Chapter 10

Random Variables: Moment Generating Functions

10.1 Introduction

If one knows what the probability mass function (PMF) is; probability density function (PDF) is; or, if one knows what the cumulative distribution function (CDF) is then one is completely versed in all aspects of the random variable.

The other type of function that if known to a person then the person is completely versed in all aspects of the random variable is the moment generating function (MGF) which we will now consider.

10.2 MGFs

Definition 10.2.1. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability mass function (PMF) of the discrete random variable X. The **moment generating function** (MGF) of X (where it exists) is defined where $\exists\ \varepsilon > 0 \ \ni \forall\ t \in (-\varepsilon, \varepsilon)$

$$M_X(t) = E[e^{tX}] = \sum_x (e^{tX} \cdot f_X(x))$$

$$M_X : \mathbb{R} \longrightarrow \mathbb{R}$$

Theorem 10.2.1. *Let X be a discrete random variable where $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}$, $\exists\ \varepsilon > 0 \ \ni \forall\ t \in (-\varepsilon, \varepsilon)$ $M_X(t)$ exists.*

1. $M_{(X+\alpha)}(t) = e^{(t\alpha)} \cdot M_X(t)$

2. $M_{(\beta X)}(t) = M_X(\beta t)$

3. $M_{(\frac{X+\alpha}{\beta})}(t) = e^{(\frac{\alpha t}{\beta})} \cdot M_X(\beta t)$

Definition 10.2.2. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability mass function (PMF) of the discrete random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X] = M'_X(t)$ at $t = 0$ (alternately $E[X] = \frac{d}{dt}(M_X(t))$ at $t = 0$ or $E[X] = M'_X(0)$).

Definition 10.2.3. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability mass function (PMF) of the discrete random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X^2] = M''_X(t)$ at $t = 0$ (alternately $E[X^2] = \frac{d^2}{d^2t}(M_X(t))$ at $t = 0$ or $E[X] = M''_X(0)$).

Definition 10.2.4. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability mass function (PMF) of the discrete random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X^n] = M_X^{(n)}(t)$ at $t = 0$ (alternately $E[X^n] = \frac{d^n}{d^nt}(M_X(t))$ at $t = 0$ or $E[X] = M_X^{(n)}(0)$).

Theorem 10.2.2. *Let X be a discrete random variable Then $\exists$ a unique function, $f_X(x)$, which is its probability mass function where such exists; $\exists$ a unique function, $F_X(x)$, which is its cumulative distribution function where such exists; and, $\exists$ a unique function, $M_X(t)$, which is its moment generating function where such exists.*

We claim there are some Special MGFs associated with discrete random variables:

Let $X \sim Bin(x, p, n)$ $\quad M_X(t) = ((1-p) + pe^t)^n$

Let $X \sim Pois(x, \lambda)$ $\quad M_X(t) = e^{(\lambda(e^t-1))}$

Let $X \sim Geo(x, p)$ $\quad M_X(t) = \dfrac{pe^t}{(1 - e^t(1-p))} \ni t < -ln(1-p)$.

10.3 Exercises: Moment Generating Functions

Exercise 10.3.1. Let $X \sim Pois(x, \lambda = 4)$.
A. Use the MGF for $X \sim Pois(x, 4)$ to find μ
B. Use the MGF for $X \sim Pois(x, 4)$ to find σ^2
C. Use the MGF for $X \sim Pois(x, 4)$ to find η_3

Exercise 10.3.2. Let $X \sim DisUni(x, \theta)$. $\theta \in \mathbb{N}$

$$DisUni(x,\theta) = f(x) = \begin{cases} \dfrac{1}{\theta}, & x \in \mathbb{N}_\theta \\ \\ 0 & else \end{cases} \qquad\qquad f : \mathbb{R} \longrightarrow \mathbb{R}$$

