MATH 403 WORKSHEET 0 NAME:______________________________

(PLEASE PRINT LEGIBLY)

Twenty "significantly interesting" problems to do for the first day of class.

1. Let $n > 1$ and $n \in \mathbb{N}$ Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from some probability density function, $X \sim f_X(x)$, and let $S_X^2 = \sum_{i=1}^{n} \frac{(X-\overline{X})^2}{n-1}$. $S_X^2 = \hat{\sigma}_X^2$. It is the case that

$$S_X^2 = \frac{1}{(n-1)}\left(\sum_{i=1}^{n}(X_i^2) - n(\overline{X})^2\right).$$ Prove or disprove the claim.

2. Let $X \sim$, $N(x, \mu_X, \sigma_X)$ such that $\sigma_X^2 = 25$. Find the approximate probability the sample variance of a random sample of size 5 from this normal population is between 20 and 30.

3. Let $X_1, X_2, X_3, X_4, \ldots, X_n$ be a random sample from a Bernoulli distribution with mean of θ.
Let $n > 1$ and $n \in \mathbb{N}$

Let $\hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n-1}$ and note it is a biased estimator. Find the bias in (of) $\hat{\theta}$.

4. Let $X, Y \sim h((x, y))$ such that $$h(x,y) = \begin{cases} \frac{36xy}{25} & 0 < x < \frac{1}{6} \quad 0 < y < 10 \\ 0 & \text{else} \end{cases}$$

A. Find $\Pr(X < Y)$.
B. Find $E[Y]$
C. Find $E[X \mid Y = 1]$
D. Find ρ_{XY}

5. Let X, Y be jointly distributed such that the joint probability density function, $f_{xy}(x, y) = \frac{6(x^2 + \frac{xy}{2})}{7}$

$\ni x \in (0, 1)\ y \in (0, 2)$.
A. Find the marginal probability density function $f_X(x)$
B. Find $\Pr(X > Y)$
C. Find $\Pr(Y > ½ \mid X < ½)$

6. Let $\alpha, \beta, \delta \in \mathbb{R}$ and all be positive. Let X, Y be jointly distributed such that the joint probability density function,

$$f_{xy}(x, y) = \frac{x^{(\alpha-1)}y^{(\delta-1)}}{\Gamma(\alpha)\Gamma(\delta)\beta^{(\alpha+\delta)}e^{\left(\frac{x+y}{\beta}\right)}} \ni x \in (0, \infty)\ y \in (0, \infty).$$

Find the joint probability density function of the random variables $U = X + Y$ and $V = \dfrac{X}{X+Y}$

7. Suppose $X \sim \Gamma(x, 3, 5)$.
Let X_1, X_2, X_3, X_4 be a random sample from a $\Gamma(x, 3, 5)$ population.
Approximate $\Pr\left(\sum_{i=1}^{4} X_i > 100\right)$.

8. Suppose Z_1, Z_2, Z_3, Z_4, Z_5 is a random sample from a $N(z, \mu_Z = 0, \sigma_Z = 1)$
Let $V = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2$ Find the approximate probability that $\Pr(12 < V \le 16)$

9. Let X_1, X_2, X_3, X_4, X_5 be a random sample from a $N(x, \mu_X, \sigma_X)$ such that $\mu_X = -40$ and $\sigma_X = 3$.
We find an $\overline{X}$ and a S_X. Approximate $\Pr(\overline{X} > S_X - 40)$.

