Class Notes

Math 301
Probability and Statistics I

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Chapter 1

Preliminaries

1.1 Introduction

Little (if any) lectures will take place.

The work done is by the students.

Prove means to rigorously prove a claim is so from the premises of the claim to the conclusion if the claim is a universal claim or produce an example to prove an existential claim is true Recall this also means that the claim is conditioned on a set of axioms. The axioms we assume for Math 301 are the axioms of Set Theory, the Peano axioms, the axioms of the Reals, and the Kolmorgorov axioms of Probability Theory (next chapter).

Show means to outline a proof of a claim – from the premises of the claim to the conclusion. Recall this also means that the claim is conditioned on a set of axioms but such might not be explicitly shown. Show is a weaker term than prove and some steps are usually skipped.

Demonstrate means to work out a problem as was done in Math 181 – 283. Demonstrate is a weaker term than show or prove, many steps are usually skipped, and it is assumed that the justification is inferred but not shown. However, Over my 30+ years of teaching and 50+ years of schooling, I have found it is advisable to write for myself next to that which I am demonstrating reasons (if not obvious).

Disprove means to rigorously produce a counterexample to a universal claim or prove the negation of an existential claim is true.

Comment means think about it and see if you can discern the validity or lack thereof for the claim. Do not bother to prove or disprove.

Notation is standard, unless otherwise noted. Go to my web-site for Math 224 and use those notes for background definitions, notation, etc.[1]

Some problems (marked exercises usually) are simple computations and unworthy of presentation at this level of college. Other problems are worthy of presentation due to the nature of the technique used, insight gained, or some other wonderful reason.

Most problems (marked claim or proposition usually) are not just simple computations but claims to prove or disprove; hence, worthy of presentation at this level of college. True claims (marked lemma, theorem, or corollary usually) are claims to prove; hence, most are worthy of presentation at this level of college.

We will use the $\mathbb{B}$ notation to mean the problem is board-worthy and the notation $\mathbb{D}$ to suggest difficulty level (based on my experience proving or disproving such along with my experience of watching students attempt to and in some cases actually prove or disprove said claims[2]).

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[1] Those who did not have 224 with me, use your book.

[2] Almost all the problems I proved under a Moore Method class or simply through interest. Others, I saw colleagues prove; and, in rare instances I read the proof in a book.

1.1.1 Dr. M^{c}Loughlin's Math 301 Board-work Guidelines

[3] Questions should satisfy two criteria. The first criterion is mathematical. Questions should be well-posed, clearly stated, and address one specific point. The second criterion is social. Questions should be politely presented and the speaker should be allowed time to respond before the poser or other members speak. Here are some guidelines that Dr. Ted Mahavier of Lamar University drew up which I think are quite helpful.[4]

Presenter Guidelines: DOs:

1. Before you offer to present an argument, go over every line of the proof and ask yourself: why do I think this line is correct? and what are they going to ask me about this line? Is everything in this line defined?
2. Before you offer to present an argument, read the argument aloud youll be surprised at what you catch when you slow down to read it to yourself.
3. As you present, offer the class the opportunity to address each statement or line by saying, is it clear that Ive defined a, b, and p at this point? Youll feel more confident with each correct line and youll get a feel for any confusion from the audience early in the presentation.

Presenter Guidelines: DONTs:

1. Ask vague questions like is everybody OK to here or challenging questions like does anybody have a problem with this?
2. Use words like obvious or trivial.
3. Dont use concepts or notation at the board that we have not defined unless you are prepared to define them.
4. Dont be upset when you make a mistake brush it off and learn from it.

Questioning a Presenter Guidelines: DOs:

1) First stop the speaker politely by either raising your hand to be recognized or by a phrase such as excuse me, or pardon me
2) Ask a specific question such as on the second line, you wrote, why is $Pr(E) >$? or if you cant be that specific perhaps can you say more about the second line?
3) Before you ask a question, ask yourself, Do I have a specific question? and Can I ask it politely? If the answer is no, dont speak.

Questioning a Presenter Guidelines: DONTs:

1) Make statements such as I dont think that is right because or I dont think that is a proof because Statements are not questions.
2) Suggest solutions or alternatives like couldnt we just define the event E so that This is not your proof and demonstrating to me or the class that you can solve it, once someone has put an idea into your head, is not relevant to the presentation.

[3] Adapted, with permission, from Dr. Ted Mahavier's (Lamar University) Guidelines
[4] Dr. McLoughlin Guidelines DOs: 1) Be vicious; awful; mean, and, confusing.
Dr. McLoughlin Guidelines DONTs: 1) Be nice; kind; or, encouraging.

1.2 Functions and Sequences

Let U be a well defined universe and W be a well defined universe.
For my sanity, let $U \neq \emptyset$ and $W \neq \emptyset$ Let $V = U \times W$.

Definition 1.2.1. Let $D \subseteq U$ and $C \subseteq W$.
A **relation** f from D to C is a subset of $D \times C$.

