27}\right] \cup \left[\frac{11}{27}, \frac{12}{27}\right] \cup \left[\frac{15}{27}, \frac{16}{27}\right] \cup \left[\frac{17}{27}, \frac{18}{27}\right] \cup \left[\frac{18}{27}, \frac{21}{27}\right] \cup \left[\frac{24}{27}, \frac{27}{27}\right]$$

$\ell(B_{(2,D)}) = \left(\ell(B_{(1,A)}) - \frac{8}{27}\right) = 4 \cdot \frac{3}{27} + 4 \cdot \frac{1}{27} = \frac{16}{27}$

Now we add 'small' (next level size smaller) Cantor sets to the removed segments to create $B_{(2,A)} =$

$$\left[\frac{0}{27}, \frac{3}{27}\right] \cup \left[\frac{3}{27}, \frac{4}{27}\right] \cup \left[\frac{5}{27}, \frac{6}{27}\right] \cup \left[\frac{6}{27}, \frac{9}{27}\right] \cup \left[\frac{9}{27}, \frac{10}{27}\right] \cup \left[\frac{30}{81}, \frac{31}{81}\right] \cup$$
$$\left[\frac{32}{81}, \frac{33}{81}\right] \cup \left[\frac{11}{27}, \frac{12}{27}\right] \cup \left[\frac{36}{81}, \frac{37}{81}\right] \cup \left[\frac{44}{81}, \frac{45}{81}\right] \cup \left[\frac{15}{27}, \frac{16}{27}\right] \cup \left[\frac{48}{81}, \frac{49}{81}\right] \cup$$
$$\left[\frac{50}{81}, \frac{51}{81}\right] \cup \left[\frac{17}{27}, \frac{18}{27}\right] \cup \left[\frac{18}{27}, \frac{21}{27}\right] \cup \left[\frac{21}{27}, \frac{22}{27}\right] \cup \left[\frac{23}{27}, \frac{24}{27}\right] \cup \left[\frac{24}{27}, \frac{27}{27}\right]$$

[4]McLoughlin wanted this name since it is the 'close friend' of the Krizan set since Krizan created the Bubba set, also. In the South, the term 'Bubba' is oft used as a non-specific term of endearment.

[5]Formally, the removed segment being of length $\frac{1}{3^j}$ for some $j \in \mathbb{N}$ implies that the 'added' intervals are of length $\frac{1}{3^{(j+1)}}$.

$$\ell(B_{(2,A)}) = \frac{9}{81} + 2\cdot\frac{3}{81} + \frac{9}{81} + \frac{3}{81} + 2\cdot\frac{1}{81} + \frac{3}{81} + 2\cdot\frac{1}{81} + \frac{3}{81} + 2\cdot\frac{1}{81} + \frac{3}{81} + \frac{9}{81} + 2\cdot\frac{3}{81} + \frac{9}{81} = \frac{66}{81}$$

Now we delete middle third segment from each interval to yield $B_{(3,D)} =$

$$\left[\frac{0}{243},\frac{9}{243}\right] \cup \left[\frac{18}{243},\frac{27}{243}\right] \cup \left[\frac{27}{243},\frac{30}{243}\right] \cup \left[\frac{33}{243},\frac{36}{243}\right] \cup \left[\frac{45}{243},\frac{48}{243}\right] \cup \left[\frac{51}{243},\frac{54}{243}\right] \cup \left[\frac{54}{243},\frac{63}{243}\right] \cup$$
$$\left[\frac{72}{243},\frac{81}{243}\right] \cup \left[\frac{81}{243},\frac{84}{243}\right] \cup \left[\frac{87}{243},\frac{90}{243}\right] \cup \left[\frac{90}{243},\frac{91}{243}\right] \cup \left[\frac{92}{243},\frac{93}{243}\right] \cup \left[\frac{96}{243},\frac{97}{243}\right] \cup \left[\frac{98}{243},\frac{99}{243}\right] \cup$$
$$\left[\frac{99}{243},\frac{102}{243}\right] \cup \left[\frac{105}{243},\frac{108}{243}\right] \cup \left[\frac{108}{243},\frac{109}{243}\right] \cup \left[\frac{110}{243},\frac{111}{243}\right] \cup \left[\frac{132}{243},\frac{133}{243}\right] \cup \left[\frac{134}{243},\frac{135}{243}\right] \cup \left[\frac{135}{243},\frac{138}{243}\right] \cup$$
$$\left[\frac{141}{243},\frac{144}{243}\right] \cup \left[\frac{144}{243},\frac{145}{243}\right] \cup \left[\frac{146}{243},\frac{147}{243}\right] \cup \left[\frac{150}{243},\frac{151}{243}\right] \cup \left[\frac{152}{243},\frac{153}{243}\right] \cup \left[\frac{153}{243},\frac{156}{243}\right] \cup \left[\frac{159}{243},\frac{162}{243}\right] \cup$$
$$\left[\frac{162}{243},\frac{171}{243}\right] \cup \left[\frac{180}{243},\frac{189}{243}\right] \cup \left[\frac{189}{243},\frac{192}{243}\right] \cup \left[\frac{195}{243},\frac{198}{243}\right] \cup \left[\frac{207}{243},\frac{210}{243}\right] \cup \left[\frac{213}{243},\frac{216}{243}\right] \cup \left[\frac{216}{243},\frac{225}{243}\right] \cup$$
$$\left[\frac{235}{243},\frac{243}{243}\right]$$

$$\ell(B_{(3,D)}) = \frac{132}{243}$$

Now we add the next 'smallest' Cantor sets to the removed segments to create $B_{(3,A)} =$

