Chapter 9

Random Variables: Continuous

Note: The basic building block of probability is set theory: Suppose we have a well defined sample space S and events E_1, E_2, etc. (like when we talk about a well defined universe U and sets X_1, X_2, etc.), yada, yada, yada. The basic ideas are grounded in the sets!

Moreover, much of the really powerful theorems and concepts are grounded in Set Theory, Real Analysis, and Topology. However, we have not studied Real Analysis to the extent (e.g.: no one has completed Math 351, 352 and I doubt many of us have completed Math 431) that we can ground fully all of the discussion of continuous random variables.

Suffice it to say, we considered cases of random variables such that, for example:

9.1 Continuous Random Variables

Definition 9.1.1. If $\exists$ a non-negative function, f, defined from the domain[1] $(-\infty, \infty)$ to the codomain $(-\infty, \infty)$ meaning $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and if $\exists$ a positive measurable set[2] $A \subseteq \mathbb{R}$ where $Pr\,(X \in A) = \displaystyle\int_{x \in A} (f(x))\,dx$, then X is said to be a **continuous random variable**.

One can weaken it to require if $\exists$ a non-negative function, f, defined $\forall x \in (-\infty, \infty)$ such that for any subset, B of $\mathbb{R}$ that can be constructed from a countable number of set

[1]Note: Most statistics texts call the domain of f(x)*the range... don't ask me why, they just do; however, we will use the correct terminology in this class.*

[2]This is where we get into a problem. In order to really understand this one needs to take Math 351 & 352 and a couple of graduate level Analysis classes (perhaps even more). Suffice it to say that we will only consider sets of positive measure in this class.

operations, $Pr\,(X \in B) = \int_{x \in B} (f(x))\,dx$, then X is said to be an [absolutely] continuous random variable.

Definition 9.1.2. If X is a continuous random variable, the function given f in definition 9.1.1 exists for each x in the domain of the function is called the **probability density function (p. d. f.).**

Note 9.1.1. *Let X be a continuous random variable with probability density function $f(x)$. We write $X \sim f(x)$ to denote that X is 'distributed with a p.d.f. that is $f(x)$.'*

Theorem 9.1.1. *If X is a continuous random variable such that the function given by f from definition 11.1 exists for each x in the domain of the function, and $a \in \mathbb{R}$, then $Pr\,(X = a) = 0$*

Theorem 9.1.2. *If X is a continuous random variable such that the function given by f from definition 11.1 exists for each x in the domain of the function, and A is a set such that $|A| = n \ni n \in \mathbb{N}$, then $Pr\,(X \in A) = 0$*

Theorem 9.1.3. [3] *If X is a continuous random variable such that the function given by f from definition 11.1 exists for each x in the domain of the function, and and A is a set such that $|A| = \aleph_0$ $Pr\,(X \in A) = \sum_{x \in A} Pr\,(X = x) = 0$*

Theorem 9.1.4. [4] *If X is a continuous random variable such that the function given by f from definition 11.1 exists for each x in the domain of the function, and and A is a set such that the set A has Lebesgue measure zero, then $Pr\,(X \in A) = 0$*

Theorem 9.1.5. *A function serves as a p. d. f. of a continuous random variable iff the values of f satisfies:*

1. $f(x) \geq 0 \quad \forall x \in dom(f)$
2. $\int\limits_{x} f(x) = 1.$

Definition 9.1.3. Let X be a continuous random variable and the function f is the p. d. f. of X, then

$$Pr(a < X < b) = Pr(a < X \leq b) = Pr(a \leq X < b) = Pr(a \leq X \leq b) = \int_{a}^{b} (f(x))\,dx$$

[3]We will not prove this theorem in this class. More prerequisite knowledge and experience is necessary.
[4]We will not prove this theorem in this class. More prerequisite knowledge and experience is necessary.

Definition 9.1.4. If X is a continuous random variable, the function given by $F(x) = Pr(X \leqslant x)$ for each x in the domain of the function is called the **probability distribution function** or **cumulative distribution function (c. d. f.)**

$$F(x) = Pr(X \leq x) = \int\limits_{t \leq x} f(t)dt = \int\limits_{t<x} f(t)dt = \int\limits_{-\infty}^{x} f(t)dt \quad \forall x \in (-\infty, \infty)$$

Theorem 9.1.6. *A function serves as a c. d. f. of a continuous random variable iff its values $F(x)$ satisfies:*

1. $F(x) \longrightarrow 0$ *as* $x \longrightarrow -\infty$ *or*
   $\exists\ x_1 \in \mathbb{R} \ni F(x_1) = 0 \quad (\wedge\ F(x) = 0\ \forall x < x_1)$
2. $F(x) \longrightarrow 1$ *as* $x \longrightarrow \infty$ *or* $\exists\ x_2 \in \mathbb{R} \ni F(x_2) = 1 \quad (\wedge\ F(x) = 1\ \forall x > x_2)$
3. $F(x)$ *is non-decreasing* $\forall x \in \mathbb{R}$.

Definition 9.1.5. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x, then the **expected value (or mean)** of X is $E[X] = \int\limits_{x} (x \cdot f(x))dx = \int\limits_{-\infty}^{\infty} (x \cdot f(x))dx$.

Notation 9.1.1. $E[X] = \mu_X = \mu = \mu'_1$.

Definition 9.1.6. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x, then the r^{th} moment about the origin of X is $E[X^r] = \int\limits_{x} (x^r \cdot f(x))dx = \int\limits_{-\infty}^{\infty} (x^r \cdot f(x))dx$.

Notation 9.1.2. $E[X^r] = \mu'_r$.

Definition 9.1.7. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x and $E[X]$ exists, then **the variance (or second moment about the mean) of X** is
$Var[X] = \int\limits_{x} \left((x - E[X])^2 \cdot f(x)\right) dx$.

Notation: $Var[X] = E[(X - \mu)^2] = \mu_2 = \sigma^2 = \sigma_X^2$.

Definition 9.1.8. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x and $E[X]$ exists, then **the standard deviation of X** is $\sqrt{Var[X]} = \sqrt{\left(\int_x ((x - E[X])^2 \cdot f(x))dx\right)}$

Notation 9.1.3. $SD[X] = \sqrt{\left(E[(X-\mu)^2]\right)} = \sigma = \sigma_X$

Definition 9.1.9. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x and $E[X]$ exists, then **the r$^{\mathbf{th}}$ moment about the mean of X** is
$E[(X-\mu)^r] = \int_x ((x - E[X])^r \cdot f(x))dx$

Notation 9.1.4. $E[(X-\mu)^r] = \mu_r$

Theorem 9.1.7. *If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x, then*
$Var[X] = \mu_2' - \mu^2 = E[X^2] - (E[X])^2$.

Definition 9.1.10. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x, then **the coefficient of skewness**, η_3, is $\frac{\mu_3}{\sigma^3} = \frac{\mu_3}{\left(\sqrt{\mu_2}\right)^3}$.

Definition 9.1.11. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x, then **the coefficient of kurtosis**, η_4, is $\frac{\mu_4}{\sigma^4} = \frac{\mu_4}{(\mu_2)^2}$.