Definition 1.2.2. Let $D \subseteq U$ and $C \subseteq W$. Consider the relation $f \subseteq D \times C$.
f is a **function** from D to C *iff*:

1. $D \neq \emptyset$;
2. $dom(D) = cor(D)$[5]; and,
3. $(x, y) \in f \ \wedge \ (x, z) \in f \implies y = z$.

Definition 1.2.3. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **injective** *iff* $(x, y) \in f \ \wedge \ (w, y) \in f \implies x = w$.

Definition 1.2.4. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **surjective** *iff* $ran(f) = cod(f)$.

Definition 1.2.5. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **bijective** *iff* it is injective and surjective.

Theorem 1.2.1. *Let $D \subseteq U$ and $C \subseteq W$.*
Consider the bijective function, f, from D to C.
The relation f^{-1} from C to D is a bijective function also.

Definition 1.2.6. A function f for $D \subseteq U$, $C \subseteq W$, and $f \subseteq D \times C$ which satisfies the previous theorem is said to be an **invertible** function.

Definition 1.2.7. Let $V = U \times \mathbb{R}$. Let $D \subseteq U$ and $C \subseteq \mathbb{R}$.
Consider the well defined function $f \subseteq D \times C$. f is called a **real-valued function** simply because the codomain is a subset of $\mathbb{R}$.

Definition 1.2.8. Let $V = \mathbb{R} \times \mathbb{R}$. Let $D = \mathbb{N}$ and $C \subseteq \mathbb{R}$.
Consider the well defined function $f \subseteq D \times C$. f is called a **sequence** simply because the domain is $\mathbb{N}$.

1.3 Foundations and Calculus Pre-requisite Concepts

Recall:
We will let $U = \mathbb{R}$ for the line and let $U = \mathbb{R} \times \mathbb{R}$ for the plane.

Definition 1.3.1. $\mathbb{N}^* = \{0, 1, 2, 3, \ldots, (k-1), k, \ldots\}$

Definition 1.3.2. $\mathbb{N} = \{1, 2, 3, \ldots, (k-1), k, \ldots\}$

[5] The corange has to be “completely used.”

Definition 1.3.3. $\mathbb{N}_k^* = \{0, 1, 2, 3, \ldots, (k-1), k\}$

Definition 1.3.4. $\mathbb{N}_k = \{1, 2, 3, \ldots, (k-1), k\}$

Definition 1.3.5. Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that $a \leq b$:
a **segment** is (a, b)
an **interval** is $[a, b]$.
a **half-segment or half-interval** is $(a, b]$ or $[a, b)$.

Definition 1.3.6. Let $a \in \mathbb{R}$. $\emptyset$ is (a, a)

Definition 1.3.7. Let $a \in \mathbb{R}$. A singleton point-set is $[a, a] = \{a\}$

Assignment Due Mon., 23 Jan. 2017[6]

Exercise 1.3.1. (Math 181). Consider $k : \mathbb{R} \longrightarrow \mathbb{R} \ni k(x) = 5 \cdot x^2 \cdot \arctan x$ which is a well defined function. Find $k'(x)$

Exercise 1.3.2. $\mathbb{B}$. (Math 181). Let $h : \mathbb{R} \longrightarrow \mathbb{R} \ni h(x) = \displaystyle\int_{-\infty}^{x} e^{(t^2)} dt$ is a well defined function and find $\frac{dh}{dx}$

Exercise 1.3.3. (Math 181). By definition of the area between curves (Riemann Sums) show that the region, R, defined as the area between the curves
$f : \mathbb{R} \longrightarrow \mathbb{R} \quad \ni \quad f(x) = x^2, y = 0, x = -1, x = 2$ has the property that said area exists (and is therefore obviously finite). Find the area of R.

Exercise 1.3.4. (Math 181) Find the absolute maxima, minima, relative maxima, minima for $f(x) = (3x - 5)^2 \cdot (2x + 5)^3$ and $f : [-2, 5] \longrightarrow \mathbb{R}$.

Exercise 1.3.5. (Math 182). By definite integration find the area of the region K where K is defined as the area between the curves
$f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : [0, \infty) \longrightarrow \mathbb{R} \quad \ni \quad f(x) = x^2, g(x) = \sqrt{x}, x = 0, x = 4$. Note K has the property that said area exists (and is therefore obviously finite).

Exercise 1.3.6. (Math 182). By definite integration find the area of the region M where M is defined as the area between the curves $f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : [0, \infty) \longrightarrow \mathbb{R} \ \ni \ f(x) = \frac{1}{x^3},\ g(x) = 0$ to the right of the line $x = 2$ never-ending (on and on to the right). Note M has the property that said area exists (and is therefore obviously finite).

Exercise 1.3.7. (Math 182). Explain why the area of the region L where L is defined as the area between the curves $f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : [0, \infty) \longrightarrow \mathbb{R} \ \ni \ f(x) = \frac{1}{x},\ g(x) = 0$ to the right of the line $x = 2$ never-ending (on and on to the right) does not exist (is not a real number obviously).

Exercise 1.3.8. (Math 182). By the integral test show that $\displaystyle\sum_{i=1}^{\infty} i^{-3}$ exists (and is therefore obviously finite).

