

MATH 302 PROBABILITY AND STATISTICS II

DR. McLOUGHLIN'S CLASS

SOME USEFUL DISTRIBUTION RESULTS

HANDOUT 3

A whole point to studying probability and statistics is the analysis of data. So, when one is studying variables it is important to understand that there are different types of variables in applied studies and different types of applications.

Let's take pause and gather together all that we have proven, argued, discussed, or reviewed and add some other useful results to our list. For any of the results in this list that have not been proven - - they are 'easy' to prove.

Let $n, p, \alpha, \beta, \sigma, \mu, \gamma, \theta, \lambda, c$ be constants.

Theorem 1 (DeMoivre - Laplace): Let $X \sim \text{Bin}(x, n, p)$. Let $Y = \frac{X - np}{\sqrt{np(1-p)}}$.

As $n \longrightarrow \infty$, it is the case that $Y \longrightarrow Z$ where $Z \sim \text{Nor}(z, 0, 1)$.

Theorem 2: Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let Y be a linear transformation of X (i.e.: $Y = \alpha X + \beta$) it is the case that $Y \sim \text{Nor}(y, \alpha \cdot \mu + \beta, \alpha \cdot \sigma)$.

Theorem 3: Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let $Z = \frac{X - \mu}{\sigma}$ it is the case that $Z \sim \text{Nor}(z, 0, 1)$.

Theorem 4: Let $Z \sim \text{Nor}(z, 0, 1)$. Let $Y = Z^2$ it is the case that $Y \sim \chi_1^2$.

Theorem 5: Let Z_1, Z_2, \dots, Z_n be independent, identically distributed random variables from $Z \sim \text{Nor}(z, 0, 1)$. Let $U = Z_1^2 + Z_2^2 + \dots + Z_n^2 = \sum_{j=1}^n Z_j^2$ it is the case that $U \sim \chi_n^2$.

Theorem 6: Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$ Let $Y = \sum_{i=1}^n X_i$ it is the case that

$Y \sim \text{Nor}(y, n \cdot \mu, \sqrt{n \cdot \sigma^2})$.

Theorem 7: Let $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let $Y = \bar{X}$ it is the case that $E[\bar{X}] = \mu_X$. So, $\mu_{\bar{X}} = \mu_X$.

Theorem 8: Let $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let $Y = \bar{X}$ it is the case that

$$\text{Var}[Y] = \frac{\sigma_X^2}{n}. \text{ So, } \sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n}.$$

Theorem 9: Let $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let $Y = s^2 = \sum_{i=1}^n \frac{(X - \bar{X})^2}{n-1} = \frac{1}{n-1} \sum_{i=1}^n (X - \bar{X})^2$.

$$E[Y] = \sigma_X^2. \text{ So, } E[s^2] = \sigma_X^2.$$

Theorem 10: Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$ Let $Y = \bar{X}$

it is the case that $Y \sim \text{Nor}(y, \mu, \frac{\sigma}{\sqrt{n}})$.

Theorem 11: Let $X \sim \Gamma(x, \alpha, \beta)$. Let $Y_1 = a \cdot X$ it is the case that $Y_1 \sim \Gamma(y_1, \alpha, a \cdot \beta)$;

furthermore, let $Y_2 = \frac{2X}{\beta}$ it is the case that $Y_2 \sim \Gamma(y_2, \alpha, 2) \equiv \chi_{2\alpha}^2$.

Theorem 12: Let $X \sim \Gamma(x, \alpha, \beta)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \Gamma(x, \alpha, \beta)$.

(1) Let $Y_1 = \sum_{i=1}^n X_i$ it is the case that $Y_1 \sim \Gamma(y_1, n\alpha, \beta)$.

(2) Let $Y_2 = \bar{X}$ it is the case that $Y_2 \sim \Gamma(y_2, n\alpha, \frac{\beta}{n})$

(3) Let $Y_3 = \frac{2}{\beta} \sum_{i=1}^n X_i$ it is the case that $Y_3 \sim \Gamma(y_3, n\alpha, 2) \equiv \chi_{2n\alpha}^2$.

Theorem 13: Let $X \sim \text{Pois}(x, \lambda)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Pois}(x, \lambda)$. Let $Y = \sum_{i=1}^n X_i$ it is the case that $Y \sim \text{Pois}(y, n\lambda)$.

Theorem 14: Let $X \sim \text{Ber}(x, p)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Ber}(x, p)$. Let $Y = \sum_{i=1}^n X_i$ it is the case that $Y \sim \text{Bin}(y, n, p)$.

Theorem 15: Let $X \sim \text{Bin}(x, m, p)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Bin}(x, m, p)$. Let $Y = \sum_{i=1}^n X_i$ it is the case that $Y \sim \text{Bin}(y, n \cdot m, p)$.