Theorem 9.1.8. *If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x and $g(X)$ is a function of X, then the expected value (or mean) of $g(x)$ is $E[g(X)] = \int_x (g(X) \cdot f(x))dx$.*

Theorem 9.1.9. *If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x, the expected value of X exists, and T is a linear transformation of X, meaning $T = \alpha \cdot X + \beta$ such that $\alpha \in \mathbb{R} \wedge \beta \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\alpha \cdot X + \beta] = \alpha \cdot E[X] + \beta$.*

Corollary 9.1.1. *If X is a continuous random variable, the expected value of X exists, and the function given by f for each x in the domain of the function is the p.d.f. at x and T is a zero linear transformation of X, meaning $T = \alpha \cdot X$ such that $\alpha \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\alpha \cdot X] = \alpha \cdot E[X]$.*

Corollary 9.1.2. *If X is a continuous random variable, the expected value of X exists, and the function given by f for each x in the domain of the function is the p.d.f. at x and T is a constant transformation of X, meaning $T = \beta$ such that $\beta \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\beta] = \beta$.*

Theorem 9.1.10. *If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x and T is a linear transformation of X, meaning $T = \alpha \cdot X + \beta$ such that $\alpha \in \mathbb{R} \wedge \beta \in \mathbb{R}$, the expected value of X exists, and the standard deviation of X exists and is not zero, then the variance of T is $Var[T] = E[(T - \mu_T)^2] = Var[\alpha \cdot X + \beta] = \alpha^2 \cdot Var[X]$.*

Lemma 9.1.1. *(Markov's Inequality) If X is a continuous random variable and the function given by f for each x in the domain of the function is the p. d. f. at x and X has fixed μ, then for any $k > 0 \quad (k \in \mathbb{R})$ the probability that X is greater than or equal to k is less than or equal to the mean divided by k. $Pr\,(X \geq k) \leq \dfrac{E[X]}{k}$.*

Theorem 9.1.11. *(Chebyshev's Theorem or Tchebyshev's Theorem) If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x and μ and σ are the mean and standard deviation of X ($\sigma \neq 0$), then for any $k > 0 \quad (k \in \mathbb{R})$ the probability is at least $1 - \dfrac{1}{k^2}$ that X will take on a value within k standard deviations of the mean.*

The meaning of Tchebyshev's Theorem is that when we have a well defined continuous random variable, X, with fixed μ and σ ($\sigma \neq 0$), then $Pr\,(|X - \mu| < k \cdot \sigma) \geq 1 - \dfrac{1}{k^2}$.

9.2 PDF - CDF Exercises 1

Exercise 9.2.1. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that g is defined as

$$g(x) = \begin{cases} \dfrac{6-x^2}{15}, & x \in [-2, 1] \\ \\ 0, & else \end{cases}$$

Assume g is a well defined relation.
A. Prove or disprove that g is a well defined function.
B. Prove or disprove that g is a well defined probability mass function.
C. Prove or disprove that g is a well defined probability density function.
D. Find the CDF.

Exercise 9.2.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f is defined as

$$f(x) = \begin{cases} \dfrac{3}{4}, & x \in [-\pi, 0) \\ \\ \dfrac{1}{1+x^2}, & x \in [0, 1) \\ \\ 0, & else \end{cases}$$

Assume f is a well defined function.

A. Prove or disprove that f is a well defined probability density function.

B. If f is a well defined probability density function find $Pr(X \leq \dfrac{1}{\sqrt{3}})$

C. If f is a well defined probability density function find $Pr(-2 \leq X \leq 1)$

D. If f is a well defined probability density function find $Pr(|X| < 1)$

Exercise 9.2.3. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that h is defined as

$$h(x) = \begin{cases} \dfrac{1}{5}, & x \in [3, 8] \\ \\ 0 & else \end{cases}$$

A. Find $Pr(X \geq 1)$
B. Find $Pr(X < 4)$
C. Find $Pr(X > 4)$
D. Find $Pr(X = 4)$
E. Find $Pr(\pi < X < 2\pi)$
F. Find $Pr(4 < X < 7)$
G. Find $Pr(|X - 5| \geq 1)$
H. Find the CDF
I. Find μ
J. Find σ^2
K. Find η_3
L. Find η_4

Exercise 9.2.4. Let $j : \mathbb{R} \longrightarrow \mathbb{R}$ such that j is defined as

$$j(x) = \begin{cases} \frac{1}{3}e^{-(x/3)}, & x \in [0, \infty) \\ 0 & else \end{cases}$$

A. Find $E[X]$ using the definition of expected value B. Find $Pr(X < 4)$
C. Find $Pr(X > e)$ D. Find $Pr(X > 1)$
E. Find $Pr(1 < X < 4)$ F. Find $Pr(X < 4 \mid X < 1)$

Exercise 9.2.5. Suppose $X \sim Pois(x, \lambda)$ with the property that $Pr(X = 1) = Pr(X = 3)$. Find $Pr(X = 5)$.

Exercise 9.2.6. Let $k : \mathbb{R} \longrightarrow \mathbb{R}$ such that k is defined as

$$k(x) = \begin{cases} 10xe^{-5x^2} & \text{x } \in (0,\infty) \\ 0 & else \end{cases}$$

A. Find $Pr(X < 1)$ B. Find $Pr(X \geq 1)$ C. Find $Pr(X < 4)$
D. Find $Pr(1 < X < 4)$ E. Find $Pr(|X - 2.5| \geq 1.5)$ F. Find $Pr(X \geq 4)$

Exercise 9.2.7. Prove theorem 9.1.7.

Exercise 9.2.8. Let X be a randomly selected real from the segment $(0, 3)$. Consider the expression $2X^2 - 5X + 2$. What is the probability that $2X^2 - 5X + 2 > 0$?

Exercise 9.2.9. Prove theorem 9.1.9.

Exercise 9.2.10. Prove theorem 9.1.10.

Exercise 9.2.11. Suppose B is a randomly selected real from the segment $(-3, 3)$. Find the probability that $x^2 + Bx + 1$ has at least one real root.

Exercise 9.2.12. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that F is a CDF defined as

$$F(x) = \frac{1}{2} + \frac{1}{\pi}arctan(\frac{x}{2})$$

A. Find $Pr(X > 2)$ B. Find $Pr(X \leq 0)$ C. Find $Pr(X < 2\sqrt{3})$
D. Find the PDF for X.

Exercise 9.2.13. Let $G : \mathbb{R} \longrightarrow \mathbb{R}$ such that G is a CDF defined as

$$G(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

A. Find $Pr(\frac{1}{2} < X < \frac{3}{4})$ B. Find the PDF for X.

Exercise 9.2.14. Let $j : \mathbb{R} \longrightarrow \mathbb{R}$ such that j is a PDF defined as

$$j(x) = \begin{cases} \frac{1}{3}e^{-(x/3)}, & x \in [0, \infty) \\ 0 & else \end{cases}$$

Find the CDF, J, for the random variable.

Exercise 9.2.15. Let $j : \mathbb{R} \longrightarrow \mathbb{R}$ such that j is a PDF defined as

$$j(x) = \begin{cases} \frac{1}{3}e^{-(x/3)}, & x \in [0, \infty) \\ 0 & else \end{cases}$$

Find the value, ℓ, such that $Pr(X \leq \ell) = \frac{1}{2}$.

Exercise 9.2.16. Let $w : \mathbb{R} \longrightarrow \mathbb{R}$ such that w is a function defined as

$$w(x) = \begin{cases} \frac{1}{4} \cdot x \cdot e^{-(x^2/8)}, & x \in (0, \infty) \\ 0 & else \end{cases}$$

A. Show that w is a well-defined PDF (e.g.: show that it satisfies theorem 9.1.5)
B. Find the value, m, such that m is the mode of the distribution.
C. Find μ.