[6] These exercises touch upon concepts that will be used during the semester and, therefore, might be of use. However, of what use shall not be divulged – it is your to discover during the course - it could be that how I conceive to 'how to do a proof' is not the only way; so, think about each claim without interference. You may do something in a way no one has ever previously thought to do it (and be correct (even maybe better)).

Exercise 1.3.9. (Math 182). By the integral test show that $\sum_{i=1}^{\infty} i^{-\frac{3}{4}}$ does not exist (and is therefore obviously a divergent series).

Exercise 1.3.10. (Math 224) Prove or disprove the following claim.
Claim: Let $U = \mathbb{R}$. $\forall n \in \mathbb{N}$ it is the case that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Exercise 1.3.11. $\mathbb{B}$ (Math 224) Prove or disprove the following claim.
Claim: Let U be a well defined universe. Let A and B be sets. $A = (A \cap B) \cup (A \cap B^C)$.

Exercise 1.3.12. $\mathbb{B}$ (Math 224) Prove or disprove the following claim.
Claim: Let U be a well defined universe. Let A, B, and C be sets. If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Exercise 1.3.13. $\mathbb{B}$ (Math 224) Prove or disprove the following claim.
Claim: Let U be a well defined universe. Let A, B, and C be sets. If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Exercise 1.3.14. (Math 224) Prove or disprove the following claim.
Claim: Let U be a well defined universe, sets A, B, and C non-empty subsets of that universe, and $A \cap C \neq \emptyset$. It is the case that $(A \cap C \subseteq C - B)$ and $(A \cap B \subseteq C)$ if and only if $(A \cap B = \emptyset)$

Exercise 1.3.15. $\mathbb{B}$ (Math 224) Prove or disprove the following claim.
$h : \mathbb{N} \longrightarrow \mathbb{R} \ni h(x) = 3 \cdot x + 5$.
A. Claim: h is a well defined function.
B. Claim: h is injective from $\mathbb{N}$ to $\mathbb{R}$.
C. Claim: h is surjective from $\mathbb{N}$ to $\mathbb{R}$.

Exercise 1.3.16. (Math 181) Consider the following claim.
$h : \mathbb{N} \longrightarrow \mathbb{R} \ni h(x) = 3 \cdot x + 5$.
$h'(x) = 3$. Comment[7] please on this claim.

Exercise 1.3.17. (Math 224) Prove or disprove the following claim.
$|\mathbb{N}^*| = \aleph_0$.

Exercise 1.3.18. (Math 283). Consider the solid formed from the x-y plane region restricted as the area between the curves $f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = x^2,\ g(x) = 4$ and then above the x-y plane defined by the function $w : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \ni w((x, y)) = (4y^3 \cdot x^2)$ and call it W. Find the volume of W.

Exercise 1.3.19. (Math 182). Recall the Maclaurin series for $\ell : \mathbb{R} \longrightarrow \mathbb{R} \ni \ell(x) = e^x$. What is its interval (or segment or half interval) of convergence?

Exercise 1.3.20. (Math 182). Find the Maclaurin series for $g : \mathbb{R} \longrightarrow \mathbb{R} \ni g(x) = e^{-x}$. What is its interval (or segment or half interval) of convergence?

Exercise 1.3.21. (Math 182). Find the area of the region L where L is defined as the region between the curves $j : \mathbb{R} \longrightarrow \mathbb{R} \wedge k : \mathbb{R} \longrightarrow \mathbb{R} \ni j(x) = e^{2x},\ k(x) = 0, x = 0, x = 3$.

[7] Comment means think about it and see if you can discern the validity or lack thereof for the claim. Do not bother to prove or disprove.

Exercise 1.3.22. $\mathbb{B}$ (Math 182). Note the area of the region M where M is defined as the region between the curves $f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2},\ g(x) = 0, x = 0, x = 3$ exists but is non-computable (meaning we cannot get the real number that is the value of the area (in units squared)).

1. Find a Taylor series (show at least the first 5 non-zero terms) for $f : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2}$.

2. Find the Taylor polynomial centred at $x = 0$ of degree 10 that approximates $f : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2}$.

3. Find the Maclaurin polynomial of degree 6 that approximates $f : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2}$.

4. Find the degree of a Maclaurin polynomial that is used to approximate, with $f : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2}$, $f(3) = e^9$ such that the error is less than 0.00001.

Exercise 1.3.23. (Math 182). Approximate the area of the region M where M is defined as the region between the curves $f : \mathbb{R} \longrightarrow \mathbb{R} \wedge g : \mathbb{R} \longrightarrow \mathbb{R} \ni f(x) = e^{x^2},\ g(x) = 0, x = 0, x = 3$ such that the approximation is within 0.00001 of the actual area.

Welcome back! Get started on some authentically interesting material - math.

