Chapter 7

Random Variables: Discrete

Note: The basic building block of probability is set theory: Suppose we have a well defined sample space S and events E_1, E_2, etc. (like when we talk about a well defined universe U and sets X_1, X_2, etc.), yada, yada, yada. The basic ideas are grounded in the sets!

Moreover, much of the really powerful theorems and concepts are grounded in Set Theory, Real Analysis, and Topology. However, we have not studied Real Analysis to the extent (e.g.: no one has completed Math 351, 352 and I doubt many of us have completed Math 430) that we can ground fully all of the discussion of continuous random variables.

Suffice it to say, we considered cases of random variables such that, for example:

7.1 Discrete Random Variables

Definition 7.1.1. If $\exists$ a non-negative function, f, defined from the domain[1] $(-\infty, \infty)$ to the codomain $(-\infty, \infty)$ meaning $f : \mathbb{R} \longrightarrow \mathbb{R}$ and if $\exists$ a set A which is the set of all $x \in dom(f)$ where $Pr(X = x) > 0$ and that A has Lesbegue measure 0 [2], then X is said to be a **discrete random variable**.

Definition 7.1.2. If X is a discrete random variable, the function given $f(x) = Pr(X = x)$ for each x in the domain of the function is called the **probability mass function (p. m. f.).**

Note 7.1.1. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the probability mass function and* A *is the set of all* $x \in dom(f)$ *where* $Pr(X = x) > 0$*. One can consider the function* $f|_A : A \longrightarrow \mathbb{R}$ *as the probability mass function since* $\forall x \in A^C \quad Pr(X = x) = 0$*. So, we have a condition that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is considered as*

$$f(x) = \begin{cases} Pr(X = x) & x \in A \\ 0 & x \notin A \end{cases}$$

whilst it can also be considered as $f : A \longrightarrow \mathbb{R}$ and all points of $\mathbb{R}$ not in the set A are not even mentioned. So, without loss of generality we will say many times $f : A \longrightarrow \mathbb{R}$ is the probability

[1] Note: Most statistics texts call the domain of $f(x)$ *the range... don't ask me why, they just do; however, we will use the correct terminology in this class.*

[2] Really all we need is for A to be finite or denumerable for our purposes. Therefore, $|A| \leqslant \aleph_0$. Recall $|A| < \aleph_0$ means A is finite and $|A| = \aleph_0$ means A is denumerable. So, $|A| \leqslant \aleph_0$ means A is countable.

mass function and A is the set $dom(f)$ so $Pr(X = x) > 0$ for all $x \in A$ where $A = dom(f)$ and not mention all the reals where $Pr(X = x) = 0$.

Note 7.1.2. *Let X be a discrete random variable with probability mass function $f(x)$. We write $X \sim f(x)$ to denote that X is 'distributed with a p.m.f. that is $f(x)$.'*

Theorem 7.1.1. *A function serves as a p.m.f. of a discrete random variable iff its values, $f(x)$, satisfy:*

1. $f(x) \geq 0 \quad \forall x \in dom(f)$
2. $\sum\limits_x f(x) = 1$.

Definition 7.1.3. If X is a discrete random variable, the function given by $F(x) = Pr(X \leqslant x)$ for each x in the domain of the function is called the **probability distribution function** or **cumulative distribution function (c. d. f.)**

$$F(x) = Pr(X \leq x) = \sum_{t \leq x} f(t) \quad \forall x \in (-\infty, \infty)$$

Theorem 7.1.2. *A function serves as a c. d. f. of a discrete random variable iff its values $F(x)$ satisfy:*

1. $F(x) \longrightarrow 0$ *as* $x \longrightarrow -\infty$ *or* $\exists \ x_1 \in \mathbb{R} \ni F(x_1) = 0 \quad (\wedge \ F(x) = 0 \ \ \forall x < x_1)$
2. $F(x) \longrightarrow 1$ *as* $x \longrightarrow \infty$ *or* $\exists \ x_2 \in \mathbb{R} \ni F(x_2) = 1 \quad (\wedge \ F(x) = 1 \ \ \forall x > x_2)$
3. $F(x)$ *is non-decreasing* $\forall x \in \mathbb{R}$.