9.3 PDF - CDF Exercises 2

Exercise 9.3.1. Let $X \sim Pois(x, 2)$.
A. Find $Pr(X > 3)$ B. Find $Pr(X = 3)$ C. Find $Pr(X < 0)$ D. Find $Pr(X \in \Delta)$ where $\Delta = \{x|x = 2p, p \in \mathbb{N}_3\}$

Exercise 9.3.2. Let $X \sim Bin(x, 10, \frac{1}{3})$.
A. Find $Pr(X > 3)$ B. Find $Pr(X = \mu)$ C. Find $Pr(X < \mu)$ D. Find $Pr(X \in \Xi)$ where $\Xi = \{x|x = 2p - 1, p \in \mathbb{N}_2\}$

Exercise 9.3.3. Let $X \sim Exp(x, \theta = 4)$.
A. Find $Pr(X = 5)$ B. Find $Pr(X = 4)$
C. Find $Pr(X < 2)$ D. Find $Pr(X \geq 3)$
E. Find $Pr(X > 3)$ F. Find $Pr(X = 3)$
G. Find $Pr(X < 0)$ H. Find $Pr(X \in (e, \pi))$

Exercise 9.3.4. Let $X \sim Uni(x, \alpha = 4, \beta = 14)$.
A. Find $Pr(X = 5)$ B. Find $Pr(X \geq 4)$
C. Find $Pr(X < 12)$ D. Find $Pr(X \geq 13)$
E. Find $Pr(X > 13)$ F. Find $Pr(X < 13)$
G. Find $Pr(|X - 6| < 2)$

Exercise 9.3.5. Let $X \sim Erl(x, \alpha = 1, \beta = 4)$.
A. Find $Pr(X = 5)$ B. Find $Pr(X \geq 4)$
C. Find $Pr(X < 2)$ D. Find $Pr(X \geq 3)$

Exercise 9.3.6. Let $X \sim Wei(x, \alpha = 4, \beta = 1)$.
A. Find $Pr(X = 5)$ B. Find $Pr(X \geq 4)$
C. Find $Pr(X < 2)$ D. Find $Pr(X \geq 3)$

Exercise 9.3.7. Let $X \sim Exp(x, \theta = 5)$.
A. Find $Pr(X > 5)$ B. Find $Pr(X \geq 5)$
C. Find $Pr(X \geq 7)$ D. Find $Pr(X \geq 2)$
E. Find $Pr(X \geq 5|X \geq 7)$ F. $Pr(X \geq 7|X \geq 5)$
G. $Pr(X \geq 7|X \geq 5)$

9.4 Important Continuous Probability Density Functions and Moments

Definition 9.4.1. If X is a continuous random variable and the function given by f for each x in the domain of the function is the p.d.f. at x, and there is one (or more) real numbers that are fixed which define a particular case of the random variable we call that constant **a parametre** of the p.d.f.

1. The **Uniform** random variable is a probabilistic (or stochastic) experiment that can have any outcome on an interval (or segment) of $\mathbb{R}$ such that it can be $\beta(X = 1), \alpha(X = 0)$, or any value between such the probability is one divided by $\beta - \alpha$ for any of these values; the probability is zero otherwise. So, the parameters are α and β. An application of the uniform random variable is the round off difference between recorded and true values of physical quantities. It provides (oft times) a fair approximation over a relatively narrow range for a random variable whose distribution is not uniform. $x \in [\alpha, \beta]$ where $\beta > \alpha$ (it can also be $x \in (\alpha; \beta]\,; x \in [\alpha, \beta)$; or, $x \in (\alpha, \beta)$ for some problems).

$$Uni(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & x \in [\alpha, \beta] \\ \\ 0 & else \end{cases}$$

Claims about the moments of a Uniform random variable[5]

1. $\mu = \dfrac{\beta + \alpha}{2}$
2. $\mu'_r = \dfrac{\beta^{r+1} - \alpha^{r+1}}{(\beta - \alpha)(r + 1)}$
3. $\sigma^2 = \dfrac{(\beta - \alpha)^2}{12}$
4. $\eta_3 = 0$
5. $\eta_4 = 1.8$

6. $\mu_r = \begin{cases} 0 & r \in O \\ \\ \frac{(\beta-\alpha)^r}{2^r \cdot (r+1)} & r \in E \end{cases}$

[5]Such need to proven; but, for computational problems you may assume said. Also, let $E = \{x\,|x = 2 \cdot j, j \in \mathbb{N}\}\,, O = \{x\,|x = 2 \cdot k + 1, k \in \mathbb{N}\}$.

2. The **Normal or Gaussian**[6] random variable is a probabilistic (or stochastic) random variable that is the **most** important of all the random variables (or so it would seem). The normal variable is a probabilistic (or stochastic) experiment that can have any outcome on $(-\infty, \infty)$. The parameters are μ and σ (or μ and σ^2 and it **must** always be clearly delineated whether the parameters are the mean and variance or the mean and standard deviation).

Thus, it is defined by its mean and standard deviation. Its applications are many and its uses quite important. A substantial number of empirical studies have indicated that the normal function provides an adequate representation of, or at least a decent approximation to, the distributions of many physical, mental, economic, biological, and social variables. (for example: meteorological data (temperature and rainfall), measurements of living organisms (height or weight of humans, populations, etc.), scores on aptitude tests, physical measurements of manufactured parts, instrumental errors, deviations from social norms, and above all else as we take means of means the limiting distribution is normal.).
$x \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma \in \mathbb{R} \ni \sigma \in (0, \infty)$.[7]

$$Nor(x, \mu, \sigma) = \frac{e^{\frac{-(x-\mu)^2}{2 \cdot \sigma^2}}}{\sqrt{2 \cdot \pi \cdot \sigma^2}} \quad \forall x \in \mathbb{R}$$

$$Nor(x, \mu, \sigma) = \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma^2} \cdot e^{\frac{(x-\mu)^2}{2 \cdot \sigma^2}}} \quad \forall x \in \mathbb{R}$$

Claims about the moments of a Normal random variable[8]

1. $E[X] = \mu$
2. The median is μ.
3. The mode is μ.
4. $\mu_r = \begin{cases} 0 & r \in O \\ \frac{\sigma^r \cdot r!}{2^{r/2}(\frac{r}{2})!} & r \in E \end{cases}$
5. $\sigma_{X^2} = \sigma^2$

---

[6] Gaussian after K. F. Gauss, 'the Prince of Mathematics,' who laid much of the ground work for Modern Probability & Statistics (1809 and later). The Normal curve family of functions was first considered by DeMoivre (1733). Modern Probability & Statistics, however, was formalised by R. A. Fisher & K. Pearson.

[7] We need , $\sigma \neq 0$ for if it were zero it would mean there is no variation and all values are the mean which implies the integral across all reals is not 1 (contradicting the Kolmogorov axioms).

[8] Such need to proven; but, for computational problems you may assume said.