Chapter 2

Factoral & The Gamma Function

2.1 Factoral

Before discussing the gamma function, let us consider that which we are are all familiar with – the following sequence, f, $f(y) : \mathbb{N}^* \longrightarrow \mathbb{R} \ni f(y) = \prod_{j=1}^{y}(j), y \geq 1$ and $f(0) = 1$ which is a well defined function over the cardinal naturals and is called the factoral sequence or simply factoral. Many just think of it as (in notational form):
$0! = 1, \quad 1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24, \ldots,$
$n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1) \cdot n$

So, compute $5!$, $6!$, & $7!$. Note the inductive pattern formed.
Also note each factoral is the number of ways to line up that number of elements, so, for example, three factoral is understood as take three distinct things and line all of them up. This can be done six ways (I opine $0!$ is best understood as the number of ways to line nothing up; so, it is 1).
Let $U = \mathbb{R}$. Let $A = \{1, 4, 11\}$. The number of ways to line up these elements is:
1, 4, 11   1, 11, 4   4, 1, 11   4, 11, 1   11, 4, 1   and 11, 1, 4.

Claim 2.1.1. Let $n \in \mathbb{N}^*$. It is the case that $n! = n \cdot (n-1)!$ Prove or disprove the claim.

Claim 2.1.2. Let $n \in \mathbb{N}$. It is the case that $n! = n \cdot (n-1)!$ Prove or disprove the claim.

Claim 2.1.3. Let $n, m \in \mathbb{N}^*$. It is the case that $(n+m)! = n! + m!$ Prove or disprove the claim.

Claim 2.1.4. Let $n \in \mathbb{N}$ such that $n \geq 4$. It is the case that $2^n \leq n!$ Prove or disprove the claim.

2.2 The Gamma Function

We will look at the Gamma Function. The source for this discussion is *Einführung in die Theorie Gammafunktion,'* 1931, Artin, E. (translation by Butler, M.)[1]

Recall (see your Calculus II book) The gamma function is defined as
$\Gamma : (0, \infty) \longrightarrow \mathbb{R} \ni \Gamma(x) = \int_0^\infty e^{-t}(t^{(x-1)})dt$
So, $\Gamma : (0, \infty) \longrightarrow \mathbb{R}$ well defined function over the positive reals.

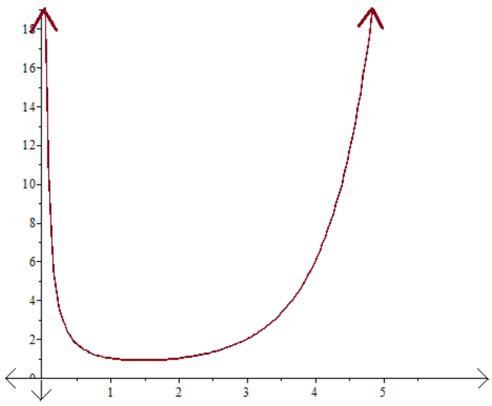

Figure 2.1: Γ, The Gamma Function

Exercises:[2]

Exercise 2.2.1. Find $\Gamma(1)$ and $\Gamma(2)$.

Exercise 2.2.2. Find $\Gamma(3)$

Exercise 2.2.3. Find $\Gamma(4)$

Exercise 2.2.4. Show for any $y \ni y \in (1, \infty)$ It is the case that $\Gamma(y) = (y - 1) \cdot \Gamma(y - 1)$.

Corollary 2.2.1. *For any $y \ni y \in (0, \infty)$ It is the case that $\Gamma(y + 1) = y \cdot \Gamma(y)$.*

Exercise 2.2.5. Show for any $y \in \mathbb{N}$ It is the case that $\Gamma(y) = (y - 1)!$

Theorem 2.2.1. *Let $U = \mathbb{R} \times \mathbb{R}$. It is the case that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$*

According to Artin, it (the Gamma function) was developed as a problem of generalising the factoral function.

[1] Whilst my mother taught us to speak a tad of German to our dog, I am NOT, obviously, fluent in German. Hence, the translation.

[2] See any basic text for a further discussion of factoral.

In order to prove the theorem, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, consider:

Definition 2.2.1. Let $B(x,y)$ be a function of x and y defined as $B : (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$ such that
$B(x,y) = \int_0^1 (t^{x-1}(1-t)^{y-1})dt$ which exists $\forall x \in (0,\infty) \wedge y \in (0,\infty)$

So, according to Artin:
Outline of Proof (OOPf) of the Theorem:
Assume the premises (ATP).

$$\begin{aligned} B(x,y) &= \int_0^{1/2} (t^{x-1}(1-t)^{y-1})dt + \int_{1/2}^1 (t^{x-1}(1-t)^{y-1})dt \\ &< (t^{x-1}(1-t)^{-1}) + (t^{-1}(1-t)^{y-1}) \\ &< 2(t^{x-1}) + 2((1-t)^{y-1}) \end{aligned}$$

Then note:

$$B(x+1,y) = \int_0^1 (t^{x+y-1}(\frac{t}{1-t})^x)dt$$

Integrate by parts (IBP) to get

$$\int_0^1 (t^{x+y-1}(\frac{t}{1-t})^x)dt = \int_\varepsilon^{1-\delta} (t^{x+y-1}(\frac{t}{1-t})^x)dt =$$

$$-(\frac{(1-t)^{x+y}}{x+y}(\frac{t}{1-t})^x)_\varepsilon^{1-\delta} + \int_\varepsilon^{1-\delta} (\frac{x}{x+y})(1-t)^{x+y}(\frac{t}{1-t})^{x-1}(\frac{1}{(1-t)^2})dt =$$

$$\frac{(1-\varepsilon)^y\varepsilon^x - \delta^y(1-\delta)^x}{x+y} + (\frac{x}{x+y})\int_\varepsilon^{1-\delta} (1-t)^{x+y}(\frac{t}{1-t})^{x-1}(\frac{1}{(1-t)^2})dt$$