$$\left[\frac{0}{81},\frac{3}{81}\right] \cup \left[\frac{3}{81},\frac{4}{81}\right] \cup \left[\frac{5}{81},\frac{6}{81}\right] \cup \left[\frac{6}{81},\frac{9}{81}\right] \cup \left[\frac{9}{81},\frac{10}{81}\right] \cup \left[\frac{30}{243},\frac{31}{243}\right] \cup \left[\frac{32}{243},\frac{33}{243}\right] \cup \left[\frac{11}{81},\frac{12}{81}\right] \cup$$
$$\left[\frac{36}{243},\frac{37}{243}\right] \cup \left[\frac{44}{243},\frac{45}{243}\right] \cup \left[\frac{15}{81},\frac{16}{81}\right] \cup \left[\frac{48}{243},\frac{49}{243}\right] \cup \left[\frac{50}{243},\frac{51}{243}\right] \cup \left[\frac{17}{81},\frac{18}{81}\right] \cup \left[\frac{18}{81},\frac{21}{81}\right] \cup$$
$$\left[\frac{63}{243},\frac{66}{243}\right] \cup \left[\frac{69}{243},\frac{72}{243}\right] \cup \left[\frac{24}{81},\frac{27}{81}\right] \cup \left[\frac{27}{81},\frac{28}{81}\right] \cup \left[\frac{84}{243},\frac{85}{243}\right] \cup \left[\frac{86}{243},\frac{87}{243}\right] \cup \left[\frac{29}{81},\frac{30}{81}\right] \cup$$
$$\left[\frac{90}{243},\frac{91}{243}\right] \cup \left[\frac{273}{729},\frac{274}{729}\right] \cup \left[\frac{275}{729},\frac{276}{729}\right] \cup \left[\frac{92}{243},\frac{93}{243}\right] \cup \left[\frac{279}{729},\frac{280}{729}\right] \cup \left[\frac{287}{729},\frac{288}{729}\right] \cup \left[\frac{96}{243},\frac{97}{243}\right] \cup$$
$$\left[\frac{291}{729},\frac{292}{729}\right] \cup \left[\frac{293}{729},\frac{294}{729}\right] \cup \left[\frac{98}{243},\frac{99}{243}\right] \cup \left[\frac{99}{243},\frac{102}{243}\right] \cup \left[\frac{306}{729},\frac{307}{729}\right] \cup \left[\frac{314}{729},\frac{315}{729}\right] \cup \left[\frac{105}{243},\frac{108}{243}\right] \cup$$
$$\left[\frac{108}{243},\frac{109}{243}\right] \cup \left[\frac{327}{729},\frac{328}{729}\right] \cup \left[\frac{329}{729},\frac{330}{729}\right] \cup \left[\frac{110}{243},\frac{111}{243}\right] \cup \left[\frac{333}{729},\frac{334}{729}\right] \cup \left[\frac{395}{729},\frac{396}{729}\right] \cup \left[\frac{132}{243},\frac{133}{243}\right] \cup$$
$$\left[\frac{399}{729},\frac{400}{729}\right] \cup \left[\frac{401}{729},\frac{402}{729}\right] \cup \left[\frac{134}{243},\frac{135}{243}\right] \cup \left[\frac{135}{243},\frac{138}{243}\right] \cup \left[\frac{414}{729},\frac{415}{729}\right] \cup \left[\frac{421}{729},\frac{424}{729}\right] \cup \left[\frac{141}{243},\frac{144}{243}\right] \cup$$
$$\left[\frac{144}{243},\frac{145}{243}\right] \cup \left[\frac{435}{729},\frac{436}{729}\right] \cup \left[\frac{437}{729},\frac{438}{729}\right] \cup \left[\frac{146}{243},\frac{147}{243}\right] \cup \left[\frac{441}{729},\frac{442}{729}\right] \cup \left[\frac{449}{729},\frac{450}{729}\right] \cup \left[\frac{150}{243},\frac{151}{243}\right] \cup$$
$$\left[\frac{453}{729},\frac{454}{729}\right] \cup \left[\frac{455}{729},\frac{456}{729}\right] \cup \left[\frac{152}{243},\frac{153}{243}\right] \cup \left[\frac{153}{243},\frac{156}{243}\right] \cup \left[\frac{468}{729},\frac{471}{729}\right] \cup \left[\frac{474}{729},\frac{477}{729}\right] \cup \left[\frac{159}{243},\frac{162}{243}\right] \cup$$
$$\left[\frac{162}{243},\frac{171}{243}\right] \cup \left[\frac{513}{729},\frac{522}{729}\right] \cup \left[\frac{531}{729},\frac{540}{729}\right] \cup \left[\frac{180}{243},\frac{189}{243}\right] \cup \left[\frac{63}{81},\frac{64}{81}\right] \cup \left[\frac{192}{243},\frac{193}{243}\right] \cup \left[\frac{582}{729},\frac{585}{729}\right] \cup$$
$$\left[\frac{195}{243},\frac{198}{243}\right] \cup \left[\frac{594}{729},\frac{597}{729}\right] \cup \left[\frac{618}{729},\frac{621}{729}\right] \cup \left[\frac{207}{243},\frac{210}{243}\right] \cup \left[\frac{630}{729},\frac{633}{729}\right] \cup \left[\frac{636}{729},\frac{639}{729}\right] \cup \left[\frac{213}{243},\frac{72}{81}\right] \cup$$
$$\left[\frac{72}{81},\frac{75}{81}\right] \cup \left[\frac{225}{243},\frac{228}{243}\right] \cup \left[\frac{231}{243},\frac{234}{243}\right] \cup \left[\frac{78}{81},\frac{81}{81}\right]$$

$$\ell(B_{(3,A)}) = \frac{546}{729}$$

Now we delete the middle third segments from each interval to create $B_{(4,D)} =$