6. $\eta_3 = 0$

7. $\eta_4 = 3$

A very special case of the normal family is the **standard normal** function:

$$Nor(z, 0, 1) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}} \quad \forall z \in \mathbb{R}$$

Note: We will use the notation $Z \sim Nor(z, 0, 1)$ for the standard normal.
The c. d. f. of the standard normal function[9] also called the unidimensional normal ogive function is:

$$\Phi(Z \leq z) = CDFN(z, 0, 1) = \int\limits_{-\infty}^{z} \frac{1}{\sqrt{2\pi} \cdot e^{t^2/2}} dt \quad \forall z \in \mathbb{R}$$

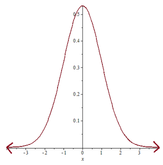

Figure 9.1: PDF of $X \sim Nor(x, 0, 1)$

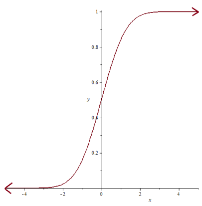

Figure 9.2: CDF of $X \sim Nor(x, 0, 1)$

Theorem 9.4.1. *(DeMoivre - Laplace): Let* $X \sim Bin(x, n, p)$*. Let* $Y = \frac{X - np}{\sqrt{np(1-p)}}$ *It is the case that as* $n \longrightarrow \infty$*, it is the case that* $Y \longrightarrow Z$ *where* $Z \sim Nor(z, 0, 1)$.

[9] Note it is a non-integrable [closed form] function.

Recall (see your Calculus II book or notes)

The **gamma function**, $\Gamma : (0, \infty) \longrightarrow \mathbb{R}$, is defined as $y \in (0, \infty) \quad \Gamma(y) = \int\limits_0^\infty e^{-x}(x^{y-1})dx$

Review Exercises[10]

1. Show for any $y \ni y \in (1, \infty) \quad \Gamma(y) = (y-1) \cdot \Gamma(y-1)$.
2. Show for any $y \ni y \in \mathbb{N} \quad \Gamma(y) = (y-1)!$.
3. Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4. Given $f(x) = \dfrac{x^2}{\Gamma(3) \cdot 2^3 \cdot e^{x/2}}$ such that $f : (0, \infty) \longrightarrow \mathbb{R}$

   A. Find $\displaystyle\int_0^\infty f(x)dx$.

   B. Find $\displaystyle\int_0^\infty x \cdot f(x)dx$.

   C. Find $\displaystyle\int_0^\infty x^2 \cdot f(x)dx$.

The gamma function is a very useful and pliable function –it is a central result that we rest upon for many of the distributions discussed in this and any latter Probability class.

[10]See the discussion on the Gamma Function [Chapter 2] for a more detailed discussion.

3. The **Beta** random variable is a probabilistic (or stochastic) experiment that can have any outcome on the segment $(0, 1)$. The parameters are α and β. It has been used to represent physical variables whose values are restricted to an interval of finite length, tolerance limits without the assumption of normality, and in Bayesian[11] statistics. $x \in \mathbb{R}$; $\alpha \in (0, \infty)$; and, $\beta \in (0, \infty)$

$$Beta(x, \alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{(\alpha-1)}(1-x)^{(\beta-1)} & \text{x} \in (0,\, 1) \\ 0 & else \end{cases}$$

Claims about the moments of a Beta random variable[12]

1. $\mu = \dfrac{\alpha}{\alpha+\beta}$

2. $\sigma^2 = \dfrac{\alpha \cdot \beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

3. $\mu'_r = \displaystyle\prod_{i=0}^{r-1} \frac{(\alpha+i)}{(\alpha+\beta+i)} \quad r \in \mathbb{N} - \{1\}$

4. $\eta_3 = \dfrac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{\sqrt{\alpha \cdot \beta}(\alpha+\beta+2)}$

5. $\eta_4 = \dfrac{3(\beta+\alpha+1)[2(\alpha+\beta)^2+\alpha\beta(\alpha+\beta-6)]}{\alpha \cdot \beta(\alpha+\beta+2)(\alpha+\beta+3)}$

[11] Bayesian statistics is a parametric procedure such that the sample drives the a priori distribution rather than the traditional a posteriori method such that a distribution a priori is hypothesised, then the sample gather and parametric statistics are employed on the sample data to infer conclusions based on the evidence.

[12] Such need to proven; but, for computational problems you may assume said.

4. The **Exponential** random variable (also known as the negative exponential distribution) is a probabilistic (or stochastic) experiment that can have any outcome on an interval a subset of $[0, \infty)$ or $(0, \infty)$. An application of the exponential random variable is the length of time until the occurrence of a first Poisson event. Therefore, the exponential distribution can model the length of time between two successive Poisson events that occur independently and at a constant rate. The exponential distribution has been used extensively as a time-to-failure model in reliability problems, and as a model for random time-length in waiting line problems. It is said that the exponential distribution exhibits the 'lack of memory' property or 'memorylessness.'[13]Therefore, the probability of an occurrence of present or future events does not depend on events that happened in the past. Thus, the probability of a unit failing during a specified time interval depends only on the length of that interval and not on how long the unit has been in operation. Its parameter is θ where $\theta \in \mathbb{R} \ni \theta \in (0, \infty)$. Theta represents the average length of time between two Poisson events.
$x \in \mathbb{R}$ and $\theta \in (0, \infty)$.

$$Exp(x, \theta) = \begin{cases} \frac{1}{\theta} e^{(-x/\theta)}, & x \in [0, \infty) \\ 0 & else \end{cases}$$

Claims about the moments of an Exponential random variable[14]

1. $\mu = \theta$
2. $\mu'_r = r! \cdot \theta^r$
3. $\sigma^2 = \theta^2$
4. $\eta_3 = 2.$
5. $\eta_4 = 9.$
6. The mode is 0.
7. The median is $\theta \cdot ln(2)$.

Note in many texts the exponential random variable is expressed slightly differently:

$$Exp(x, \lambda) = \begin{cases} \lambda e^{(-\lambda x)}, & x \in [0, \infty) \\ 0 & else \end{cases}$$

But $\lambda = \frac{1}{\theta}$; so, big whoop.

---

[13] I believe it is the only lack of memory continuous probability density. Its discrete analogue is the geometric mass function.

[14]Such need to proven; but, for computational problems you may assume said.

5. The **Gamma (Erlang)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(0, \infty)$. The parameters are α and β. It has been used to represent physical variables under stress (such as a metal specimen) such that it will break only if a certain number of stress cycles are applied. If the cycles occur independently and at a given average rate, then the length of time until the specimen breaks is a random variable that follows the gamma p. d. f. It has been used to represent random time to failure of a system that fails only if exactly α independent components fail and component failure occurs at a constant rate of $\frac{1}{\beta}$. It has been used to model waiting line problems to represent the total length of time to compleat service if service is made up of exactly α substations and completion of service at each substation occurs independently at a constant rate of $\frac{1}{\beta}$. It is also named the Erlang probability model after the Danish mathematician Erlang who established its usefulness an application to telephone traffic problems. α is referred to as the **shape parameter** and β the **scale parameter.** $x \in \mathbb{R}$; $\alpha \in (0, \infty)$; and, $\beta \in (0, \infty)$.

$$Gamma(x, \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}e^{x/\beta}} & \text{x } \in (0,\infty) \\ 0 & else \end{cases}$$

Claims about the moments of an Gamma random variable[15]

1. $\mu = \alpha \cdot \beta$
2. $\sigma^2 = \alpha \cdot \beta^2$
3. $\mu'_r = \dfrac{\beta^r \Gamma(\alpha + r)}{\Gamma(\alpha)}$
4. $\eta_3 = \dfrac{2}{\sqrt{\alpha}}$
5. $\eta_4 = 3 \cdot (1 + \dfrac{2}{\alpha})$
6. The mode is $\beta \cdot (\alpha - 1)$ when $\alpha \geq 1$; otherwise it does not exist.