Let $\varepsilon \longrightarrow 0$ and $\delta \longrightarrow 0$, thus,

$$B(x+1,y) = \frac{x}{x+y}B(x,y)$$

Hold y fixed and consider

$$\int_0^1 (t^{x-1}(1-t)^{y-1})dt$$

as a function of x.
Define

$$f(x) = B(x,y) \cdot \Gamma(x+y)$$

Therefore

$$B(1,y) = \int_0^1 ((1-t)^{y-1})dt = \frac{1}{y}$$

Thus,

$$f(1) = \frac{1}{y} \cdot \Gamma(1+y) = \Gamma(y)$$

Hence,

$$f(x) = \Gamma(x) \cdot \Gamma(y)$$

So,

$$\frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (t^{x-1}(1-t)^{y-1})dt$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{2}$

$$\frac{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(1)} = \int_0^1 (t^{-1/2}(1-t)^{-1/2})dt$$

but $\Gamma(1) = 1$ so we were dividing by zero; thus,

$$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \int_0^1 (t^{-1/2}(1-t)^{-1/2})dt$$

$$\left(\Gamma(\frac{1}{2})\right)^2 = \int_0^1 (t^{-1/2}(1-t)^{-1/2})dt$$

Let use change variables with the transformation $t = \sin^2(\theta)$
Thus,

$$\left(\Gamma(\frac{1}{2})\right)^2 = \int_0^{\pi/2} d\theta = \pi$$

Hence, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

QEDish (End of Outline of Proof)

Exercises:

Exercise 2.2.6. Find $\Gamma(\frac{3}{2})$.

Exercise 2.2.7. Find $\Gamma(\frac{5}{2})$.

Exercise 2.2.8. Find $\Gamma(\frac{7}{2})$.

Recall **the** gamma function is defined as $y \in (0,\infty)$ $\Gamma(y) = \displaystyle\int_0^{\infty} e^{-x}(x^{y-1})dx$; can we extend it?

The answer is in the affirmative.

It turns out that if y is zero or a negative integer, the gamma function does not exist. However, suppose $z = -n$ such that $n \in \mathbb{N}$. Let $y \in (-z, -z+1)$ then

$$\Gamma(y) = \frac{1}{y\cdot(y+1)\cdots(y+n-1)}\Gamma(y+n)$$

Let $A = \{w | w = 0 \vee w = -n, n \in \mathbb{N}\}$, $B = (-\infty, 0) - A$.
This iterative definition allows us to extend the gamma function to a domain $D = \mathbb{R} - A$, $\Gamma^* : D \longrightarrow \mathbb{R}$

$$\Gamma^*(x) = \begin{cases} \Gamma(x), & x \in (0,\infty) \\ \frac{1}{y\cdot(y+1)\cdots(y+n-1)}\cdot\Gamma(y+n), & x \in B \end{cases}$$

We can use this result to do the exercises on the next page.

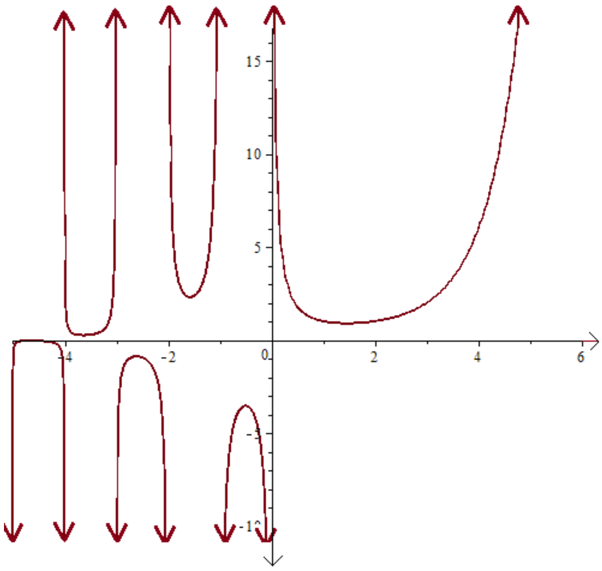

Figure 2.2: Γ^*, The Extended Gamma Function

Exercise 2.2.9. Find $\Gamma^*(-\frac{1}{2})$

Exercise 2.2.10. Find $\Gamma^*(-\frac{3}{2})$

Exercise 2.2.11. Find $\Gamma^*(\frac{3}{2})$

Exercise 2.2.12. Find $\Gamma(-\frac{5}{2})$

Exercise 2.2.13. Find $\Gamma^*(-\frac{5}{2})$

A couple of other comments:
1. $\Gamma^*(x)|_{(0,\infty)} = \Gamma(x)$.
2. Both Γ and Γ^* (within each domain, obviously) is continuous and differentiable.