$$\left[\frac{0}{81},\frac{1}{81}\right] \cup \left[\frac{2}{81},\frac{3}{81}\right] \cup \left[\frac{9}{243},\frac{10}{243}\right] \cup \left[\frac{11}{243},\frac{12}{243}\right] \cup \left[\frac{15}{243},\frac{16}{243}\right] \cup \left[\frac{17}{243},\frac{18}{243}\right] \cup \left[\frac{6}{81},\frac{7}{81}\right] \cup \left[\frac{8}{81},\frac{9}{81}\right] \cup$$
$$\left[\frac{27}{243},\frac{28}{243}\right] \cup \left[\frac{29}{243},\frac{30}{243}\right] \cup \left[\frac{90}{729},\frac{91}{729}\right] \cup \left[\frac{92}{729},\frac{93}{729}\right] \cup \left[\frac{96}{729},\frac{97}{729}\right] \cup \left[\frac{98}{729},\frac{99}{729}\right] \cup \left[\frac{33}{243},\frac{34}{243}\right] \cup$$
$$\left[\frac{35}{243},\frac{36}{243}\right] \cup \left[\frac{108}{729},\frac{109}{729}\right] \cup \left[\frac{110}{729},\frac{111}{729}\right] \cup \left[\frac{112}{729},\frac{113}{729}\right] \cup \left[\frac{114}{729},\frac{115}{729}\right] \cup \left[\frac{45}{243},\frac{46}{243}\right] \cup \left[\frac{47}{243},\frac{48}{243}\right] \cup$$
$$\left[\frac{144}{729},\frac{145}{729}\right] \cup \left[\frac{146}{729},\frac{147}{729}\right] \cup \left[\frac{150}{729},\frac{151}{729}\right] \cup \left[\frac{152}{729},\frac{153}{729}\right] \cup \left[\frac{51}{243},\frac{52}{243}\right] \cup \left[\frac{53}{243},\frac{54}{243}\right] \cup \left[\frac{54}{243},\frac{57}{243}\right] \cup$$
$$\left[\frac{60}{243},\frac{63}{243}\right] \cup \left[\frac{63}{243},\frac{64}{243}\right] \cup \left[\frac{65}{243},\frac{66}{243}\right] \cup \left[\frac{69}{243},\frac{70}{243}\right] \cup \left[\frac{71}{243},\frac{72}{243}\right] \cup \left[\frac{24}{81},\frac{25}{81}\right] \cup \left[\frac{26}{81},\frac{27}{81}\right] \cup \ldots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\ldots \left[\frac{189}{243},\frac{190}{243}\right] \cup \left[\frac{191}{243},\frac{192}{243}\right] \cup \left[\frac{596}{729},\frac{597}{729}\right] \cup \left[\frac{598}{729},\frac{599}{729}\right] \cup \left[\frac{582}{729},\frac{583}{729}\right] \cup \left[\frac{584}{729},\frac{585}{729}\right] \cup \left[\frac{195}{243},\frac{196}{243}\right] \cup$$
$$\left[\frac{197}{243},\frac{198}{243}\right] \cup \left[\frac{594}{729},\frac{595}{729}\right] \cup \left[\frac{596}{729},\frac{597}{729}\right] \cup \left[\frac{618}{729},\frac{619}{729}\right] \cup \left[\frac{620}{729},\frac{621}{729}\right] \cup \left[\frac{207}{243},\frac{208}{243}\right] \cup \left[\frac{209}{243},\frac{210}{243}\right] \cup$$
$$\left[\frac{630}{729},\frac{631}{729}\right] \cup \left[\frac{632}{729},\frac{633}{729}\right] \cup \left[\frac{636}{729},\frac{637}{729}\right] \cup \left[\frac{638}{729},\frac{639}{729}\right] \cup \left[\frac{213}{243},\frac{214}{243}\right] \cup \left[\frac{215}{243},\frac{216}{243}\right] \cup \left[\frac{72}{81},\frac{73}{81}\right] \cup$$
$$\left[\frac{74}{81},\frac{75}{81}\right] \cup \left[\frac{225}{243},\frac{226}{243}\right] \cup \left[\frac{227}{243},\frac{228}{243}\right] \cup \left[\frac{231}{243},\frac{232}{243}\right] \cup \left[\frac{233}{243},\frac{234}{243}\right] \cup \left[\frac{78}{81},\frac{79}{81}\right] \left[\frac{80}{81},\frac{81}{81}\right]$$

$$\ell(B_{(4,D)}) = \frac{1092}{2187}$$

We continue $\forall n \in \mathbb{N}$ constructing $B_{n,D}$ then $B_{n,A}$ and $\overset{n\to\infty}{\longrightarrow} \mathcal{B}$

4. Proofs on Some of the Properties of the Krizan Set

Theorem 4.01: $\mathcal{K}$ is compact.
Proof: Assume the premises.
By definition of $\mathcal{K}$, $\mathcal{K}$ is a generalized intersection of intervals.
Since intervals are closed, by theorem 1.04, $\mathcal{K}$ is closed.
Further, $\mathcal{K} \subseteq [0,1]$.
$[0,1]$ is bounded below by -1 and above by 2. Thus $\mathcal{K}$ is bounded above and below.
Hence $\mathcal{K}$ is bounded.
Therefore, $\mathcal{K}$ is compact by the Heine-Borel Theorem (theorem 1.05).

Q.E.D.

Theorem 4.02: $\forall b \in \mathcal{K}, \quad b$ is not an isolated point.
Proof: Assume the premises.
Suppose $\exists b \in \mathcal{K}$ such that b is an isolated point.
Then $\exists \varepsilon_1 > 0$ such that $(b - \varepsilon_1, b + \varepsilon_1) \cap \mathcal{K} = \{b\}$.
Since $\varepsilon_1 > 0$, $\frac{1}{\varepsilon_1}$ exists and is greater than 0.
$\exists m \in \mathbb{N}$ such that $(m-1) \leqslant \frac{1}{\varepsilon_1} < m \Rightarrow \frac{1}{m} \in (0, \varepsilon_1)$.
On the Z_m level each interval is of length less than $\frac{1}{3^m}$.
Further, it is ther case that b is in one of theses intervals.
Since $\frac{1}{3^m} < \frac{1}{m} < \varepsilon_1$, the interval containing b of Z_m is contained in $(b - \varepsilon_1, b + \varepsilon_1)$, and it contains a point of $\mathcal{K}$ distinct from b. #!
Therefore, b is not an isolated point of $\mathcal{K}$.

Q.E.D.

Corollary 4.02: $\mathcal{K}$ is perfect. It follows from theorem 4.01 and theorem 4.02.

Theorem 4.03: Every endpoint of an interval that is a subset of Z_m $(m \in \mathbb{N})$ in the construction of $\mathcal{K}$ is a point of $\mathcal{K}$.
Proof: Assume the premises.
$0 \in \mathcal{K}$ and $1 \in \mathcal{K}$, $[0,1] = Z_0$. Let $p \in Z_k$ for some $k \in \mathbb{N}$ such that p is an endpoint of an interval in Z_k.Clearly $p \in Z_0$. Consider Z_{k+1}, none of the $\frac{k+1}{2\cdot(k+1)+1}$ segments deleted from Z_k contain p. Therefore, $p \in Z_{k+1}$.

$$\Rightarrow p \in Z_n \forall n \in \mathbb{N} \Rightarrow p \in \mathcal{K}$$

Q.E.D.