[15]Such need to proven; but, for computational problems you may assume said. The first is probably my favourite claim. I love proving the claim about the mean of a Gamma random variable.

6. The **Chi-square (or chi-squared)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(0, \infty)$. The parameter is ν (nu). It is a special case of the Gamma variate (you will (hopefully) prove this later). The parameter, ν, is called the 'degrees of freedom' (referenced quite frequently in applied probability and statistics). It plays a significant role in inferential statistics and sampling theory, especially with regard to inferences about variances.[16] It also is highly useful in 'non-parametric' statistics (for example, in tests of the lack of fit of data and frequency problems).[17] $x \in \mathbb{R}$ whist $\nu \in \mathbb{N}$.

$$Chi(x,\nu) = \begin{cases} \frac{x^{(\nu-2)/2}}{2^{\nu/2}\Gamma(\nu/2)e^{(x/2)}} & \text{x } \in (0,\infty) \\ \\ 0 & else \end{cases}$$

The standard notation is $Chi(x, \nu) = \chi^2_\nu$.

Claims about the moments of a Chi-squared random variable[18]

1. $\mu = \nu$
2. $\sigma^2 = 2\nu$
3. $\eta_3 = \dfrac{4}{\sqrt{2\nu}}$
4. $\eta_4 = 3(1 + \dfrac{4}{\nu})$
5. The mode is ν - 2 when $\nu \geq 2$
6. $\mu'_r = 2^r \displaystyle\prod_{i=0}^{r-1} [i + (\frac{\nu}{2})] = \frac{2^r \cdot \Gamma(r + \frac{\nu}{2})}{\Gamma(\frac{\nu}{2})}$ where $r \in \mathbb{N} - \{1\}$

Heuristic[19] Claim: The median is approximately $\nu - \frac{2}{3}$ when ν is 'large.' (corrected by Mr. Bales Math 403 Fall 2013.)

---

[16] Which is discussed in Math 302. It turns out that if $X \sim Nor(x, \mu = 0, \sigma = 1)$, then $X^2 = Y \sim Chi(y, 1)$.

[17] Which is discussed in Math 302.

[18] Such need to proven; but, for computational problems you may assume said.

[19] A heuristic is as we have noted a 'rule of thumb' and is not a theorem nor is it proven. Empirical evidence suggests such.

7. The **Weibull** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(0, \infty)$. The parameters are α and β. It has been used to represent physical variables such that it models the strength of many materials. It has also been used extensively as a time-to-failure model for a wide variety of mechanical and electrical components. It is named after the Swedish physicist Weibull who demonstrated empirically its usefulness for modelling strength of materials. α is referred to as the 'shape' parameter and β the 'scale' parameter (note the similarity of language to the Gamma distribution).
$x \in \mathbb{R}$; $\alpha \in (0, \infty)$; and, $\beta \in (0, \infty)$.

$$Wei(x, \alpha, \beta) = \begin{cases} \frac{\beta x^{\beta-1}}{\alpha^\beta} e^{-(\frac{x}{\alpha})^\beta} & \text{x } \in (0{,}\infty) \\ 0 & else \end{cases}$$

Claims about the moments of a Weibull random variable[20]

1. $\mu = \alpha \cdot \Gamma(1 + \dfrac{1}{\beta})$
2. Either $\mu_r = \alpha^r \cdot \Gamma(1 + \dfrac{r}{\beta})$ or $\mu_r' = \alpha^r \cdot \Gamma(1 + \dfrac{r}{\beta})$
3. $\sigma^2 = \alpha^2 \cdot \left(\Gamma\left(1 + \frac{2}{\beta}\right) - \left(\Gamma\left(1 + \frac{1}{\beta}\right)\right)^2\right)$
4. $\eta_3 = \frac{\Gamma\left(1+\frac{3}{\beta}\right) - 3\left(\Gamma\left(1+\frac{1}{\beta}\right)\left(\Gamma\left(1+\frac{2}{\beta}\right)\right) + 2\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^3\right.}{\left(\left(\Gamma(1+\frac{2}{\beta}) - \left(\Gamma(1+\frac{1}{\beta})\right)^2\right)\right)^{3/2}}$

Note: η_4 is even more hideous.

Let m_O be the mode and m_d be the median. $m_O = \begin{cases} \alpha(1 - \frac{1}{\beta})^{1/\beta} & \beta \geqslant 1 \\ 0 & \beta < 1 \end{cases}$ $m_d = \alpha\sqrt{\ln(2)}$

Some Nifty Claims About Some of the Continuous Random Variable Functions' Relationships:

Claim 9.4.1. $Gamma(x, 1, \theta) \equiv Exp(x, \theta)$

Claim 9.4.2. $Gamma(x, 1/2, 2) \equiv Chi(x, 1)$

Claim 9.4.3. $Gamma(x, \nu/2, 2) \equiv Chi(x, \nu)$

Claim 9.4.4. $Wei(x, \theta, 1) \equiv Exp(x, \theta)$

Claim 9.4.5. $Beta(x, \alpha, \beta) \equiv Beta(1 - x, \beta, \alpha)$

Claim 9.4.6. $Beta(x, \alpha, \beta)$ is symmetric if $\alpha = \beta$

[20] Such need to proven; but, for computational problems you may assume said.

8. The **Pareto** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(1, \infty)$. The parameter is α.

$$Par(x, \alpha) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{x } \in (1,\infty) \\ 0 & else \end{cases}$$

Claims about the moments of a Pareto random variable[21]

1. $\mu = \dfrac{\alpha}{\alpha - 1}$ only if $\alpha > 1$
2. μ_r' exists only if $r < \alpha$.

Another Nifty Claims About Some of the Continuous Random Variable Functions' Relationships:

Claim 9.4.7. Let $X \sim Uni(x, 0, 1)$. Let $Y = X^{-1/\alpha}$ such that $\alpha > 0$. It is the case that $Y \sim Par(y, \alpha)$.

9. The **Generalised Pareto** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $[\delta, \infty)$. The parameters are δ and γ.

$$GenPar(x, \delta, \gamma) = \begin{cases} \gamma\frac{\delta^{\gamma}}{x^{\delta+1}} & \text{x } \in [\delta,\infty) \\ 0 & else \end{cases}$$

Claims about the moments of a Generalised Pareto random variable[22]

1. $\mu = \gamma\dfrac{\delta}{\gamma - 1}$ only if $\gamma > 1$.
2. $\mu_r' = \gamma\dfrac{\delta^r}{(\gamma - r)}$ only if $r < \gamma$.

[21] Such need to proven; but, for computational problems you may assume said.
[22] Such need to proven; but, for computational problems you may assume said.

10. The **Rayleigh** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(0, \infty)$. The parameter is α. One amusing application is that if one is aiming at a target in the plane, if the horizontal and vertical miss distances are independent, then the absolute value of the error is Rayleigh distributed.

$$Ray(x, \alpha) = \begin{cases} 2\alpha x e^{-\alpha x^2} & \text{x } \in (0,\infty) \\ \\ 0 & else \end{cases}$$

Claims about the moments of a Rayleigh random variable[23]

1. $\mu = \sqrt{\dfrac{\pi}{4\alpha}}$

2. $\sigma^2 = \dfrac{1}{\alpha} - \dfrac{\pi}{4\alpha}$

3. Let m_O be the mode and m_d be the median. $m_O = \frac{1}{\sqrt{2\alpha}}$ and $m_d = \sqrt{\frac{\log 4}{2\alpha}}$

11. The **Hazard Rate (or Failure Rate)** variate is a general notion set of functions where a continuous random variable X is the lifetime of some unit such that X has density f and cumulative distribution F. The hazard rate function is therefore λ such that $\lambda(t) = \frac{f(t)}{1-F(t)}$ for time t.