Theorem 16: Let $X \sim \text{Uni}(x, \alpha, \beta)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Uni}(x, \alpha, \beta)$. Let $Y_1 = a \cdot X$ it is the case that $Y_1 \sim \text{Uni}(y_1, a \cdot \alpha, a \cdot \beta)$.

Special note: let $Y_2 = \sum_{i=1}^n X_i$ it is **not** the case that Y_2 is distributed uniformly.

Theorem 17¹: Let $X \sim \text{Exp}(x, \theta)$. Let $Y_1 = a \cdot X$ such that $a \in \mathbb{R}$ and $a > 0$ it is the case that

$Y_1 \sim \text{Exp}(y_1, a \cdot \theta)$. Let $Y_2 = \frac{2X}{\theta}$ it is the case that $Y_2 \sim \text{Exp}(y_2, 2) \equiv \Gamma(y_2, 1, 2) \equiv \chi_2^2$.

Theorem 18: Let $X \sim \text{Exp}(x, \theta)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Exp}(x, \theta)$.

(1) Let $Y_1 = \sum_{i=1}^n X_i$ it is the case that $Y_1 \sim \Gamma(y_1, n, \theta)$.

(2) Let $Y_2 = \bar{X}$ it is the case that $Y_2 \sim \Gamma(y_2, n, \frac{\theta}{n})$

(3) Let $Y_3 = \frac{2}{\theta} \sum_{i=1}^n X_i$ it is the case that $Y_3 \sim \Gamma(y_3, n, 2) \equiv \chi_{2n}^2$.

Theorem 19: Let $U \sim \chi_n^2$ and $V \sim \chi_m^2$ and U and V be independent. Let $Y = U + V$ it is the case that $Y \sim \chi_{(n+m)}^2$.

Theorem 20: Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$. Let $Y = \frac{(n-1)s^2}{\sigma^2}$ it is the case that $Y \sim \chi_{(n-1)}^2$.

Theorem 21 (Gossett): Let $Z \sim \text{Nor}(z, 0, 1)$, $U \sim \chi_m^2$, and Z and U be independent.

Let $W = \frac{Z}{\sqrt{U/m}}$ it is the case that $W \sim t_m$ (t with m degrees of freedom or t with $df = m$).

Theorem 9 (restated): Let $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu_X, \sigma_X)$

it is the case that $\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \sim \text{Nor}(z, 0, 1)$.

Theorem 22 (Gossett): Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let X_1, X_2, \dots, X_n be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$

¹ Thanks to Ms. R. Poniktera for noting the a was not properly well defined.

it is the case that $\frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{(n-1)}$.

Theorem 23 (Fisher – Snedecor): Let $U \sim \chi_n^2$ and $V \sim \chi_m^2$ and U and V be independent. Let $Y = \frac{U/n}{V/m}$ it is the case that $Y \sim F_{n, m}$.

Theorem 24: Let $U \sim \chi_n^2$ and $V \sim \chi_m^2$ and U and V be independent. Let $Y = \frac{U/n}{V/m}$.

Let $W = \frac{1}{Y}$ it is the case that $W \sim F_{m, n}$.

Theorem 25: Let $X_{11}, X_{12}, \dots, X_{1n_1}$ be independent, identically distributed random variables from $X_1 \sim \text{Nor}(x_1, \mu_1, \sigma_1)$. Let $X_{21}, X_{22}, \dots, X_{2n_2}$ be independent, identically distributed random variables from $X_2 \sim \text{Nor}(x_2, \mu_2, \sigma_2)$.

Let $Y = \frac{s_1^2 \sigma_2^2}{s_2^2 \sigma_1^2}$ it is the case that $Y \sim F_{n_1-1, n_2-1}$.

Theorem 26 (Central Limit Theorem / Law of Large Numbers) :

Let $X \sim f_X(x)$. Let $E[X] = \mu_x$ and $\text{Var}[X] = \sigma_x^2$ be constant.

(1) As $n \longrightarrow \infty$, it is the case that $X \longrightarrow Y_1$ where $Y_1 \sim \text{Nor}(y_1, n \cdot E[X], \sqrt{n} \cdot \text{SD}[X])$.

(2) Let $Y_2 = \bar{X}$

As $n \longrightarrow \infty$, it is the case that $Y_2 \longrightarrow Y_3$ where $Y_3 \sim \text{Nor}(y_3, E[X], \frac{\text{SD}[X]}{\sqrt{n}})$.

Definition 1: Let $X \sim f_X(x)$. Let $\alpha, \beta, \sigma, \mu, \gamma, \lambda, \dots, \theta$ be (possible) parameters for $f_X(x)$.

We say $\hat{\alpha}$ is an estimator of α , $\hat{\beta}$ is an estimator for $\beta, \dots, \hat{\theta}$ is an estimator for θ .