Definition 7.1.4. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the **expected value (or mean)** of X is $E[X] = \sum\limits_x (x \cdot f(x))$.

Notation 7.1.1. $E[X] = \mu_X = \mu = \mu_1'$.

Definition 7.1.5. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then the r^{th} moment about the origin of X is $E[X^r] = \sum\limits_x (x^r \cdot f(x))$.

Notation 7.1.2. $E[X^r] = \mu_r'$.

Definition 7.1.6. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then **the variance (or second moment about the mean) of X** is $Var[X] = \sum\limits_x \left((x - E[X])^2 \cdot f(x)\right)$.

Notation 7.1.3. $Var[X] = E[(X - \mu)^2] = \mu_2 = \sigma^2 = \sigma_X^2$.

Definition 7.1.7. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then **the standard deviation of X** is $\sqrt{Var[X]} = \sqrt{\left(\sum\limits_x ((x - E[X])^2 \cdot f(x))\right)}$.

Notation 7.1.4. $SD[X] = \sqrt{\left(E[(X-\mu)^2]\right)} = \sigma = \sigma_X$.

Definition 7.1.8. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p. m. f. at x, then **the $\mathbf{r^{th}}$ moment about the mean of X is** $E[(X-\mu)^r] = \sum_x ((x - E[X])^r \cdot f(x))$**.**

Notation 7.1.5. $E[(X-\mu)^r] = \mu_r$.

Theorem 7.1.3. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x, then $Var[X] = \mu_2' - \mu^2$.*

Definition 7.1.9. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x, then **the coefficient of skewness**, η_3, is $\dfrac{\mu_3}{\sigma^3} = \dfrac{\mu_3}{\left(\sqrt{\mu_2}\right)^3}$.

Definition 7.1.10. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x, then **the coefficient of kurtosis**, η_4, is $\dfrac{\mu_4}{\sigma^4} = \dfrac{\mu_4}{(\mu_2)^2}$.

Theorem 7.1.4. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and $g(X)$ is a function of X , then the expected value (or mean) of $g(x)$ is $E[g(X)] = \sum_x g(X) \cdot f(x)$.*

Theorem 7.1.5. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and T is a linear transformation of X, meaning $T = \alpha \cdot X + \beta$ such that $\alpha \in \mathbb{R} \wedge \beta \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\alpha \cdot X + \beta] = \alpha \cdot E[X] + \beta$.*

Corollary 7.1.1. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and T is a zero linear transformation of X, meaning $T = \alpha \cdot X$ such that $\alpha \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\alpha \cdot X] = \alpha \cdot E[X]$.*

Corollary 7.1.2. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and T is a constant transformation of X, meaning $T = \beta$ such that $\beta \in \mathbb{R}$, then the expected value (or mean) of T is $E[T] = E[\beta] = \beta$.*

Theorem 7.1.6. *If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and T is a linear transformation of X, meaning $T = \alpha \cdot X + \beta$ such that $\alpha \in \mathbb{R} \wedge \beta \in \mathbb{R}$, then the variance of T is $Var[T] = E[(T - \mu_T)^2] = var[\alpha \cdot X + \beta] = \alpha^2 \cdot Var[X]$.*

Theorem 7.1.7. *(Chebyshev's Theorem or Tchebyshev's Theorem) If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x and μ and σ are the mean and standard deviation of X ($\sigma \neq 0$), then for any $k > 0 \quad (k \in \mathbb{R})$ the probability is at least $1 - \dfrac{1}{k^2}$ that X will take on a value within k standard deviations of the mean.*

The meaning of Tchebyshev's Theorem is that when we have a well defined discrete random variable, X, with fixed μ and σ ($\sigma \neq 0$), then $Pr\left(|X - \mu| < k \cdot \sigma\right) \geq 1 - \dfrac{1}{k^2}$.

