MATH 302 PROBABILITY AND STATISTICS II
DR. MCLOUGHLIN'S CLASS
MORE JOINT PDF OR PMF RESULTS MGF RESULTS AND SOME SAMPLING THEORY
HANDOUT II – VERSION 3

§ 1 The Basics and some reiteration of previous results.

The Normal bivariate joint random variable is a probabilistic (or stochastic) experiment that can have any outcome on $\mathbb{R}$. The parameters are μ_x and σ_x^2 (or μ_x and σ_x), μ_y and σ_y^2 (or μ_y and σ_y), and $\rho(X, Y)$ (which will be denoted as ρ_{xy}). Thus, it is defined by its means, variances, and covariance (or correlation). Its applications are many and its use *quite* important. A substantial number of empirical studies have indicated that the normal function provides an adequate representation of, or at least a decent approximation to, the distributions of many physical, mental, economic, biological, and social variables.

$x \in \mathbb{R}$, $\mu_x \in \mathbb{R}$, and $\sigma_x \in (0,\infty)$, $y \in \mathbb{R}$, $\mu_y \in \mathbb{R}$, and $\sigma_y \in (0,\infty)$, and $\rho_{xy} \in [-1, 1]$

$$\text{BivNor}((x, y), \mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho_{xy}) = \frac{e^{-\{\frac{1}{2(1-\rho^2)}[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)}{\sigma_x}\frac{(y-\mu_y)}{\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}]\}}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$\forall\ x \in (-\infty, \infty)$ $\forall\ y \in (-\infty, \infty)$

The Dirichlet bivariate joint random variable is a probabilistic (or stochastic) experiment that can have any outcome on $x \geq 0$, $y \geq 0$, $0 \leq x + y \leq 1$.
The parametres are α, β, and δ (all are members of $(0, \infty)$).

$$D((x, y),\alpha, \beta, \delta) = \frac{\Gamma(\alpha+\beta+\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta)}x^{(\alpha-1)}y^{(\beta-1)}(1-x-y)^{(\delta-1)}$$

Theorem 1. 1: If X and Y are statistically independent normally distributed random variables then, $\text{BivNor}(x, y, \mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho_{xy}) = \text{Nor}(x, \mu_x, \sigma_x^2) \cdot \text{Nor}(y, \mu_y, \sigma_y^2)$

Theorem 1.2: If X and Y are statistically independent random variables then, $\rho_{xy} = 0$.

Special note:
X and Y are random variables such that $\rho_{xy} = 0$ then X and Y are statistically independent is false.
X and Y are normal random variables such that $\rho_{xy} = 0$ then X and Y are statistically independent is true.

Theorem 1. 3 (DeMoivre - Laplace) : Let $X \sim \text{Bin}(x, n, p)$. Let $Y = \dfrac{X-np}{\sqrt{np(1-p)}}$.

As $n \longrightarrow \infty$, it is the case that $Y \longrightarrow Z$ where $Z \sim \text{Nor}(z, 0, 1)$.

Theorem 1.4 : Let $X \sim \text{Nor}(x, \mu,\sigma)$. Let $Z = \dfrac{X-\mu}{\sigma}$ it is the case that $Z \sim \text{Nor}(z, 0, 1)$.

Theorem 1.5 Let $X_1, X_2, \ldots, X_n$ be independent distributed random variables from such that $X_i \sim N(x_i, \mu_{X_i}, \sigma_{X_i})$ where μ_{X_i} and σ_{X_i} exist, and $\sigma_{X_i} \neq 0$. Let $\alpha_{X_i} \in \mathbb{R}$ $i \in \mathbb{N}_n$

Let $Y = \sum_{i=1}^{n} \alpha_{X_i} X_i$ it is the case that $E[Y] = \sum_{i=1}^{n} \left(\alpha_{X_i} \mu_{X_i} \right)$ and $Var[Y] = \sum_{i=1}^{n} \left(\alpha_{X_i}^2 \sigma_{X_i}^2 \right)$

Corollary 1.5 Let $X_1, X_2, \ldots, X_n$ be independent distributed random variables from such that $X_i \sim N(x_i, \mu_{X_i}, \sigma_{X_i})$ where μ_{X_i} and σ_{X_i} exist, and $\sigma_{X_i} \neq 0$. Let $Y = \sum_{i=1}^{n} X_i$ it is the case that

$E[Y] = \sum_{i=1}^{n} \mu_{X_i}$ and $Var[Y] = \sum_{i=1}^{n} \sigma_{X_i}^2$

Theorem 1. 6 (Central Limit Theorem / Law of Large Numbers) :

Let $X \sim f_X(x)$. Let $E[X] = \mu_x$ and $Var[X] = \sigma_x^2$ be constant.