Theorem 4.04: $\mathcal{K}$ is nowhere dense in $[0,1]$.
Proof: Assume the premises.
Suppose $\overline{\mathcal{K}}$ contains a segment.
Since $\mathcal{K}$ is closed, $\overline{\mathcal{K}} = \mathcal{K}$
Then $\exists$ a segment $S \subseteq \mathcal{K}$
Without loss of generality, let $S = (p,q)$ such that $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $0 < p < q < 1$.
Then $(p,q) \subseteq \bigcap_{j=1}^{\infty} Z_j$

Let $l = |p - q| > 0$
So $l \in (0,1) \Rightarrow \frac{1}{l} \in (1, \infty)$
$\exists m \in \mathbb{N}$ such that $(m-1) \leqslant \frac{1}{l} < m \Rightarrow \frac{1}{m} \in (0, l)$
Since $3^m > m$ and $(\prod_{i=1}^{m} 2 \cdot (i+1)) > 3^m > m \Rightarrow$
$0 < \frac{1}{(\prod_{i=1}^{m} 2 \cdot (i+1))} < \frac{1}{3^m} < \frac{1}{m} < l$
Thus, on the Z_{m+1} level of the construction of $\mathcal{K}$, each interval is $\frac{1}{(\prod_{i=1}^{m+1} 2 \cdot (i+1))}$ in length and

$$\left(\prod_{i=1}^{m+1} (2 \cdot (i+1)) \right)^{-1} < l$$

So, $S = (p, q)$ is not a subset of any interval which is a subset of Z_{m+1}. So $\exists r \in S$ such that $r \in [0,1] \setminus Z_m$
Thus $r \notin \mathcal{K} \Rightarrow r \notin \overline{\mathcal{K}}$ #!
Hence, we note that $\overline{\mathcal{K}}$ contains no segments.
Hence $\mathcal{K}$ is nowhere dense in $[0,1]$.

Q.E.D.

Theorem 4.05: $\mathcal{K}$ is totally disconnected.
Proof: Assume the premises.
Let $x \in \mathcal{K}$, $y \in \mathcal{K}$ such that $x \neq y$, without loss of generality let $x < y$.
Let $d = |x - y|$ So $d \in (0,1)$ Then $\frac{1}{d} \in (1, \infty)$ $\exists n \in \mathbb{N}$ such that $(n-1) \leqslant \frac{1}{d} < n$
So, $\frac{1}{n} \in (0, d)$ Since $3^n > n$ and $(\prod_{i=1}^{n} 2 \cdot (i+1)) > 3^n > n \Rightarrow$
$0 < \frac{1}{(\prod_{i=1}^{n} 2 \cdot (i+1))} < \frac{1}{3^n} < \frac{1}{n} < d$
Notice $Z_n = \bigcup_{i \in I} Z_{(n,i)}$ where $I = \mathbb{N}_k$ for some $k \in \mathbb{N}$ and $Z_{(n,i)} \forall i \in I$ are intervals.
Therefore, $\exists j \in I$ such that $x \in Z_{(n,j)}$ and $y \in Z_{(n,(j+p))}$, where $p \in \mathbb{N}$.
$\exists S$ where $\forall z \in S z \notin \mathcal{K}$where $\forall a \in Z_{(n,j)}$ and $b \in Z_{(n,(j+p))}$, it is the case $a < z < b$ and moreover we note that $x < z < y$.
Let $w \in S$.
Let $U = (-1, w)$, $V = (w, 2)$

$$U \cap V = \emptyset$$
$$\mathcal{K} \subseteq U \cup V$$
$$x \in U, \quad y \in V$$

Therefore, $\mathcal{K}$ is totally disconnected.

Q.E.D.

So, we do have that $\mathcal{K}$ is a perfect, compact set such that it is a totally disconnected nowhere dense subset of $[0,1]$.

Theorem 4.06: $\mathcal{K}$ is uncountable.
Proof: Assume the premises.
It behooves us to recall the Cantor set is constructed by considering the interval $C_0 = [0,1]$ and removing the middle 3^{rd} segment $\left(\frac{1}{3}, \frac{2}{3}\right)$. Then, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Next we remove the middle 3^{rd} segments from these intervals to yield $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Thus, $\mathcal{C}$ is inductively $\mathcal{C} = \bigcap_{j=1}^{\infty} C_j$.
We need also recall $\mathcal{K}$, the Krizan set is constructed by considering the interval $Z_0 = [0,1]$, we remove the middle third to yield $Z_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Next,

Z_2 is constructed by removing the two middle fifths from each interval to yield $Z_2 = \left[\frac{0}{15}, \frac{1}{15}\right] \cup \left[\frac{2}{15}, \frac{3}{15}\right] \cup \left[\frac{4}{15}, \frac{5}{15}\right] \cup \left[\frac{10}{15}, \frac{11}{15}\right] \cup \left[\frac{12}{15}, \frac{13}{15}\right] \cup \left[\frac{14}{15}, \frac{15}{15}\right]$ whilst we remove the three middle sevenths from each existing interval of Z_2 to construct Z_3 so, we have $Z_3 = \left[0, \frac{1}{105}\right] \cup \left[\frac{2}{105}, \frac{3}{105}\right] \cup \left[\frac{4}{105}, \frac{5}{105}\right] \cup \left[\frac{6}{105}, \frac{7}{105}\right] \cup \left[\frac{14}{105}, \frac{15}{105}\right] \cup \left[\frac{16}{105}, \frac{17}{105}\right] \cup \left[\frac{18}{105}, \frac{19}{105}\right] \cup \left[\frac{20}{105}, \frac{21}{105}\right] \cup \left[\frac{28}{105}, \frac{29}{105}\right] \cup \left[\frac{30}{105}, \frac{31}{105}\right] \cup \left[\frac{32}{105}, \frac{33}{105}\right] \cup \left[\frac{34}{105}, \frac{35}{105}\right] \cup \left[\frac{70}{105}, \frac{71}{105}\right] \cup \left[\frac{72}{105}, \frac{73}{105}\right] \cup \left[\frac{74}{105}, \frac{75}{105}\right] \cup \left[\frac{76}{105}, \frac{77}{105}\right] \cup \left[\frac{84}{105}, \frac{85}{105}\right] \cup \left[\frac{86}{105}, \frac{87}{105}\right] \cup \left[\frac{88}{105}, \frac{89}{105}\right] \cup \left[\frac{90}{105}, \frac{91}{105}\right] \cup \left[\frac{98}{105}, \frac{99}{105}\right] \cup \left[\frac{100}{105}, \frac{101}{105}\right] \cup \left[\frac{102}{105}, \frac{103}{105}\right] \cup \left[\frac{104}{105}, 1\right]$. We continue with removing the four middle ninths from each existing interval of Z_3 to construct Z_4; we remove the five middle elevenths from each existing interval of Z_4 to construct Z_5, etc. Thus, $\mathcal{K}$ is inductively $\mathcal{K} = \bigcap_{n=0}^{\infty} Z_n$.