7.2 Exercises: Theoretical

All problems in this section are board-worthy.

Exercise 7.2.1. Prove Theorem 7.1.3.

Exercise 7.2.2. Prove Theorem 7.1.5.

Exercise 7.2.3. Prove Theorem 7.1.6.

Exercise 7.2.4. Prove Theorem 7.1.7.

7.3 Exercises: Computational

Problems 3, 4, 5, and 7 in this section are board-worthy. If an answer DNE, explain why it does not exist.

Exercise 7.3.1. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls. One draws two balls from the urn. Let L be the random variable defined by the number of blue balls drawn from the urn. Find the probability mass function associated with L.

Exercise 7.3.2. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls. One draws three balls from the urn. Let B be the random variable defined by the number of blue balls drawn from the urn. Find the probability mass function associated with B.

Exercise 7.3.3. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls. One draws three balls from the urn. Let B be the random variable defined by the number of blue balls drawn from the urn. Find the expected value of B, the variance, coefficient of skewness, and coefficient of kurtosis.

Exercise 7.3.4. Suppose we have the following random variable X defined by the PMF

$X = x$	0	1	2	3	5	else
Pr(X = x)	0.25	0.1	0.05	0.4	0.2	0

where $f : \mathbb{R} \longrightarrow \mathbb{R}$

A. Graph the PMF. B. Find $Pr(X > 0.05)$ C. Draw the histogramme associated with the PMF
D. Find $Pr(X = 3)$ E. Find $Pr(X = 4)$ F. Find $Pr(|X - 2| > 1)$
G. Find $Pr(X > 1)$ H Find the CDF I. Graph the CDF.
J. Find μ. K. Find η_3. L. Find η_4.

Exercise 7.3.5. Consider the following function:

$$g(x) = Pr(X = x) = \begin{cases} \binom{6}{x} \cdot \left(\frac{1}{5}\right)^x \cdot \left(\frac{4}{5}\right)^{(6-x)} & x \in \mathbb{N}_6^* \\ 0 & else \end{cases}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ Prove the function is a well defined probability mass function.

Exercise 7.3.6. A pair of fair dice is tossed. Record the sum of the faces of the dice that are faced up. Call the variable X. Find the probability mass function associated with X

A. Find $Pr(X > 5)$ B. Find $Pr(X = 3)$ C. Find $Pr(X = 4)$
D. Find $Pr(X \geq 8)$ E. Find $Pr(X < 3)$ F. Find $Pr(X \geq 4)$
G. Find $Pr(X \leq 1)$ H. Find $Pr(X < -1)$ I. Find $Pr(X \geq \pi)$

Exercise 7.3.7. A pair of fair dice is tossed. Record the absolute value of the difference of the faces of the dice that are faced up. Call the variable W. Find the probability mass function associated with W and the cumulative distribution function associated with W.

A. Find $Pr(W > 5)$	B. Find $Pr(W = 3)$	C. Find $Pr(W = 4)$
D. Find $Pr(W \geq 8)$	E. Find $Pr(W < 3)$	F. Find $Pr(W \geq 4)$
G. Find $Pr(W \leq 1)$	H. Find $Pr(W < -1)$	I. Find $Pr(W \geq \pi)$

Exercise 7.3.8. Suppose we have the following random variable X defined by the PMF

$X = x$	-2	0	1	3	5	else
Pr(X = x)	0.2	0.1	0.25	0.4	0.05	0

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ (meaning the PMF is f).

A. Find $Pr(X = 3)$	B. Find $Pr(X \leq 2.05)$	C. Find $Pr(X > 2.05)$
D. Find $Pr(X > 1)$	E. Find $Pr(X = 4)$	F. Find $Pr(\lvert X - 1\rvert > 2)$
G. Find $Pr(X \leq 1)$	H Find the CDF.	I. Graph the CDF.