(1) As $n \longrightarrow \infty$, it is the case that $X \longrightarrow Y_1$ where $Y_1 \sim Nor(y_1, n \cdot E[X], \sqrt{n} \cdot SD[X])$.

(2) Let $Y_2 = \bar{X}$

As $n \longrightarrow \infty$, it is the case that $Y_2 \longrightarrow Y_3$ where $Y_3 \sim Nor(y_3, E[X], \frac{SD[X]}{\sqrt{n}})$.

Definition 1.1: Let $X \sim f_X(x)$. Let $\alpha, \beta, \sigma, \mu, \gamma, \lambda, \ldots, \theta$ be (possible) parametres for $f_X(x)$. We say $\hat{\alpha}$ is an estimator of α, $\hat{\beta}$ is an estimator for $\beta, \ldots,$ $\hat{\theta}$ is an estimator for θ.

Definition 1.2: Let X be a discrete random variable where $X \sim f_X(x)$. The moment generating function (MGF) of X [where it exists] is defined where $\exists\ \varepsilon > 0 \ni \forall\ t \in (-\varepsilon, \varepsilon)$

$M_X(t) = E[e^{tX}] = \sum_{x} e^{tx} \cdot f_X(x)$

Definition 1.3: Let X be a continuous random variable where $X \sim f_X(x)$. The moment generating function (MGF) of X [where it exists] is defined where $\exists\ \varepsilon > 0 \ni \forall\ t \in (-\varepsilon, \varepsilon)$

$M_X(t) = E[e^{tX}] = \int_{x} e^{tx} \cdot f_X(x)dx$

Theorem 1.7: Let X be a random variable where $X \sim f_X(x)$, $\alpha, \beta \in \mathbb{R}$, and

$\exists\ \varepsilon > 0 \ni \forall\ t \in (-\varepsilon, \varepsilon)$ where $M_X(t)$ exists. It is the case that

(1) $M_{(X+\alpha)}(t) = e^{t\alpha} \cdot M_X(t)$

(2) $M_{(\beta X)}(t) = M_X(\beta t)$

(3) $M_{\left(\frac{X+\alpha}{\beta}\right)}(t) = e^{\frac{\alpha t}{\beta}} \cdot M_X(\frac{t}{\beta})$

Theorem 1.8: Let Let $X_1, X_2, \ldots, X_n$ be independent random variables such that moment generating functions for each exist $X_1 \sim f_{X_1}(x_1)$ and there is a $M_{X_1}(t)$, $X_2 \sim f_{X_2}(x_2)$ and there is a $M_{X_2}(t)$, . . . $X_1 \sim f_{X_n}(x_n)$ and there is a $M_{X_n}(t)$. Let $t \in \bigcap_{i=1}^{n} dom(M_{X_i}(t))$.

Let $Y = \sum_{i=1}^{n} X_i$ It is the case that $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$

We recall there are some Special MGFs:

$X \sim Bin(x, p, n)$	$M_X(t) = \left((1-p) + pe^t\right)^n$
$X \sim Pois(x,\lambda)$	$M_X(t) = e^{\lambda(e^t - 1)}$
$X \sim Geo(x,p)$	$M_X(t) = \dfrac{pe^t}{1 - e^t(1-p)} \ni t < -\ln(1-p)$
$X \sim Uni(x, \alpha, \beta)$	$M_X(t) = \dfrac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$
$X \sim Nor(x, \mu, \sigma)$	$M_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$
$X \sim \Gamma(x, \alpha, \beta)$	$M_X(t) = (1 - \beta t)^{-\alpha} \ni t < \dfrac{1}{\beta}$
$X \sim Exp(x,\theta)$	$M_X(t) = (1 - \theta t)^{-1} \ni t < \dfrac{1}{\theta}$
$X \sim Chi(x, \nu)$	$M_X(t) = (1 - 2t)^{-(\nu/2)} \ni t < 2$

Theorem 1.9: Let $X \sim f_X(x)$ such that μ_X, σ_X exist, and $\sigma_X \neq 0$. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim f_X(x)$. Let $Y = \overline{X}$ it is the case that $E[Y] = E[\overline{X}] = \mu_X$. So, $\mu_{\overline{X}} = \mu_X$ and $Var[Y] = Var[\overline{X}] = \dfrac{\sigma_X^2}{n}$. So, $\sigma_{\overline{X}}^2 = \dfrac{\sigma_X^2}{n}$.

Corollary 1. 9: Let $X \sim Nor(x, \mu_X, \sigma_X)$. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim Nor(x, \mu_X, \sigma_X)$. Let $Y = \overline{X}$ it is the case that $\sigma_{\overline{X}} = \dfrac{\sigma_X}{\sqrt{n}}$.