$\lambda(t)$ for time t is considered as follows: Suppose a unit has survived for a time t and we wish to find the probability that it will NOT survive for an additional time Δt. So, $\lambda(t)$ represents the conditional probability intensity that a t-old unit will fail. Thus,

$$Pr(X \in (t, t+\Delta t) | X > t) = \frac{\Pr[\{X \in (t, t+\Delta t)\} \wedge (X > t)]}{\Pr(X > t)} = \frac{\Pr[X \in (t, t+\Delta t)]}{\Pr(X > t)} \approx \frac{f(t)}{1 - F(t)} \Delta t$$

[23]Such need to proven; but, for computational problems you may assume said.

12. The **Cauchy** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $((-\infty, \infty)$. The parameters are α and β.

$$Cauchy(x, \alpha, \beta) = \frac{\frac{\beta}{\pi}}{(x-\alpha)^2 + \beta^2} \quad \text{x } \in (\text{-}\infty\text{,}\infty)$$

Claims about the moments of a Cauchy random variable[24] is that μ does not exist; μ_1' does not exist, μ_2' does not exist , indeed no moments exist for this distribution!

13. The **Laplace (or double-exponential)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $((-\infty, \infty)$. The parameters are α and β.

$$Laplace(x, \alpha, \beta) = \frac{1}{2\beta}\ e^{-|x-\alpha|/\beta} \quad \text{x } \in (\text{-}\infty\text{,}\infty)$$

Claims about the moments of a Laplace random variable[25]

1. $\mu = \alpha$
2. $\sigma^2 = 2\beta^2$
3. $\mu_r = \begin{cases} 0 & r \in O \\ \\ r!\beta^r & r \in E \end{cases}$
4. $\eta_3 = 0$.
5. $\eta_4 = 6$.
6. Let m_O be the mode and m_d be the median. $m_O = \alpha$ and $m_d = \alpha$.

[24] Such need to proven; but, for computational problems you may assume said.
[25] Such need to proven; but, for computational problems you may assume said. Also, let $E = \{x\,|x = 2 \cdot j, j \in \mathbb{N}\}\,, O = \{x\,|x = 2 \cdot k + 1, k \in \mathbb{N}\}$

14. The **Student (or t or Gosset)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $((-\infty, \infty)$. The parameter is ν such that $\nu \in \mathbb{N}$. The parameter, ν, is called the 'degrees of freedom' (referenced quite frequently in applied probability and statistics).

It plays a significant role in inferential statistics and sampling theory, especially with regard to inferences about differences of the means between two groups[26] under certain conditions (which usually are not verified by the social scientists using it - aargh).

$$Stu(x, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})(1 + \frac{x^2}{\nu})^{(\nu+1)/2}} \quad x \in (-\infty, \infty)$$

Notation: $Stu(x, \nu) = t_\nu$

Claims about the moments of a Student random variable[27]

1. $\mu = 0$ when $\nu > 1$
2. $\sigma^2 = \dfrac{\nu}{\nu - 2}$ when $\nu > 2$.
3. $\eta_3 = 0$ when $\nu > 3$.
4. $\eta_4 = \dfrac{3(\nu - 2)}{\nu - 4}$ when $\nu > 4$.
5. Let m_O be the mode and m_d be the median. $m_O = 0, \quad m_d = \alpha$.
6. $\mu_r = \begin{cases} 0 & r \in O \\ AM & r \in E \end{cases}$

Note: AM denotes 'a mess' in the definition of μ_r.

[26] Which is discussed in Math 302.

[27] Such need to proven; but, for computational problems you may assume said. Also, let $E = \{x \,|x = 2 \cdot j, j \in \mathbb{N}\}, O = \{x \,|x = 2 \cdot k + 1, k \in \mathbb{N}\}$

15. The **F (variance ratio) (or Fisher - Snedecor)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $[0, \infty)$. The parameters are ν and ω (lower case omega) such that $\nu \in \mathbb{N}$ and $\omega \in \mathbb{N}$. The parameters, ν and ω, are both called the degrees of freedom (referenced quite frequently in applied probability and statistics).

It plays a significant role in inferential statistics and sampling theory, especially with regard to inferences about differences of the means between multiple groups.[28] under certain conditions (which usually are not verified by the social scientists (as with the t) using it - aargh).

$$Fish(x, \nu, \omega) = \begin{cases} \frac{\Gamma(\frac{\nu+\omega}{2})(\frac{\nu}{\omega})^{\nu/2} x^{(\nu-2)/2}}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\omega}{2})[1+\frac{\nu x}{\omega}]^{(\nu+\omega)/2}} & x \in [0, \infty) \\ 0 & else \end{cases}$$

Notation: $Fish(x, \nu, \omega) = F_{\nu,\omega}$.

1. $\mu = \dfrac{\omega}{\omega - 2}$ when $\omega > 2$
2. $\sigma^2 = \dfrac{2\omega^2(\nu + \omega - 2)}{\nu(\omega - 2)^2(\omega - 4)}$ when $\omega > 4$

[28] Which is discussed in Math 302.

16. The **Inverse Gaussian (or Wald)** random variable is a probabilistic (or stochastic) experiment that can have any outcome on $(0, \infty)$. The parameters are μ and λ such that $\mu \in (0, \infty)$ and $\lambda \in (0, \infty)$. The parameters, μ and λ, are typically called the **location** and **scale** parameters, respectively.

$$InvGauss(x, \mu, \lambda) = \begin{cases} \sqrt{\frac{\lambda}{2\pi x^3}} \cdot \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right) & x \in (0, \infty) \\ 0 & else \end{cases}$$

Claims about the moments of an Inverse Gaussian (or Wald) random variable[29]

1. $\mu_1' = \mu$
2. $\sigma^2 = \dfrac{\mu^3}{\lambda}$

Finally, it should be noted that there are so *many* more distributions (most of which I have not studied). The Dirichlet, Gumbel, Kendall, Maxwell, etc. Then there are the non-centrals, multivariates, etc. Statistics is composed of two general areas, parametric and non-parametric, of which we are just 'scratching the surface' of univariate parametric statistics.

Addendum:

The one definition of measure theory I will include is as follows. For those interested, play with it and try to grasp its meaning.

Definition 9.4.2. A set $A \subseteq \mathbb{R}$ is defined to be a set of measure zero if $\forall \varepsilon > 0$, $\exists$ a countable collection of mutually disjoint segments
$\Omega = \{S_1, S_2, \ldots, S_i, \ldots\} = \{S_j\}_{j=1}^{\infty}, |\Omega| \leq \aleph_0 \ \wedge \ \ni A \subseteq \bigcup_{j \in \mathbb{N}} S_j$ and $\sum_{j \in \mathbb{N}} |S_j| < \varepsilon$
or $\Omega = \{S_1, S_2, \ldots, S_k\} = \{S_j\}_{j=1}^{k}, |\Omega| \leq \aleph_0 \ \wedge \ \ni A \subseteq \bigcup_{j \in \mathbb{N}_k} S_j$ and $\sum_{j \in \mathbb{N}_k} |S_j| < \varepsilon$

[29]Such need to proven; but, for computational problems you may assume said.