Now, let us define an injective function from the Cantor set to the Krizan set; to do so we will define functions from the levels of construction of each set.

$$f_0 : C_0 \longrightarrow Z_0 \quad \ni \quad f_0(x) = x$$

So,

$$[0,1] \xrightarrow{f_0} [0,1]$$

Define

$$f_1 : C_1 \longrightarrow Z_1 \quad \ni \quad f_1(x) = x$$

So,

$$\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \xrightarrow{f_1} \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \quad \ni \quad f_1(x) = x$$

Since C_2 has 4 intervals and Z_2 has six intervals we can define a function that scales the first two intervals of C_2 to the first two intervals of Z_2 and the first last intervals of C_2 to the last two intervals of Z_2 (for the first interval $f_{(2,1)}(x) = \frac{3x}{5}$ and for all else simply a transformation of $f_{(2,1)}$ for the appropriate scaling an interval of length $\frac{1}{9}$ to an interval of length $\frac{1}{15}$ and recentering it.

$$\left[0, \frac{1}{9}\right] \xrightarrow{f_{(2,1)}} \left[0, \frac{1}{15}\right], \quad \left[\frac{2}{9}, \frac{1}{3}\right] \xrightarrow{f_{(2,2)}} \left[\frac{2}{15}, \frac{3}{15}\right], \left[\frac{2}{3}, \frac{7}{9}\right] \xrightarrow{f_{(2,3)}} \left[\frac{12}{15}, \frac{13}{15}\right], \quad \left[\frac{8}{9}, 1\right] \xrightarrow{f_{(2,4)}} \left[\frac{14}{15}, 1\right]$$

$$f_2 : C_2 \longrightarrow Z_2 \ni f_2 = \bigcup_{i=1}^{4} f_{(2,i)} \quad i \in \mathbb{N}_4$$

Similarly, C_3 has 8 intervals and Z_3 has 24 intervals; hence, we can define a function that scales the first four intervals of C_3 to the first four intervals of Z_3 and the last four intervals of C_3 to the last four intervals of Z_3 (for the first interval $f_{(3,1)}(x) = \frac{3^3 \cdot x}{3 \cdot 5 \cdot 7}$ and for all else simply a transformation of $f_{(3,1)}$ for the appropriate scaling an interval of length $\frac{1}{27}$ to an interval of length $\frac{1}{105}$ and recentering it.

$$\left[0, \frac{1}{27}\right] \xrightarrow{f_{(3,1)}} \left[0, \frac{1}{105}\right], \left[\frac{2}{27}, \frac{1}{9}\right] \xrightarrow{f_{(3,2)}} \left[\frac{2}{105}, \frac{3}{105}\right], \left[\frac{2}{9}, \frac{7}{27}\right] \xrightarrow{f_{(3,3)}} \left[\frac{4}{105}, \frac{5}{105}\right], \ldots,$$

$$\left[\frac{8}{9}, \frac{25}{27}\right] \xrightarrow{f_{(3,7)}} \left[\frac{102}{105}, \frac{103}{105}\right], \left[\frac{26}{27}, 1\right] \xrightarrow{f_{(3,8)}} \left[\frac{104}{105}, 1\right]$$

$$f_3 : C_3 \longrightarrow Z_3 \ni f_3 = \bigcup_{i=1}^{8} f_{(3,i)} \quad i \in \mathbb{N}_8$$

We inductively define the functions $f_n : C_n \longrightarrow Z_n$ as above $\forall n \in \mathbb{N}$. It is facile to argue f_n is a well defined injective function from C_n into Z_n. Then we define the relation $F : \mathcal{C} \longrightarrow \mathcal{K}$ such that

$$F = \bigcap_{i=0}^{\infty} f_i$$
$$F : \mathcal{C} \longrightarrow \mathcal{K}$$

The relation F is a well defined function since $\forall n \in \mathbb{N}, f_n$ is a well defined function from C_n into Z_n. Furthermore, F is injective since $\forall n \in \mathbb{N}, f_n$ is injective from C_n into Z_n. So by theorem 1.02, it is the case that $|\mathcal{C}| \leq |\mathcal{K}| \Longrightarrow \mathcal{K}$ is uncountable.

Q.E.D.

Theorem 4.07: $m(\mathcal{K}) = 0$.
Proof: Assume the premises.
Consider $\ell(Z_0) = 1$.

$$\ell(Z_1) = \frac{2}{3}$$
$$\ell(Z_2) = \frac{6}{15} = \frac{2}{3} \cdot \frac{3}{5}$$
$$\ell(Z_3) = \frac{24}{105} = \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{4}{7}$$
$$\vdots$$
$$\ell(Z_n) = \prod_{i=1}^{n} \left(\frac{i+1}{2i+1}\right)$$
$$\vdots$$

So,

$$\ell(\mathcal{K}) = \prod_{i=1}^{\infty} \left(\frac{i+1}{2i+1}\right)$$

Now, consider the sequence, $s : \mathbb{N} \to \mathbb{R}$, such that it is defined by $\left\{ s_n = \prod_{i=1}^{n} \left(\frac{i+1}{2i+1}\right) : n \in \mathbb{N} \right\}$.
We note that each term of the subsequence $\left\{ s_n = \prod_{i=6}^{n} \left(\frac{i+1}{2i+1}\right) : n \in \mathbb{N} \right\}$ has the property that $s_n \leq \frac{1}{n}$ when (obviously) $n \geqslant 6$. Quite clearly, $\lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow \lim_{n\to\infty} s_n \leq 0$. But, $\forall n \in \mathbb{N} s_n \geq 0 \Longrightarrow \lim_{n\to\infty} s_n \geq 0$.
So, by trichotomy, $\lim_{n\to\infty} s_n = 0$.
Since $\mathcal{K} \subseteq \mathbb{R} \quad \Longrightarrow m(\mathcal{K}) = 0$

Q.E.D.