§ 2 Some More Advanced Results.

A whole point to studying probability and statistics is the analysis of data. So, when one is studying variables it is important to understand that there are different types of variables in applied studies and different types of applications.

Let's take pause and gather together all that we have proven, argued, discussed, or reviewed and add some other useful results to our list. For any of the results in this list that have not been proven - - they are 'easy' to prove.

Let n, p, α, β, σ, μ, γ, θ, λ, c be constants.

Lemma 2.1 : Let X ~ Nor (x, μ,σ). Let Y be a linear transformation of X (i.e.: $Y = \alpha X + \beta$) it is the case that Y ~ Nor (y, $\alpha\cdot\mu + \beta$, $\alpha\cdot\sigma$).

Theorem 2.1 : Let X ~ Nor (x, μ_X,σ_X). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Nor (x, μ_X,σ_X). Let Y = $\bar{X}$ it is the case that $E[\bar{X}] = \mu_X$. So, $\mu_{\bar{X}} = \mu_X$.

Theorem 2.2 : Let X ~ Nor (x, μ_X,σ_X). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Nor (x, μ_X,σ_X). Let Y = $\bar{X}$ it is the case that $\text{Var}[Y] = \frac{\sigma_X^2}{n}$. So, $\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n}$.

Theorem 2.3 : Let X ~ Nor (x, μ,σ). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Nor (x, μ,σ) Let $Y = \sum_{i=1}^{n} X_i$ it is the case that

Y ~ Nor (y, $n\cdot\mu$, $\sqrt{n\cdot\sigma^2}$) .

Theorem 2.4 : Let X ~ Nor (x, μ_X,σ_X). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Nor (x, μ_X,σ_X) Let Y = $\bar{X}$ it is the case that

Y ~ Nor (y, μ_X, $\frac{\sigma_X}{\sqrt{n}}$) .

Theorem 2.5 (Central Limit Theorem / Law of Large Numbers) :

Let X ~ $f_X(x)$. Let $E[X] = \mu_x$ and Var [X] = σ_x^2 be constant.

(1) As $n \longrightarrow \infty$, it is the case that $X \longrightarrow Y_1$ where Y_1 ~ Nor(y_1, n·E[X], $\sqrt{n}\cdot$SD[X]).

(2) Let $Y_2 = \bar{X}$

As $n \longrightarrow \infty$, it is the case that $Y_2 \longrightarrow Y_3$ where Y_3 ~ Nor(y_3, E[X], $\frac{SD[X]}{\sqrt{n}}$).

Theorem 2.5 (condensed): Let X ~ Nor (x, μ,σ). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ f_X (x) (a well defined pdf) and μ exists, σ exists and $\sigma \neq 0$

Let Y = $\bar{X}$ it is the case that $Y \longrightarrow$ Nor (y, μ, $\frac{\sigma}{\sqrt{n}}$) as $n \longrightarrow \infty$.

Theorem 2.6 : Let Z ~ Nor (z, 0, 1). Let Y = Z^2 it is the case that Y ~ χ_1^2.

Theorem 2.7 : Let $Z_1, Z_2, \ldots, Z_n$ be independent, identically distributed random variables from Z ~ Nor(z, 0, 1). Let U = $Z_1^2 + Z_2^2 + \ldots + Z_n^2 = \sum_{j=1}^{n} Z_j^2$ it is the case that U ~ χ_n^2.

Theorem 2.8: Let X ~ Nor (x, μ_X,σ_X). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Nor (x, μ_X,σ_X). Let Y = s^2 =

$\sum_{i=1}^{n} \frac{(X-\overline{X})^2}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(X-\overline{X})^2$. E[Y] = σ_X^2. So, E[s^2] = σ_X^2.

Theorem 2.9: Let X ~ Γ(x, α, β). Let $Y_1 = a \cdot X$ it is the case that Y_1 ~ Γ(y_1, α, a·β); furthermore, let $Y_2 = \frac{2X}{\beta}$ it is the case that Y_2 ~ Γ(y_2, α, 2) ≡ $\chi_{2\alpha}^2$.

Theorem 2.10: Let X ~ Γ(x, α, β). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Γ(x, α, β).

(1) Let $Y_1 = \sum_{i=1}^{n} X_i$ it is the case that Y_1 ~ Γ(y_1, nα, β).

(2) Let $Y_2 = \overline{X}$ it is the case that Y_2 ~ Γ(y_2, nα, $\frac{\beta}{n}$)

(3) Let $Y_3 = \frac{2}{\beta}\sum_{i=1}^{n} X_i$ it is the case that Y_3 ~ Γ(y_3, nα, 2) ≡ $\chi_{2n\alpha}^2$.