J. Define μ as $\sum_{x \in D}(x \cdot f(x))$ where $D = \{x|f(x) > 0\}$. Find μ.
K. Define d as $d \in \mathbb{R} \ni Pr(X \leq d) = Pr(X \geq d) = \frac{1}{2}$ Find d.
L. Define m as $m \in \mathbb{R} \ni Pr(X = m) \geq Pr(X = q)$
$\forall\ q \in \mathbb{R}\ \wedge\ \exists\ p \in \mathbb{R} \ni Pr(X = m) > Pr(X = p)$. Find m.

Exercise 7.3.9. Consider the following function:

$$g(x) = Pr(X = x) = \begin{cases} \binom{5}{x} \cdot (\frac{1}{2})^x \cdot (\frac{1}{2})^{(5-x)} & x \in \mathbb{N}_5^* \\ 0 & else \end{cases}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$

A Draw the histogramme for the PMF
B. Find the CDF. C. Graph the CDF.
D. Define μ as $\displaystyle\sum_{x \in D}(x \cdot g(x))$ where $D = \{x|g(x) > 0\}$. Find μ.
E. Define d as $d \in \mathbb{R} \ni Pr(X \leq d) = Pr(X \geq d) = \frac{1}{2}$ Find d.
F. Define m as $m \in \mathbb{R} \ni Pr(X = m) \geq Pr(X = q)$
$\forall\ q \in \mathbb{R}\ \wedge\ \exists\ p \in \mathbb{R} \ni Pr(X = m) > Pr(X = p)$. Find m.
G. Define σ^2 as $\displaystyle\sum_{x \in D}((x - \mu)^2 \cdot g(x))$ where $D = \{x|g(x) > 0\}$. Find σ^2.

7.4 Important Discrete Probability Mass Functions and Moments

Definition 7.4.1. If X is a discrete random variable and the function given by $f(x) = Pr(X = x)$ for each x in the domain of the function is the p.m.f. at x, and there is one (or more) real numbers that are fixed which define a particular case of the discrete random variable we call that constant **a parametre** of the p.m.f.

1. The **Bernoulli** random variable (or sometimes we say trial) is a probabilistic (or stochastic) experiment that can have one of two outcomes, success ($X = 1$) or failure ($X = 0$) in which the probability of success is p. The parametre is p. We call the variable X a Bernoulli (discrete) random variable. $p \in (0, 1)$[3] and $X \in \{0, 1\}$

$$Ber(x,p) = Pr(X = x) = \begin{cases} p, & x = 1 \\ 1-p & x = 0 \\ 0 & else \end{cases}$$

Claims about the moments of a Bernoulli random variable[4]

1. $\mu = p$
2. $\mu'_r = p \quad \forall r \in \mathbb{N}$
3. $\sigma^2 = p \cdot (1 - p)$

2. The **Binomial** random variable or variate) is the number (x) of successes in n-independent finitely many Bernoulli trials where the probability of success at each trial is p. The parametres are p and n (the number of trials).We call the variable X a binomial random variable. $p \in (0, 1)$[5] and $n \in \mathbb{N}$. $x \in \mathbb{N}^*_n$ meaning, of course, $x \in \{0, 1, 2, ..., (n - 1), n\}$

$$Bin(x,p,n) = Pr(X = x) = \begin{cases} \left(\begin{array}{c} n \\ x \end{array} \right) \cdot p^x \cdot (1-p)^{(n-x)} & x = 0, 1, 2, ..., n \\ 0 & else \end{cases}$$

Claims about the moments of a Binomial random variable[6]

1. $\mu = n \cdot p$
2. $\mu'_2 = n \cdot p \cdot (n \cdot p + (1 - p))$
3. $\mu'_3 = n \cdot p \cdot \big((n - 1) \cdot (n - 2) \cdot p^2 + 3 \cdot p \cdot (n - 1) + 1\big)$
4. $\sigma^2 = n \cdot p \cdot (1 - p)$
5. $\mu_3 = n \cdot p \cdot (1 - p) \cdot ((1 - p) - p)$
6. $\mu_4 = n \cdot p \cdot ((1 + 3 \cdot p \cdot (1 - p) \cdot (n - 2))$

[3]It is really $p \in [0, 1]$ but the trivial cases are not interesting.
[4]Such need to proven; but, for computational problems you may assume said.
[5]Once again it is really $p \in [0, 1]$ but the trivial cases are not interesting.
[6]Such need to proven; but, for computational problems you may assume said.