A. Create the moment generating function for X.
B. Use the MGF for $X \sim DisUni(x, \theta = 2)$ to find σ^2

10.4 More MGFs

Definition 10.4.1. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability density function (PDF) of the continuous random variable X. The **moment generating function** (MGF) of X (where it exists) is defined where $\exists\ \varepsilon > 0\ \ni\ \forall\ t \in (-\varepsilon, \varepsilon)$

$$M_X(t) = E[e^{tX}] = \int_x (e^{tX} \cdot f_X(x))dx$$

$$M_X : \mathbb{R} \longrightarrow \mathbb{R}$$

Theorem 10.4.1. *Let X be a continuous random variable where $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}$, $\exists\ \varepsilon > 0\ \ni\ \forall\ t \in (-\varepsilon, \varepsilon)$ $M_X(t)$ exists.*

1. $M_{(X+\alpha)}(t) = e^{(t\alpha)} \cdot M_X(t)$
2. $M_{(\beta X)}(t) = M_X(\beta t)$
3. $M_{(\frac{X+\alpha}{\beta})}(t) = e^{(\frac{\alpha t}{\beta})} \cdot M_X(\beta t)$

Definition 10.4.2. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability density function (PDF) of the continuous random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X] = M'_X(t)$ at $t = 0$ (alternately $E[X] = \frac{d}{dt}(M_X(t))$ at $t = 0$ or $E[X] = M'_X(0)$).

Definition 10.4.3. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability density function (PDF) of the continuous random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X^2] = M''_X(t)$ at $t = 0$ (alternately $E[X^2] = \frac{d^2}{d^2t}(M_X(t))$ at $t = 0$ or $E[X] = M''_X(0)$).

Definition 10.4.4. Let $X \sim f_X(x)$ where $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the probability density function (PDF) of the continuous random variable X and let $M_X(t)$ where $M_X : \mathbb{R} \longrightarrow \mathbb{R}$ be the moment generating function (MGF) of X. $E[X^n] = M_X^{(n)}(t)$ at $t = 0$ (alternately $E[X^n] = \frac{d^n}{d^n t}(M_X(t))$ at $t = 0$ or $E[X] = M_X^{(n)}(0)$).

Theorem 10.4.2. *Let X be a continuous random variable Then $\exists$ a unique function, $f_X(x)$, which is its probability density function where such exists; $\exists$ a unique function, $F_X(x)$, which is its cumulative distribution function where such exists; and, $\exists$ a unique function, $M_X(t)$, which is its moment generating function where such exists.*

We claim there are some Special MGFs associated with continuous random variables:

Let $X \sim Uni(x, \alpha, \beta)$	$M_X(t) = \dfrac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$
Let $X \sim Nor(x, \mu, \sigma)$	$M_X(t) = e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$
Let $X \sim \Gamma(x, \alpha, \beta)$	$M_X(t) = (1 - \beta t)^{-\alpha} \quad \ni \quad t < \dfrac{1}{\beta}$
Let $X \sim Exp(x, \theta)$	$M_X(t) = (1 - \theta t)^{-1} \quad \ni \quad t < \dfrac{1}{\theta}$
Let $X \sim Chi(x, \alpha, \nu)$	$M_X(t) = (1 - 2t)^{-\nu/2} \quad \ni \quad t < 2$

10.5 More Exercises: Moment Generating Functions

Exercise 10.5.1. Let $X \sim Exp(x, \theta)$.
A. Use the MGF for $X \sim Exp(x, \theta)$ to find μ
B. Use the MGF for $X \sim Exp(x, \theta)$ to find σ
C. Use the MGF for $X \sim Exp(x, \theta)$ to find η_3

Exercise 10.5.2. Let $X \sim Chi(x, \nu)$.
A. Use the MGF for $X \sim Chi(x, \nu)$ to find μ
B. Use the MGF for $X \sim Chi(x, \nu)$ to find σ

Exercise 10.5.3. Let $X \sim \Gamma(x, \alpha, \beta)$.
A. Use the MGF for $X \sim \Gamma(x, \alpha, \beta)$ to find μ
B. Use the MGF for $X \sim \Gamma(x, \alpha, \beta)$ to find σ