Also, $\Gamma(y) = \lim\limits_{n \longrightarrow \infty} \dfrac{n^y n!}{y \cdot (y-1) \cdots (y+n)}$

Let $\Gamma_n(y) = \dfrac{n^y n!}{y \cdot (y-1) \cdots (y+n)}$ such that $n \in \mathbb{N}$

Hence, $\Gamma_n(y+1) = y \cdot \Gamma_n(y) \cdot \frac{n}{y+n+1}$

Also, $\Gamma_n(y) = e^{y(\log(n) - \frac{1}{1} - \frac{1}{2} - \cdots - \frac{1}{n})} \cdot \dfrac{1}{y} \cdot \dfrac{e^{y/1}}{1+\frac{y}{1}} \cdot \dfrac{e^{y/2}}{1+\frac{y}{2}} \cdots \dfrac{e^{y/n}}{1+\frac{y}{n}}$

Chapter 3

A Tad of Counting Theory

3.1 The Principle of Counting, Permutations, and Combinations

3.1.1 Gamma Function

Definition 3.1.1. Let $x \in (0, \infty)$ **Gamma of x** is defined as the real number that is

$$\int_0^\infty t^{(x-1)} \cdot e^{-t} dt$$

Definition 3.1.2. The **Gamma function** is defined as

$$\Gamma : (0, \infty) \longrightarrow \mathbb{R}$$

such that

$$\Gamma(x) = \int_0^\infty t^{(x-1)} \cdot e^{-t} dt$$

Theorem 3.1.1. *Let* $\alpha > 1$. $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$.

3.1.2 Factoral

Definition 3.1.3. Let $n \in \mathbb{N}^*$ **n factoral** is defined as the integer that is

$$\int_0^\infty t^{(n)} \cdot e^{-t} dt$$

and the notation is $n!$

Definition 3.1.4. Let $n \in \mathbb{N}^*$. **n factoral** is defined as the natural number that is $\prod_{k=1}^{n} k$ when $n \in \mathbb{N}$;
is 1 when $n = 0$; and, the notation is $n!$

Definition 3.1.5. Let $n \in \mathbb{N}$. The factoral function, f, is $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that $f(n) = (n-1)!$ It is the restriction function (the sequence) $\Gamma\Big|_{\mathbb{N}}$ and the notation is $(n-1)!$

Theorem 3.1.2. *Let* $n \in \mathbb{N}$. $n! = n \cdot (n-1)!$.

3.1.3 The Finite Principle of Counting

Theorem 3.1.3. *Let U be a well-defined universe such that U is finite. Consider the disjoint sets A and B such that the cardinality of A is α and the cardinality of B is β. It is the case that $|A \cup B| = \alpha + \beta$.*

Theorem 3.1.4. *(Ordered Counting of Elements From Two Sets)*
Let U be a well-defined universe such that U is finite. Let the disjoint sets A and B be ordered such that the cardinality of A is α and the cardinality of B is β. It is the case that the number of ways an element from A then B is chosen in order is $\alpha \cdot \beta$.

Theorem 3.1.5. *(The Finite Principle of Counting)*
Let U be a well-defined universe such that U is finite. Let the non-empty sets $A_1, A_2, A_3, \ldots, A_k$ be chosen in order such that the cardinality of A_1 is α_1, the cardinality of A_2 is α_2, ..., the cardinality of A_k is α_k.[1] *It is the case that the number of ways an element from each set is chosen in order is $\prod_{m=1}^{k} \alpha_m = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k$.*

Example 3.1.1. *Let us have an urn containing 7 tokens, a box containing 12 bobby-pins, and a can containing 3 tennis balls. The number of ways a person can choose a token from the urn, a bobby-pin from the box, and then a tennis ball from the can is* $7 \cdot 12 \cdot 3$.

Theorem 3.1.6. *(The Generalised Principle of Counting)*
Let U be a well-defined universe. Let the sets $A_1, A_2, A_3, \ldots, A_k$ be chosen in order such that the cardinality of A_1 is α_1, the cardinality of A_2 is α_2, ..., the cardinality of A_k is α_k.[2] *It is the case that the number of ways an element from each set may be chosen in order is*
$\prod_{m=1}^{k} \alpha_m = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k$.

Note if any set is infinite; then the number of ways to do such is infinite (not interesting to us yet); whilst if any set is empty; then the number of ways to do such is zero (also dull).

3.1.4 Permutations

Definition 3.1.6. Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$. The **permutations** n things **ordered** k at a time is defined as the integer

$$\frac{n!}{(n-k)!}$$

and the notation is ${}_nP_k \equiv P(n,k)$

Example 3.1.2. *Let U be a well defined universe and A be a set such that $|A| = m$ where $m \in \mathbb{N}$. Let w be a non-negative integer such that $w \leq m$. The number of ways to order w elements from the set A is $m \cdot (m-1) \cdot (m-2) \ldots \cdot (m-(w-1))$ which is equal to ${}_mP_w = \frac{m!}{(m-w)!}$.*

[1] Meaning each set A_i has α_i distinct elements.
[2] Meaning each set A_i has α_i distinct elements.