3. The **Geometric** random variable is the trial number (x) at which the first success occurs in independent Bernoulli trials where the probability of success at each trial is p. The parametre is p. We call the variable X a geometric random variable. $p \in (0, 1)$[7] and $x \in \mathbb{N}$.

$$Geo(x, p) = Pr(X = x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, ... \\ 0 & else \end{cases}$$

Claims about the moments of a Geometric random variable[8]

1. $\mu = \dfrac{1}{p}$
2. $\sigma^2 = \dfrac{1-p}{p^2}$
3. $\mu_3 = (1-p) \cdot (2-p) \cdot \dfrac{1}{p^3}$
4. $\mu_4 = \dfrac{9 \cdot (1-p)^2}{p^4} + \dfrac{1-p}{p^2}$

4. The **Negative Binomial (or Pascal)** random variable or variate) is the trial number (x) at which the r^{th} success occurs in independent Bernoulli trials where the probability of success at each trial is p. The parametres are p and r. We call the variable X a negative binomial random variable or a Pascal variable. $p \in (0, 1)$[9] and $x \in \mathbb{N}$ whilst $r \in \mathbb{N}_x$ meaning that $r \leq x$.

$$NegBin(x, p, r) = Pr(X = x) = \begin{cases} \begin{pmatrix} x-1 \\ r-1 \end{pmatrix} \cdot p^r \cdot (1-p)^{(x-r)} & x = 1, 2, ... \quad \wedge \quad r \leqslant x \\ 0 & else \end{cases}$$

Claims about the moments of a Pascal random variable[10]note: in summation form for proofs the summation begins at r.

1. $\mu = \dfrac{r}{p}$
2. $\sigma^2 = \dfrac{r \cdot (1-p)}{p^2}$
3. $\mu_3 = r \cdot (1-p) \cdot (2-p) \cdot p^3$
4. $\mu_4 = \dfrac{r \cdot (1-p)}{p^4} \cdot \left(3 \cdot r \cdot p + 6 \cdot (1-p) + p^2\right)$

Theorem 7.4.1. *Let X be a discrete random variable $X \sim NegBin(x, p, 1)$. It is therefore the case that $X \sim Geo(x, p)$*

[7]Once again it is really $p \in [0, 1]$ but the trivial cases are not interesting.
[8]Such need to proven; but, for computational problems you may assume said.
[9]Boy is this footnote getting old - - $p \in [0, 1]$ but the trivial cases are not interesting.
[10]Such need to proven; but, for computational problems you may assume said.

5. The **Hypergeometric** random variable or variate) is the number of success balls (x) selected when n balls are randomly chosen from an urn that contains N balls with m success balls. The parametres are n, N and m.[11] We call the variable X a hypergeometric random variable.
$N \in \mathbb{N}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \quad \ni \quad m \leq M, n \leq N$.

$$HypGeo(x, N, n, m) = Pr(X = x) = \begin{cases} \dfrac{\begin{pmatrix} m \\ x \end{pmatrix}\begin{pmatrix} N-m \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} & x = 0, 1, 2, ..., \min\{m, n\} \\ 0 & else \end{cases}$$

Claims about the moments of a Hypergeometric random variable[12]:

1. $\mu = \dfrac{n \cdot m}{N}$

2. $\sigma^2 = \dfrac{N-n}{N-1} \cdot \dfrac{nm}{N} \cdot (1 - \dfrac{m}{N})$

3. $\mu_3 = \dfrac{\left(1 - \frac{2m}{N}\right) \cdot (N-n) \cdot (N-2n)}{(N-1)\,(N-2)}$