We end this section with some comments on various interesting properties of the Krizan set. We note that it is not a fractal in the classical sense since it is not self-similar. However, it is symmetric about $\frac{1}{2}$.

5. Results on Some of the Properties of the Bubba Set

All the analagous theorems for the Krizan set follow for the Bubba set easily (more directly than $\mathcal{K}$) since we have the simple:
Lemma 5.01: $\mathcal{C} \subset \mathcal{B}$.
Theorem 5.01: $\mathcal{B}$ is compact.
Theorem 5.02: $\forall b \in \mathcal{B}, \quad b$ is not an isolated point.
Corollary 5.02: $\mathcal{B}$ is perfect. It follows from theorem 5.01 and theorem 5.02.
Theorem 5.04: $\mathcal{B}$ is nowhere dense in $[0, 1]$.
Theorem 5.05: $\mathcal{B}$ is totally disconnected.
Theorem 5.06: $\mathcal{B}$ is uncountable.
When we constructed $\mathcal{B}$ we noted that the idea was that as $n \longrightarrow \infty$ to derive:
$B_{n,A} \searrow$
. $\mathcal{B}$
$B_{n,D} \nearrow$
Indeed an interesting sequence is the sequence of lengths of the stages of the construction of the Bubba set:

$$1, \frac{2}{3}, \frac{8}{9}, \frac{16}{27}, \frac{66}{81}, \frac{132}{243}, \frac{546}{729}, \frac{1092}{2187}, \ldots$$

$\ell(B_{n,A}) \searrow$
. $\ell(\mathcal{B})$
$\ell(B_{n,D}) \nearrow$
But a careful look at the lengths yields a view of the subsequences of $\ell(B_{n,A})$ and $\ell(B_{n,D})$ finds
$1, \quad , \frac{8}{9}, \quad , \frac{66}{81}, \quad , \frac{546}{729}, \quad , \ldots \searrow$
. $\ell(\mathcal{B})$
. $\frac{2}{3}, \quad , \frac{16}{27}, \quad , \frac{132}{243}, \quad , \frac{1092}{2187}, \ldots \nearrow$

However, **both** subsequnces converge to 0; hence, we note that the Bubba set has the aspect that is the case that:

Theorem 5.07: $m(\mathcal{B}) = 0$.
<u>Proof</u>: Assume the premises.
Note $B_{(1,D)} = \left[\frac{0}{3}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right]$ and that $\forall i \in \mathbb{N}$ it is the case that
$\mathcal{C} \subseteq B_{(i,D)} \wedge \mathcal{C} \subseteq B_{(i,A)}$.
Note $B_{(1,A)} = \left[\frac{0}{9}, \frac{3}{9}\right] \cup \left[\frac{3}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{6}{9}\right] \cup \left[\frac{6}{9}, \frac{9}{9}\right]$
So, and that $\forall i \in \mathbb{N}$ it is the case that there are two sets equipollent with
$\mathcal{C}$ that are subsets of $B_{(i+1,D)} \wedge B_{(i,A)}$ (the sets that are created from the Cantor set produced by the addition of the $\left[\frac{3}{9}, \frac{4}{9}\right] \wedge \left[\frac{5}{9}, \frac{6}{9}\right]$).
Now, it is the case that inductively for the deletion steps ($\forall i \in \mathbb{N}, \quad B_{(i,D)}$) the construction produces classic Cantor sets and that it is the case that inductively for the addition steps ($\forall i \in \mathbb{N}, \quad B_{(i,A)}$) the construction produces $2^{(i+1)}$ homomorphic copies of the classic Cantor set.
Let $i \in \mathbb{N}$, so define $A_{(i,p)}$ as each of the $2^{(i+1)}$ homomorphic copies of the classic Cantor set that were produced from the i^{th} addition step in the construction of the Bubba set, where boviously $p \in \mathbb{N} \wedge p \leq 2^{(i+1)}$.
$m(\mathcal{C}) = 0$ and $m\left(A_{(i,p)}\right) = 0$ for all $A_{(i,p)}$ that are homomorphic copies of $\mathcal{C}$.

For ease let

$$\mathcal{A} = \bigcup A_{(i,p)} \quad \forall i \in \mathbb{N}, p \in \mathbb{N} \wedge p \leq 2^{(i+1)}$$

Thus,

$$\mathcal{B} = \mathcal{C} \cup \mathcal{A}$$

Therefore, it follows that $\mathcal{B}$ is a coutable union of sets homomorphic to the classic Cantor set all of which have the property that their measure is zero.
Hence, $m(\mathcal{B}) = 0$.

Q.E.D.

Just as with the Krizan set; we note that it is not a fractal in the classical sense since it is not self-similar. However, $\mathcal{B}$ is symmetric about $\frac{1}{2}$.

6. Conjecture as to The Nature of the Dimension of the Krizan Set

It is clear that the topological dimension of the Krizan set is not 0 and since it is a subset of $\mathbb{R}$ it is 1. As for the Hausdorff dimension, since the Krizan set is a subset of $\mathbb{R}$ it must be in $(0,1]$.

Let us consider our construction of $\mathcal{K}$ again, like the Cantor set, define the first level Z_0 is the interval, $[0,1]$ and we create $Z_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. If we from this point onward do the construction as with the Cantor set, then we would have $\mathcal{C}$ whose Hausdorff dimension is $\frac{\ln(2)}{\ln(3)}$. Of course, that was not done so we continue with the construction of $\mathcal{K}$. Let is call $\mathcal{C}$, H_1 for the discussion in this section.

We construct the level Z_2 by removing the two middle fifths from each interval similar to the construction of $C_{\frac{2}{5}}$.

$$Z_2 = \left[0, \frac{1}{15}\right] \cup \left[\frac{2}{15}, \frac{3}{15}\right] \cup \left[\frac{4}{15}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{11}{15}\right] \cup \left[\frac{12}{15}, \frac{13}{15}\right] \cup \left[\frac{14}{15}, 1\right]$$

If we from this point onward do the construction as with the Cantoresque set, $C_{\frac{2}{5}}$, then we would have a set homomorphic to $C_{\frac{2}{5}}$, call it H_2, thus its Hausdorff dimension is $\frac{\ln(3)}{\ln(5)}$. Once more, of course, that was not done so we continue with the construction of $\mathcal{K}$.