9.5 Exercises: Computational 3

Exercise 9.5.1. Let $X \sim Ray(x, \alpha = 3)$. Find the probability that $X < \mu$.

Exercise 9.5.2. Let $X \sim Ray(x, \alpha = 3)$. Let the median of the distribution be d. Find d.

Exercise 9.5.3. Let $X \sim Ray(x, \alpha = 3)$. Let the median of the distribution be d. Find the probability that $X < d$.

Exercise 9.5.4. Let $X \sim Beta(x, \alpha = 4, \beta = 2)$. Find $Pr(X > 2)$.

Exercise 9.5.5. Let $X \sim Erl(x, \alpha = 3, \beta = 2)$. Find $Pr(X < 1)$.

Exercise 9.5.6. Let $X \sim Erl(x, \alpha = 3, \beta = 2)$. Find $Pr(X < 1)$.

Exercise 9.5.7. Let $X \sim Wei(x, \alpha = 2, \beta = 3)$. Find $Pr(X \geq 3)$.

Exercise 9.5.8. Let $X \sim Bin(x, 7, \frac{1}{2})$.
A. Find $Pr(X < 1|X < 3)$.
B. Find $Pr(X \geq 5|X > 3)$.
C. Find $Pr(X > 3|X \geq 6)$.

Exercise 9.5.9. $\mathbb{B}$ Let $X \sim f(x)$ for some well defined probability density function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Let the cumulative distribution function $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$F(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - e^{-\arctan(1) \cdot x}, & x \in [0, \infty] \end{cases}$$

Find the PDF and identify what distribution it is and its parameter(s).

Exercise 9.5.10. $\mathbb{B}$ Let us consider the polynomial, p, to be $5x^2 + b \cdot x + 5$. Let $B \sim Exp(x, 20)$. Find the probability the polynomial has at least one real root.

9.6 Approximating Probability (Normal RV) Exercises

Exercise 9.6.1. Let $X \sim Nor\left(x, \mu = 100, \sigma = 15\right)$.

1. Approximate $Pr\left(X \in (75, 82]\right)$
2. Approximate $Pr\left(X = 82\right)$
3. Approximate $Pr\left(X > 130\right)$
4. Approximate $Pr\left(X \leq 130\right)$
5. Approximate $Pr\left(94 \leq X < 110\right)$

Exercise 9.6.2. Let $X \sim Nor\left(x, \mu = 100, \sigma = 30\right)$.

1. Approximate $Pr\left(X \leq 130\right)$
2. Approximate $Pr\left(110 < X < 155\right)$
3. Approximate $Pr\left(94 \leq X < 110\right)$

Exercise 9.6.3. Let $X \sim Nor\left(x, \mu = 100, \sigma^2 = 15\right)$.

1. Approximate $Pr\left(X \leq 130\right)$
2. Approximate $Pr\left(110 < X < 155\right)$
3. Approximate $Pr\left(94 \leq X < 110\right)$

9.6.1 Approximating Probability of a Binomial with A Poisson PMF or Normal PDF

Exercise 9.6.4. Let $X \sim Bin\left(x, 5, \frac{1}{2}\right)$.

1. Evaluate $Pr\left(X \in \{1, 2\}\right)$.
2. Approximate $Pr\left(X \in \{1, 2\}\right)$ with $X_1 \sim Pois\left(x_1, \lambda = \frac{5}{2}\right)$.
3. Approximate $Pr\left(X \in \{1, 2\}\right)$ with $X_2 \sim Nor\left(x_2, \mu = \frac{5}{2}, \sigma = \sqrt{\frac{5}{4}}\right)$.

Exercise 9.6.5. Let $X \sim Bin\left(x, 5, \frac{1}{10}\right)$.

1. Evaluate $Pr(X \in \{1,2\})$.
2. Approximate $Pr(X \in \{1,2\})$ with $X_3 \sim Pois\left(x_3, \lambda = \frac{1}{2}\right)$.
3. Approximate $Pr(X \in \{1,2\})$ with $X_4 \sim Nor\left(x_4, \mu = \frac{1}{2}, \sigma = \sqrt{\frac{9}{20}}\right)$.

Exercise 9.6.6. Let $X \sim Bin\left(x, 20, \frac{1}{4}\right)$.

1. Evaluate $Pr(X \in \{1,2\})$.
2. Approximate $Pr(X \in \{1,2\})$ with $X_5 \sim Pois(x_5, \lambda = 5)$.
3. Approximate $Pr(X \in \{1,2\})$ with $X_6 \sim Nor\left(x_6, \mu = 5, \sigma = \sqrt{\frac{15}{4}}\right)$.

Exercise 9.6.7. Let $X \sim Bin\left(x, 40, \frac{1}{20}\right)$. Approximate $Pr(X \in \{0,2,5\})$ with an appropriate Poisson or Normal distribution; leave the denominator of the solution in exponential form.

Exercise 9.6.8. Let $X \sim Bin\left(x, 40, \frac{1}{20}\right)$. Set up the formula to approximate $Pr(X \in \{x|15 < X \leq 30\})$ with an appropriate Poisson or Normal distribution.

Exercise 9.6.9. Let $X \sim Bin\left(x, 40, \frac{1}{2}\right)$. Approximate $Pr(X \in \{x|15 < X \leq 30\})$ with an appropriate Poisson or Normal distribution.

9.6.2 Other Computational Problems of Probabilities for Random Variables with Well Defined PDFs or PMFs

Exercise 9.6.10. Let $X \sim Ray(x, 50)$.

1. Evaluate $Pr(X \in (0,2))$.
2. Evaluate $Pr(X \in (0,25))$.
3. Evaluate $Pr(X < 100)$.

Exercise 9.6.11. Let $X \sim Par(x, 5)$.

1. Evaluate $Pr(X \in (0,2))$.
2. Evaluate $Pr(X \in [1,5))$.

3. Evaluate $Pr(X > 10)$.

Exercise 9.6.12. Let $X \sim Wei(x, \alpha = 2, \beta = 5)$.

1. Evaluate $Pr(X \in (0, 2))$.
2. Evaluate $Pr(X < 5)$.
3. Evaluate $Pr(X > 10)$.

Exercise 9.6.13. Let $X \sim Wei(x, \alpha = 5, \beta = 2)$.

1. Evaluate $Pr(X \in (0, 2))$.
2. Evaluate $Pr(X < 5)$.
3. Evaluate $Pr(X > 10)$.

Exercise 9.6.14. Let $X \sim Gamma(x, \alpha = 2, \beta = 2)$.

1. Evaluate $Pr(X < 5)$.
2. Evaluate $Pr(X > 8)$.

Exercise 9.6.15. Let $X \sim Gamma(x, \alpha = 5, \beta = 2)$.

1. Evaluate $Pr(X < 5)$.
2. Evaluate $Pr(X > 8)$.

Exercise 9.6.16. Let $X \sim Erl(x, \alpha = 2, \beta = 5)$.

1. Evaluate $Pr(X < 5)$.
2. Evaluate $Pr(X > 8)$.

Exercise 9.6.17. Let $X \sim Beta(x, \alpha = 2, \beta = 5)$.