4. $$\mu_4 = \frac{\frac{nm}{N} \cdot (1 - \frac{m}{N}) \cdot (N-n)}{(N-1)(N-2)(N-3)} \cdot \left(N(N+1) - 6n(N-n) + \frac{3m}{N}(1 - \frac{m}{N})[n(N-n)(N+6) - 2N^2] \right)$$

6. The **Poisson** random variable or variate) is as follows. The parametre is λ where $\lambda \in (0, \infty)$. We call the variable X a Poisson random variable and $x \in \mathbb{N}^*$

$$Pois(x, \lambda) = Pr(X = x) = \begin{cases} \dfrac{e^{-\lambda} \cdot \lambda^x}{x!} & x = 0, 1, 2, ... \\ 0 & else \end{cases}$$

Claims about the moments of a Poisson random variable[13]:

1. $\mu = \lambda$
2. $\mu_2' = \lambda + \lambda^2$
3. $\mu_3' = \lambda \left((\lambda + 1)^2 + \lambda \right)$
4. $\mu_4' = \lambda \left(\lambda^3 + 6\lambda^2 + 7\lambda + 1 \right)$
5. $\sigma^2 = \lambda$
6. $\mu_3 = \lambda$

[11] m can be bigger than n or n can be bigger than m or they can be the same - - - the only condition that is of import is that they *both must be less than or equal* to N.
[12] Such need to proven; but, for computational problems you may assume said.
[13] Such need to proven; but, for computational problems you may assume said.

7. $\mu_4 = \lambda \cdot (1 + 3\lambda)$
8. $\mu_5 = \lambda \cdot (1 + 10\lambda)$
9. $\mu_6 = \lambda \cdot \left(1 + 25\lambda + 15\lambda^2\right)$

Also, if a student is thinking about becoming an actuary or taking the actuarial exams the probability mass functions have to be memorized for the tests (the definitions are not provided); the claims of at least the mean and variance of each has to be memorized for the tests; and, have many of the theorems have to be memorized for the tests. I cannot say why I can only say it is so.

7.5 Exercises: Theoretical

All problems in this section are board-worthy.
Prove the claims about specific discrete random variable. Some of the claims are facile; others not. Some of the claims become repetitious; so, I may 'pull' some claims from the list of things to present because of the repetitiveness (but such can be submitted in writing for reduced credit – going to the board is a must in this class).

Exercise 7.5.1. Let $X \sim Ber(x, p)$ $\quad \mu = p$

Exercise 7.5.2. Let $X \sim Ber(x, p)$ $\quad \mu'_r = p \quad \forall r \in \mathbb{N}$

Exercise 7.5.3. Let $X \sim Ber(x, p)$ $\quad \sigma^2 = p \cdot (1 - p)$

Exercise 7.5.4. Let $X \sim Bin(x, p, n)$ $\quad \mu = n \cdot p$

Exercise 7.5.5. Let $X \sim Bin(x, p, n)$ $\quad \mu'_2 = n \cdot p \cdot (n \cdot p + (1 - p))$

Exercise 7.5.6. Let $X \sim Bin(x, p, n)$ $\quad \mu'_3 = n \cdot p \cdot \left((n - 1) \cdot (n - 2) \cdot p^2 + 3 \cdot p \cdot (n - 1) + 1\right)$

Exercise 7.5.7. Let $X \sim Bin(x, p, n)$ $\quad \sigma^2 = n \cdot p \cdot (1 - p)$

Exercise 7.5.8. Let $X \sim Bin(x, p, n)$ $\quad \mu_3 = n \cdot p \cdot (1 - p) \cdot ((1 - p) - p)$

Exercise 7.5.9. Let $X \sim Bin(x, p, n)$ $\quad \mu_4 = n \cdot p \cdot ((1 + 3 \cdot p \cdot (1 - p) \cdot (n - 2))$