10. Suppose $X \sim \text{Nor}(x, \mu_X, \sigma_X)$ and $W \sim \text{Nor}(w, \mu_W, \sigma_W)$
Let $X_1, X_2, X_3, X_4, \ldots, X_6$ be a random sample from that $N(x, \mu_X, \sigma_X)$ population where $\sigma_X \ne 0$.
Let $W_1, W_2, W_3, \ldots, W_7$ be a random sample from that $N(w, \mu_W, \sigma_W)$ population where $\sigma_W \ne 0$.
Suppose we assume $3\sigma_X = 2\sigma_W$. Find the approximate probability that $\Pr\left(\frac{S_X^2}{S_W^2} < 8.75\right)$

11. Let X_1, X_2, X_3, X_4 be a random sample from a **discrete** uniform distribution,

$$\Pr(X = k) = \begin{cases} \dfrac{1}{2\theta+1} & k = 0, 1, -1, 2, -2, 3, -3, \ldots, \theta, -\theta \\ 0 & \text{else} \end{cases} \quad \text{where } \theta \in \mathbb{N}$$

Let $X_1 = 2$, $X_2 = 1$, $X_3 = -3$, $X_4 = 1$ be an observed sample.
Find the maximum likelihood estimate of θ (for this observed sample).

12. Let $X_1, X_2, X_3, X_4, \ldots, X_n$ be a random sample from an gamma distribution with parametres α and β. Using the method of moments find estimators, $\hat{\alpha}$ and $\hat{\beta}$, for α and β respectively, such that $\hat{\alpha}$ and $\hat{\beta}$ are independent of each other (but of course each are sample dependent separately). [Hint: you need at least two moments since you are estimating 2 parametres]

13. Suppose five passengers set sail for a three hour tour on a boat with a crew of two. Suppose the probability that any one person on a cruise boat that was on a three hour tour (the S.S. Minnow) comes down with the Padraig virus is distributed as a Poisson random variable such that $\lambda = 4$. Suppose we wish to determine the probability that 5 or more of 7 people on the boat gets the disease. Find the probability that 5 or more of 7 people aboard gets the disease. Then, approximate the solution to four decimal places.

14 Suppose the probability an item produced by a certain machine will be defective is 0.25.
We know this can be modelled with a binomial random variable model.
A. Find the probability that a sample of 80 items will contain at most 12 defective items (leave the solution in sigma notation form and do not bother to compute this).
B. Choose another probability mass function (p. m. f.) or probability density function (p.d.f.) to approximate the solution in part A and justify its use.
C. Approximate the answer in part A to four significant digits.

15. Suppose $X \sim Ray\,(x, \alpha) \ni \alpha > 0$. Prove or disprove $\mu = \sqrt{\frac{\pi}{4\alpha}}$

16. Suppose $X \sim \mathrm{Wei}(x, \alpha, \beta) \ni \alpha > 0 \wedge \beta > 0$ Prove or disprove that $\mu = \alpha\,\Gamma(1 + \beta^{-1})$

17. Suppose $X \sim \mathrm{Nor}\,(x, \mu,\sigma)$. Prove or disprove that $E[X] = \mu$
(hint: $M_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$)

18. Let $X_1, X_2, X_3, X_4, \ldots, X_{16}$ is an i.i.d. from a $N(x, \mu_X = 184.09, \sigma_X = \sqrt{40})$ distribution and
$Y_1, Y_2, Y_3, \ldots, Y_{16}$ is an i.i.d. from a $N(y, \mu_Y = 171.93, \sigma_Y = \sqrt{40})$ distribution
Approximate $\Pr(\overline{X} - \overline{Y} \geq 9.12)$ A to four significant digits.

19. Let $X_1, X_2, X_3, X_4, \ldots, X_{16}$ is an i.i.d. from a $N(x, \mu_X = 50, \sigma_X = 10)$ distribution
Approximate $\Pr\left(796.2 \leq \sum_{i=1}^{16} (X_i - \mu_X)^2 \leq 2630\right)$ A to four significant digits.

20. Let $\overline{X}$ and S_X^2 be the sample mean and sample variance associated with a random sample of size 16 from a $N(x, \mu_X, \sigma_X = 15)$ population.
Find a constant $k, k \in \mathbb{R}$, where $\Pr\left(-k \leq \frac{\overline{X} - \mu_X}{S_X} \leq k\right) \approx 0.95$