1. Evaluate $Pr(X < 5)$.
2. Evaluate $Pr(X > 8)$.

Exercise 9.6.18. Let $X \sim Beta(x, \alpha = 2, \beta = 5)$.

1. Evaluate $Pr(X < \frac{1}{2})$.
2. Evaluate $Pr(\frac{1}{3} < X < \frac{5}{6})$.

9.7 Exercises: Theoretical

All problems in this section are board-worthy.
Prove the claims about specific discrete random variable. Some of the claims are facile; others not. Some of the claims become repetitious; so, I may 'pull' some claims from the list of things to present because of the repetitiveness (but such can be submitted in writing for reduced credit – going to the board is a must in this class).

Exercise 9.7.1. Let $X \sim Uni(x, \alpha, \beta)$ $\mu = \dfrac{\beta + \alpha}{2}$

Exercise 9.7.2. Let $X \sim Uni(x, \alpha, \beta)$ $\sigma^2 = \dfrac{(\beta - \alpha)^2}{12}$

Exercise 9.7.3. Let $X \sim Uni(x, \alpha, \beta)$ $\eta_3 = 0$

Exercise 9.7.4. Let $X \sim Nor(x, \mu, \sigma)$ $E[X] = \mu$

Exercise 9.7.5. Let $X \sim Nor(x, \mu, \sigma)$ The median is μ.

Exercise 9.7.6. Let $X \sim Nor(x, \mu, \sigma)$ The mode is μ.

Exercise 9.7.7. Let $X \sim Nor(x, \mu, \sigma)$ $\eta_3 = 0$

Exercise 9.7.8. Let $X \sim Beta(x, \alpha, \beta)$ $\mu \dfrac{\alpha}{\alpha + \beta}$

Exercise 9.7.9. Let $X \sim Beta(x, \alpha, \beta)$ $\sigma^2 = \dfrac{\alpha \cdot \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$

Exercise 9.7.10. Let $X \sim Exp(x, \theta)$ $\mu = \theta$

Exercise 9.7.11. Let $X \sim Exp(x, \theta)$ $\sigma^2 = \theta^2$

Exercise 9.7.12. Let $X \sim Exp(x, \theta)$ $\eta_3 = 2.$

Exercise 9.7.13. Let $X \sim Exp(x, \theta)$ The mode is 0.

Exercise 9.7.14. Let $X \sim Exp(x, \theta)$ The median is $\theta \cdot ln(2)$.

Exercise 9.7.15. Let $X \sim Erlang(x, \alpha, \beta)$ $\mu = \alpha \cdot \beta$

Exercise 9.7.16. Let $X \sim Erlang(x, \alpha, \beta)$ $\sigma^2 = \alpha \cdot \beta^2$

Exercise 9.7.17. Let $X \sim Chi(x, \nu)$ $\mu = \nu$

Exercise 9.7.18. Let $X \sim Chi(x, \nu)$ $\sigma^2 = 2\nu$

Exercise 9.7.19. Let $X \sim Chi(x, \nu)$ $\eta_3 = \frac{4}{\sqrt{2\nu}}$

Exercise 9.7.20. Let $X \sim Wei(x, \alpha, \beta)$ It wobbles but it doesn't fall down (this is non-board worthy).

Exercise 9.7.21. Let $X \sim Wei(x, \alpha, \beta)$ $\mu = \alpha \cdot \Gamma(1 + \frac{1}{\beta})$

Exercise 9.7.22. Let $X \sim Wei(x, \alpha, \beta)$ $\sigma^2 = \alpha^2 \cdot \left(\Gamma\left(1 + \frac{2}{\beta}\right) - \left(\Gamma\left(1 + \frac{1}{\beta}\right)\right)^2\right)$

Exercise 9.7.23. Let $X \sim Pareto(x, \alpha)$ $\mu = \frac{\alpha}{\alpha - 1}$ only if $\alpha > 1$.

Exercise 9.7.24. Let $X \sim Ray(x, \alpha)$ $\mu = \sqrt{\frac{\pi}{4\alpha}}$

Exercise 9.7.25. Let $X \sim Ray(x, \alpha)$ $\sigma^2 = \frac{1}{\alpha} - \frac{\pi}{4\alpha}$

Exercise 9.7.26. Let $X \sim Ray(x, \alpha)$ Let m_O be the mode . $m_O = \frac{1}{\sqrt{2\alpha}}$

Exercise 9.7.27. Let $X \sim Ray(x, \alpha)$ Let m_d be the median. $m_d = \sqrt{\frac{\log 4}{2\alpha}}$

Exercise 9.7.28. Let $X \sim Stu(x, \alpha)$ $\mu = 0$ when $\nu > 1$

Exercise 9.7.29. Let $X \sim Stu(x, \alpha)$ $\sigma^2 = \frac{\nu}{\nu - 2}$ when $\nu > 2$.

Exercise 9.7.30. Let $X \sim Stu(x, \alpha)$ $\eta_3 = 0$ when $\nu > 3$.

Exercise 9.7.31. Let $X \sim Stu(x, \alpha)$ $\eta_4 = \frac{3(\nu - 2)}{\nu - 4}$ when $\nu > 4$.

Exercise 9.7.32. Let $X \sim Fish(x, \nu, \omega)$ $\mu = \frac{\omega}{\omega - 2}$ when $\omega > 2$

Exercise 9.7.33. Let $X \sim Fish(x, \nu, \omega)$ $\sigma^2 = \frac{2\omega^2(\nu + \omega - 2)}{\nu(\omega - 2)^2(\omega - 4)}$ when $\omega > 4$

Exercise 9.7.34. Let $X \sim InvGauss(x, \mu, \lambda)$ $\mu_1' = \mu$

Exercise 9.7.35. Let $X \sim InvGauss(x, \mu, \lambda)$ $\sigma^2 = \frac{\mu^3}{\lambda}$

Exercise 9.7.36. Let $X \sim Cauchy(x, \alpha, \beta)$ μ does not exist.

Exercise 9.7.37. $Gamma(x, 1, \theta) \equiv Exp(x, \theta)$

Exercise 9.7.38. $Gamma(x, 1/2, 2) \equiv Chi(x, 1)$

Exercise 9.7.39. $Gamma(x, \nu/2, 2) \equiv Chi(x, \nu)$

Exercise 9.7.40. $Wei(x, \theta, 1) \equiv Exp(x, \theta)$

Exercise 9.7.41. $Beta(x, \alpha, \beta) \equiv Beta(1 - x, \beta, \alpha)$

Exercise 9.7.42. $Beta(x, \alpha, \beta)$ is symmetric if $\alpha = \beta$

Exercise 9.7.43. $\exists$ a well defined continuous random variable such that it has the memoryless property (meaning X is a well defined random variable with a probability density function of f and $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $\beta > 0$
$Pr((X > \alpha + \beta)\,|(X > \alpha)) = Pr(X > \beta)\,)$

Exercise 9.7.44. $\exists$ a well defined discrete random variable such that it has the memoryless property (meaning X is a well defined random variable with a probability density function of f and $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $\beta > 0$
$Pr((X > \alpha + \beta)\,|(X > \alpha)) = Pr(X > \beta)\,)$

Special thanks to Mr. Albert Pegus for his dedicated and able proofing of drafts of this text and Mr. Hasani M^c^Honey for detecting some additional errors that needed to be addressed.
Last revised 27 Mar. 2017