Exercise 7.5.10. Let $X \sim Geo(x, p)$ $\quad \mu = \dfrac{1}{p}$

Exercise 7.5.11. Let $X \sim Geo(x, p)$ $\quad \sigma^2 = \dfrac{1 - p}{p^2}$

Exercise 7.5.12. Let $X \sim Geo(x, p)$ $\quad \mu_3 = (1 - p) \cdot (2 - p) \cdot \dfrac{1}{p^3}$

Exercise 7.5.13. Let $X \sim Geo(x, p)$ $\quad \mu_4 = \dfrac{9 \cdot (1 - p)^2}{p^4} + \dfrac{1 - p}{p^2}$

Exercise 7.5.14. Let $X \sim Pascal(x, p, r)$ $\quad \mu = \dfrac{r}{p}$

Exercise 7.5.15. Let $X \sim Pascal(x, p, r)$ $\quad \sigma^2 = \dfrac{r \cdot (1 - p)}{p^2}$

Exercise 7.5.16. Let $X \sim Pascal(x, p, r)$ $\quad \mu_3 = r \cdot (1-p) \cdot (2-p) \cdot p^3$

Exercise 7.5.17. Let $X \sim Pascal(x, p, r)$ $\quad \mu_4 = \dfrac{r \cdot (1-p)}{p^4} \cdot \left(3 \cdot r \cdot p + 6 \cdot (1-p) + p^2\right)$

Exercise 7.5.18. Let $X \sim HypGeo(x, N, n, m)$ $\quad \mu = \dfrac{n \cdot m}{N}$

Exercise 7.5.19. Let $X \sim HypGeo(x, N, n, m)$ $\quad \sigma^2 = \dfrac{N-n}{N-1} \cdot \dfrac{nm}{N} \cdot (1 - \dfrac{m}{N})$

Exercise 7.5.20. Let $X \sim HypGeo(x, N, n, m)$ $\quad \mu_3 = \dfrac{\left(1 - \frac{2m}{N}\right) \cdot (N-n) \cdot (N-2n)}{(N-1)\,(N-2)}$

Exercise 7.5.21. Let $X \sim HypGeo(x, N, n, m)$

$$\mu_4 = \frac{\frac{nm}{N} \cdot (1 - \frac{m}{N}) \cdot (N-n)}{(N-1)(N-2)(N-3)} \cdot \left(N(N+1) - 6n(N-n) + \frac{3m}{N}(1 - \frac{m}{N})[n(N-n)(N+6) - 2N^2]\right)$$

Exercise 7.5.22. Let $X \sim Pois(x, \lambda)$ $\quad \mu = \lambda$

Exercise 7.5.23. Let $X \sim Pois(x, \lambda)$ $\quad \mu_2' = \lambda + \lambda^2$

Exercise 7.5.24. Let $X \sim Pois(x, \lambda)$ $\quad \mu_3' = \lambda \left((\lambda+1)^2 + \lambda\right)$

Exercise 7.5.25. Let $X \sim Pois(x, \lambda)$ $\quad \mu_4' = \lambda \left(\lambda^3 + 6\lambda^2 + 7\lambda + 1\right)$

Exercise 7.5.26. Let $X \sim Pois(x, \lambda)$ $\quad \sigma^2 = \lambda$

Exercise 7.5.27. Let $X \sim Pois(x, \lambda)$ $\quad \mu_3 = \lambda$

Exercise 7.5.28. Let $X \sim Pois(x, \lambda)$ $\quad \mu_4 = \lambda \cdot (1 + 3\lambda)$

Exercise 7.5.29. Let $X \sim Pois(x, \lambda)$ $\quad \mu_5 = \lambda \cdot (1 + 10\lambda)$

Exercise 7.5.30. Let $X \sim Pois(x, \lambda)$ $\quad \mu_6 = \lambda \cdot \left(1 + 25\lambda + 15\lambda^2\right)$