3.1.5 Combinations

Definition 3.1.7. Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.
The **combinations** n things **chosen** k at a time is defined as the integer

$$\frac{n!}{k! \cdot (n-k)!}$$

and the notation is ${}_nC_k \equiv \binom{n}{k}$

Example 3.1.3. *Let U be a well defined universe and A be a set such that $|A| = m$ where $m \in \mathbb{N}$. Let w be a non-negative integer such that $w \leq m$. The number of ways to choose w elements from the set A is* $\dfrac{m \cdot (m-1) \cdot (m-2) \ldots \cdot (m-(w-1))}{w!}$ *which is equal to* $\binom{m}{w} = \dfrac{m!}{(m-w)! \cdot w!}$.

Theorem 3.1.7. *$n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.*

$$P(n,k) \geq \binom{n}{k}$$

Theorem 3.1.8. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n > k$.*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Theorem 3.1.9. *$n \in \mathbb{N}^*$, $k \in \mathbb{N}^*$, and $n \geq k$.*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Theorem 3.1.10. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{k} = \binom{n}{n-k}$$

Theorem 3.1.11. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{0} = 1$$

Theorem 3.1.12. *$n \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \geq k$.*

$$\binom{n}{n} = 1$$

Theorem 3.1.13. *$n \in \mathbb{N}$.*

$$\binom{n}{1} = n$$

Theorem 3.1.14. *$n \in \mathbb{N}$.*

$$\binom{n}{n-1} = n$$

None of these theorems are board-worthy in MAT 301 (MAT 321 or MAT 123 maybe but not here).

3.2 Computational Exercises

Exercise 3.2.1. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert –in that order. Find the number of ways a meal consists of a Caesar salad, pork entrée, and then custard).

Exercise 3.2.2. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert –in that order. Find the number of ways to create a meal of a salad, then an entrée, then a dessert –in that order.

Exercise 3.2.3. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, custard). You eat a meal consisting of a salad, then an entrée, then a dessert. Find the number of ways a meal consists of a Caesar salad, not a pork entrée, and then cake or pie).

Exercise 3.2.4. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing exactly two heads and then twice exactly no heads.

Exercise 3.2.5. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing exactly two heads amongst the tosses.

Exercise 3.2.6. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing at least two heads.

Exercise 3.2.7. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing no heads.

Exercise 3.2.8. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the number of ways of tossing at most two heads.

Exercise 3.2.9. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 4 red balls.

Exercise 3.2.10. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 4 red balls amongst the six balls chosen.

Exercise 3.2.11. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing at most two heads.[3]

Exercise 3.2.12. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

Exercise 3.2.13. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; then we draw another ball from the urn; and, finally we draw another ball from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

[3] Yes, this is the question.

Exercise 3.2.14. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball from the urn; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; then we draw another ball from the urn; ; put it back; shake up the urn; and, finally we draw another ball from the urn. Find the number of ways of choosing exactly 2 red balls, exactly 2 white balls, and exactly 2 blue balls.

Exercise 3.2.15. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing a red ball; then a blue ball; then exactly 2 white balls, and finally exactly 2 blue balls.

Exercise 3.2.16. Suppose there is an urn (snore) that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing 2 red balls then at least 3 blue balls; then a green ball.

Exercise 3.2.17. Suppose there is an urn (ugh) that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing 2 red balls initially and then at least 2 blue balls of the last four remaining.

Exercise 3.2.18. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out six balls from the urn. Find the number of ways of choosing at least 2 red balls amongst the six chosen.

Exercise 3.2.19. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw out two balls from the urn. Find the number of ways of choosing 2 blue balls.

Exercise 3.2.20. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball then another ball from the urn. Find the number of ways of choosing a red ball and then a white ball.

Exercise 3.2.21. Suppose there is an urn that contains 5 red, 4 white, and 11 blue balls. We draw a ball look at its colour. Put it back shake up the urn and then draw another ball from the urn. Find the number of ways of choosing a red ball and then not drawing a white ball as the second ball.

Exercise 3.2.22. Suppose there is an urn (Lord help us all – McLoughlin is sure adept at copying and pasting) that contains 5 red, 4 white, and 11 blue balls. We draw a ball. We replace the ball shake up the urn and then draw a ball from the urn. Find the number of ways of choosing a blue ball and a white ball.

Exercise 3.2.23. Suppose there is an urn (Please enough!) that contains 5 red, 4 white, and 11 blue balls. We draw a ball. We replace the ball shake up the urn and then draw a ball from the urn. Find the number of ways of choosing at least one red ball.

With dice (unless otherwise stated) we toss a pair of dice and sum up the number of dots of the up-turned faces.

Exercise 3.2.24. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a sum of seven or an eleven.

Exercise 3.2.25. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a seven or an eleven.

Exercise 3.2.26. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a sum of five or better.

Exercise 3.2.27. Suppose we toss a pair of fair six sided dice. We view the up-turned faces of the dice. Find the number of ways of tossing a product of the upturned faces of a twelve.