So, we construct the level Z_3 by removing the three middle seventh segments from each existing interval of Z_2 yielding $Z_3 = \left[0, \frac{1}{105}\right] \cup \left[\frac{2}{105}, \frac{3}{105}\right] \cup \left[\frac{4}{105}, \frac{5}{105}\right] \cup \left[\frac{6}{105}, \frac{7}{105}\right] \cup \left[\frac{14}{105}, \frac{15}{105}\right] \cup \left[\frac{16}{105}, \frac{17}{105}\right] \cup \left[\frac{18}{105}, \frac{19}{105}\right] \cup \left[\frac{20}{105}, \frac{21}{105}\right] \cup \left[\frac{28}{105}, \frac{29}{105}\right] \cup \left[\frac{30}{105}, \frac{31}{105}\right] \cup \left[\frac{32}{105}, \frac{33}{105}\right] \cup \left[\frac{34}{105}, \frac{35}{105}\right] \cup \left[\frac{70}{105}, \frac{71}{105}\right] \cup \left[\frac{72}{105}, \frac{73}{105}\right] \cup \left[\frac{74}{105}, \frac{75}{105}\right] \cup \left[\frac{76}{105}, \frac{77}{105}\right] \cup \left[\frac{84}{105}, \frac{85}{105}\right] \cup \left[\frac{86}{105}, \frac{87}{105}\right] \cup \left[\frac{88}{105}, \frac{89}{105}\right] \cup \left[\frac{90}{105}, \frac{91}{105}\right] \cup \left[\frac{98}{105}, \frac{99}{105}\right] \cup \left[\frac{100}{105}, \frac{101}{105}\right] \cup \left[\frac{102}{105}, \frac{103}{105}\right] \cup \left[\frac{104}{105}, 1\right]$

If we from this point onward do the construction as with the Cantoresque set, $C_{\frac{3}{7}}$, then we would have a set homomorphic to $C_{\frac{3}{7}}$,, call it H_3, thus its Hausdorff dimension is $\frac{\ln(4)}{\ln(7)}$. As was the case before, that was not done so we continue with the construction of $\mathcal{K}$.

So, it is clear that we shall then repeat this pattern by removing the four middle ninth segments from the remaining intervals to constructZ_4, the five middle eleventh segments from the remaining intervals to construct Z_5, the six middle thirteenth segments from the remaining intervals to construct Z_6, etc. So, when we construct the level Z_n we remove n segments of length $\frac{1}{2n+1}$ from the remaining interval from the level $Z_{(n-1)}$.

Now the Krizan set, $\mathcal{K}$, is the generalised intersection of the sets (the levels), $Z_n \ni n \in \mathbb{N}^*$.

$$\mathcal{K} = \bigcap_{n=0}^{\infty} Z_n$$

Considering , H_1, has Hausdorff dimension $\frac{\ln(2)}{\ln(3)}$,
$C_{\frac{2}{5}}$ and H_2 have Hausdorff dimension $\frac{\ln(3)}{\ln(5)}$,
$C_{\frac{3}{7}}$ and H_3 have Hausdorff dimension $\frac{\ln(4)}{\ln(7)}$,
$C_{\frac{4}{9}}$ and H_4 have Hausdorff dimension $\frac{\ln(5)}{\ln(9)}$, etc.

So, we conjecture that one could produce a 'Hausdorffesque' dimension for a quasi-Cantoresque set which is $\lim_{n\to\infty} d(H_n)$ where $d(H_n)$ is the Hausdorff dimension of H_n.

Hence, we have inductively, the 'Hausdorffesque' dimension of $\mathcal{K}$ to be $\lim_{n\to\infty} d(H_n)$. Let $f|^{[1,\infty)} : [1,\infty) \longrightarrow \mathbb{R} \quad \ni f|^{[1,\infty)}(x) = \left(\frac{\ln(x+1)}{\ln(2x+1)}\right)$ be the extension function of the sequence $f : \mathbb{N} \longrightarrow \mathbb{R} \ni f(n) = \left(\frac{\ln(n+1)}{\ln(2n+1)}\right)$. We, therefore, note that:

$$\lim_{x\to\infty}\left(\frac{\ln(x+1)}{\ln(2x+1)}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{2}{2x+1}}\right) = \lim_{x\to\infty}\frac{2x+1}{2\cdot(x+1)} = \lim_{x\to\infty} 1 = 1$$

by a simple application of L'Hôpital's rule to determine the $\lim_{x\to\infty}\left(f|^{[1,\infty)}(x)\right)$; which clearly implies that

$$\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(2n+1)}\right) = 1$$

7. Discussion

We introduced the quasi-Cantoresque sets that Krizan created, the Krizan Set, $\mathcal{K}$, and the Bubba Set, $\mathcal{B}$. We explored propoerties of, aspects of, and proved some nice results on each of the aforementioned quasi-Cantoresque sets. We opined as to the nature of a generalised Hausdorff dimension of $\mathcal{K}$ and we offered a limiting argument for said.

During the research the primary author conducted the probe of the topic and the secondary author assisted. Our paper is the but a brief point in the research programme of the primary author. Krizan continues to explore, discover, conjecture, hypothesise, produce thesis, antitheses, and synthesis of ideas that he creates. Krizan has speculated as to the construction, properties, and aspects of other Cantor sets, Cantoresque sets, and quasi-Cantoresque sets. The direction of the research was done by the second author using the pseudo-Socratic modified Moore method; but, the formal student-professor relationship ended when the first author graduated in 2008. It is now the case that the authors are colleagues and the research that Krizan has started shall not stop.

The Krizan set seems to be much more natural than a fractal since self-similarity does not seem to be a hallmark of nature but nature seems to create processes that are quasi-self similar (as is the case with the Krizan set). However, if one were interested in modelling nature, would it not behoove one to introduce some form of probability into the process?

We conclude with some conjectures and discuss where our research is leading. We note that there are evidently more questions raised by the research than questions answered by the results.