Theorem 2.11: Let X ~ Pois(x, λ). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Pois(x, λ). Let Y = $\sum_{i=1}^{n} X_i$ it is the case that Y ~ Pois(y, nλ).

Theorem 2.12: Let X ~ Ber(x, p). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Ber(x, p). Let Y = $\sum_{i=1}^{n} X_i$ it is the case that Y ~ Bin(y, n, p).

Theorem 2.14: Let X ~ Bin(x, m, p). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Bin(x, m, p). Let Y = $\sum_{i=1}^{n} X_i$ it is the case that Y ~ Bin(y, n·m, p).

Theorem 2.15: Let X ~ Uni(x, α, β). Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from X ~ Uni(x, α, β). Let $Y_1 = a \cdot X$ it is the case that Y_1 ~ Uni(y_1, a·α, a·β).

Special note: let $Y_2 = \sum_{i=1}^{n} X_i$ it is **not** the case that Y_2 is distributed uniformly.

Theorem 2.16: Let $X \sim \text{Exp}(x, \theta)$. Let $Y_1 = a \cdot X$ it is the case that $Y_1 \sim \text{Exp}(y_1, a \cdot \theta)$. Let $Y_2 = \frac{2X}{\theta}$ it is the case that $Y_2 \sim \text{Exp}(y_2, 2) \equiv \Gamma(y_2, 1, 2) \equiv \chi^2_2$.

Theorem 2.17: Let $X \sim \text{Exp}(x, \theta)$. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim \text{Exp}(x, \theta)$.

(1) Let $Y_1 = \sum_{i=1}^{n} X_i$ it is the case that $Y_1 \sim \Gamma(y_1, n, \theta)$.

(2) Let $Y_2 = \bar{X}$ it is the case that $Y_2 \sim \Gamma(y_2, n, \frac{\theta}{n})$

(3) Let $Y_3 = \frac{2}{\theta}\sum_{i=1}^{n} X_i$ it is the case that $Y_3 \sim \Gamma(y_3, n, 2) \equiv \chi^2_{2n}$.

Theorem 2.18: Let $U \sim \chi^2_n$ and $V \sim \chi^2_m$ and U and V be independent. Let $Y = U + V$ it is the case that $Y \sim \chi^2_{(n+m)}$.

Theorem 2.19: Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$. Let $Y = \frac{(n-1)s^2}{\sigma^2}$ it is the case that $Y \sim \chi^2_{(n-1)}$.

Theorem 2.20 (Gossett): Let $Z \sim \text{Nor}(z, 0, 1)$, $U \sim \chi^2_m$, and Z and U be independent.

Let $W = \frac{Z}{\sqrt{U/m}}$ it is the case that $W \sim t_m$ (t with m degrees of freedom or t with *df* = m).

Theorem 2.20 (restated): Let $X \sim \text{Nor}(x, \mu_X, \sigma_X)$. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu_X, \sigma_X)$

it is the case that $\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \sim \text{Nor}(z, 0, 1)$.

Theorem 2.21 (Gossett): Let $X \sim \text{Nor}(x, \mu, \sigma)$. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables from $X \sim \text{Nor}(x, \mu, \sigma)$

it is the case that $\frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{(n-1)}$.

Theorem 2.22 (Fisher – Snedecor): Let $U \sim \chi^2_n$ and $V \sim \chi^2_m$ and U and V be independent.

Let $Y = \frac{U/n}{V/m}$ it is the case that $Y \sim F_{n,m}$.

Theorem 2.23: Let U ~ χ^2_n and V ~ χ^2_m and U and V be independent. Let Y = $\frac{U/n}{V/m}$.

Let W = $\frac{1}{Y}$ it is the case that W ~ $F_{m,\,n}$.

Theorem 2.24: Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ be independent, identically distributed random variables from X_1 ~ Nor (x_1, μ_1,σ_1). Let $X_{21}, X_{22}, \ldots, X_{2n_2}$ be independent, identically distributed random variables from X_2 ~ Nor (x_2, μ_2,σ_2).

Let Y = $\frac{s_1^2\sigma_2^2}{s_2^2\sigma_1^2}$ it is the case that Y ~ F_{n_1-1,n_2-1} .

§ 3 Some Estimation Theory.

Definition 3.1: Let X ~ $f_X(x)$. Let α, β, σ, μ, γ, λ, . . ., θ be (possible) parameters for $f_X(x)$.
We say $\hat{\alpha}$ is an estimator of α, $\hat{\beta}$ is an estimator for β, . . . , $\hat{\theta}$ is an estimator for θ.

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