We have noted that it is the case that a Cantor set can be constructed such that its measure is greater than 0, we have noted that it is the case that a Cantoresque set can be constructed such that its measure is greater than 0, and it seems clear that a quasi-Cantoresque set can be constructed such that its measure is greater than 0. Then the question is, is the case that each of the afforementioned types of sets can be constructed such that its topological dimension is equal to its Hausdorff dimension, its topological dimension is greater than its Hausdorff dimension?[6] We have yet to answer many ofthe questions we have positied; for example, are there sets with Hausdorff dimension less than topological dimension such that the Hausdorff dimesion is rational?

Additionally, we opine that it may be the case that a quasi-Cantoresque set, call it $\mathcal{Q}$, constructed in a similar manner to the Krizan set will have 'Hausdorffesque' dimension 1. Let $n \in \mathbb{N}$ and let $f(n)$ be the number of intevals an interval in the $(n-1)^{th}$ stage of the construction of $\mathcal{Q}$ is decomposed into on the n^{th} stage whilst whilst $g(n)$ be the number of intevals left in in the construction of $\mathcal{Q}$ at the n^{th} stage. The sequence sequences $G : \mathbb{N} \to \mathbb{R}$, such that it is defined by $\{g(n) : n \in \mathbb{N}\}$ and $F : \mathbb{N} \to \mathbb{R}$, such that it is defined by $\{f(n) : n \in \mathbb{N}\}$ have the property that $g(n) < f(n) \quad \forall n \in \mathbb{N}$. When the ratio of the natural logarithms of the sequence values and the concept of the 'Hausdorffesque' dimension is considered we opine

[6]It seems rather obvious it cannot be the case that a set can be constructed its topological dimension is less than its Hausdorff dimension.

that we will be evaluating:

$$\lim_{n\to\infty}\left(\frac{\ln(g(n))}{\ln(f(n))}\right)$$

which is a question of a sequential limit of the sequence $L : \mathbb{N} \to \mathbb{R} \ni$

$$L(n) = \left(\frac{\ln(g(n))}{\ln(f(n))}\right).$$

Extend the sequence to $D \subseteq [1, \infty) \ni \quad \forall x \in D \quad \ln(f(x)) \neq 0 \wedge \ln(g(x)) \neq 0$ where $g(x) \wedge f(x)$ are differentiable $\forall x \in D$.

We note that we derive $L|^D : D \longrightarrow \mathbb{R} \ni$

$$L^D(x) = \left(\frac{\ln(g(x))}{\ln(f(x))}\right)$$

We, therefore, note that by simple Calculus, some basic algebra, and some well-chosen applications of L'Hôpital's rule (assuming $\exists x \in \mathbb{R} \ni$ that defines a segment (x, ∞) such that $\ln(g(x))$ and $\ln(f(x))$ are differentiable over the segment so that we can consider:

$$\lim_{x\to\infty}\left(\frac{\ln(g(x))}{\ln(f(x))}\right) = \lim_{x\to\infty}\frac{g'(x)\cdot f(x)}{f'(x)\cdot g(x)} = \lim_{x\to\infty}\frac{g'(x)}{f'(x)}\cdot\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

$$= \lim_{x\to\infty}\frac{g'(x)}{f'(x)}\cdot\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{g'(x)}{f'(x)}\cdot\frac{f'(x)}{g'(x)} = \lim_{x\to\infty} 1 = 1$$

We also wonder about allowing for Cantoresque or quasi-Cantoresque set that are not symmetric. What does the non-symmetry ('lop-sidedness') do to the nature of the set? We have yet to attack the problem but have discussed it and the idea has raised some intersting conjectures.

We have also wodered about changing the construction technique of the 'additive - deletion' quasi-Cantoresque sets. We note that there could be construction that $B_0 = [0, 1]$.

We delete the middle third segment, $\left(\frac{1}{3}, \frac{2}{3}\right)$, as with the Cantor set, so we have: $B_{(1,D)} = \left[\frac{0}{3}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right]$, so we have two intervals of length $\frac{1}{3} \Longrightarrow \ell(B_{1,D}) = \frac{2}{3}$.

However, now we *'add'* two 'small' Cantor sets within the removed segment, $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left[\frac{3}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{6}{9}\right]$ to create $B_{(1,A)} = \left[\frac{0}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{9}{9}\right]$ We delete the middle third segment from each, $\left[\frac{0}{9}, \frac{4}{9}\right] \cup \left[\frac{5}{9}, \frac{9}{9}\right]$, as with the Cantor set, so we have: $B_{(2,D)} = \left[\frac{0}{27}, \frac{4}{27}\right] \cup \left[\frac{8}{27}, \frac{12}{27}\right] \cup \left[\frac{15}{27}, \frac{19}{27}\right] \cup \left[\frac{23}{27}, \frac{27}{27}\right]$, etc. which changes the nature of the set so it is not the Bubba set (it could be named the little sister of Bubba, Bubbette).

Further, we have pondered as to the nature of a probabilistic Cantoresque or quasi-Cantoresque set. Could such not be created? We ponder the following: let us simply consider $C_0 = [0, 1]$. Now, let us create a Cantoresque set by removing middle ninety-ninth segments from each interval. Divide $C_0 = [0, 1]$ into 99 equi-length intervals. Restrict the discussion so that the first and last intervals must be subsets of the C_1 level. Let $p : \mathbb{N}_{99} \backslash (\{1\} \cup \{99\}) \longrightarrow \mathbb{R}$ be a well defined probability mass function. Let $k \in (0, 1)$ be a threshold-value. Let the m^{th} segment be removed iff $p(m) > k$ and the m^{th} segment not be removed iff $p(m) \leq k$.

So, C_1 is a union of intervals such that there are c_1 intervals $\ni c_1 \in \mathbb{N}_a$ where $2 \leq a \leq 99$. For each interval, divide each into equi-length intervals and do as before (with the same probability mass function or perhaps another; a different threshold-value; etc. What effect has said on the measure, Hausdorffesque dimension, facility of computation of the set, etc.?).

What is clear is that there are a **plethora of questions before us and this project has but scratched the surface** of what seems to us to be a very interesting category of sets. It is very apparant that our research is leading to many more questions than answers; and, is that not what we wish to have whenever we embark to do mathematical research?

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