Exercise 3.2.28. Compute (where such exists):

1. $\binom{7}{4}$
2. $\binom{7}{5}$
3. $\binom{8}{4}$
4. $\binom{8}{5}$
5. $\binom{4}{0}$
6. $\binom{4}{1}$
7. $\binom{4}{2}$
8. $\binom{4}{3}$
9. $\binom{4}{4}$
10. $\binom{4}{5}$
11. $\binom{-4}{-2}$
12. $\binom{14}{13}$

3.2.1 Temporally Distinct vs. Concurrent

So, hopefully having completed the exercises from the previous section you opined that there was an (or some) important point about the wording of a problem. It, loosely, has to do with the timing of what is being done in an experiment: concurrent timing or temporally distinct.

Example 3.2.1. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw a ball; then another; then another. The number of ways this can be done such that we draw a red ball; then a blue ball; and, then a red ball is* $5 \cdot 11 \cdot 4$.

Example 3.2.2. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw a red ball; put it back; shake up the urn. We then a blue ball; put it back; shake up the urn. We finally draw a red ball. The number of ways this can be done is* $5 \cdot 11 \cdot 5$.

Example 3.2.3. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done such that we draw two red balls and a blue ball is* $\binom{5}{2} \cdot \binom{11}{1}$

Example 3.2.4. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done is* $\binom{20}{3}$

Example 3.2.5. *We have an urn with* 5 *red,* 4 *white, and* 11 *blue balls. We draw three balls. The number of ways this can be done such that we draw a non-blue ball and two blue ball is* $\binom{9}{1} \cdot \binom{11}{2}$

Chapter 4

On the Naïve or Intuitive Idea of Probability

4.1 Introduction

Informal understanding of probability pre-dated the formal theory of probability. In this chapter we will simply look at the ideas that first were encountered.

4.1.1 Finite Sample Space

Definition 4.1.1. Let U be a well defined universe. A **sample space** is a subset of U.

Definition 4.1.2. Let U be a well defined universe. Let S be a well defined sample space. An **event** is a subset of S.

Definition 4.1.3. Let U be a well defined universe. Let S be a well defined sample space. Let A be an event. An **outcome** is an element of A.

The intuitive idea of probability of an event, E, from a sample space, S, is $\frac{|E|}{|S|}$ where S is finite (meaning $|S| < \aleph_0$).

Definition 4.1.4. Let S be a well defined sample space. An outcome is defined as **fair** *iff* the probability that an outcome in the space is picked is equal to the probability that any other outcome in the space is picked (another way to express this is that all outcomes are equiprobable of being selected).

Notation 4.1.1. *Let S be a well defined sample space. Let E be an event. The probability of E occurring is denoted as* **Pr(E)** *or* **P(E)**.

4.1.2 Exercises

Exercise 4.1.1. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing exactly two heads and then twice exactly no heads.

Exercise 4.1.2. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing two heads.

Exercise 4.1.3. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing at least two heads.

Exercise 4.1.4. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing no heads.

Exercise 4.1.5. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing at most two heads.

4.1.3 Non-finite Sample Space

An intuitive idea of probability of an event, E, from a sample space, S, is NOT $\frac{|E|}{|S|}$ because the space we have is **not** finite.

Example 4.1.1. *Let* $U = \mathbb{R}$ *and* $S = \mathbb{N}$ *whilst* $E = \{x | x = 2 \cdot a, a \in \mathbb{N}\}$. $Pr(E) = \frac{|E|}{|S|}$ *fails.*

Example 4.1.2. *Let* $U = S = \mathbb{R}$ *and* $E = \mathbb{N}$. $Pr(E) = \frac{|E|}{|S|}$ *fails.*

For a one-dimensional sample space we use length. For a two-dimensional sample space we use area. For a three-dimensional sample space we use volume.

A key is to realise that an outcome is **in** an event (an element belongs to a set) and the probability of the event is a real **number** restricted to the interval of zero to one, inclusive.

Example 4.1.3. *Suppose* $U = \mathbb{R}$ *and* $S = [0, 10]$. *Let* $E = [0, 5]$. *Let* $F = [0, 5)$. *Let* $H = (0, 4)$. *Let* $J = \{x | x \in S \wedge \exists m \in \mathbb{N} \ni x = 2 \cdot m\}$.

$$Pr(J) = \frac{\ell(J)}{\ell(S)} = \frac{0}{10}$$
$$Pr(F) = \frac{\ell(F)}{\ell(S)} = \frac{5}{10}$$
$$Pr(E) = \frac{\ell(E)}{\ell(S)} = \frac{5}{10}$$
$$Pr(H) = \frac{\ell(H)}{\ell(S)} = \frac{4}{10}$$
$$Pr(E \cap F^c) = \frac{\ell(E \cap F^c)}{\ell(S)} = \frac{0}{10}$$

4.1.4 Exercises

Exercise 4.1.6. Suppose $U = \mathbb{R}$ and $S = [0, 8]$. Let $E = [1, 3]$. Let $F = [0, 3)$. Let $H = (1, 3)$. Let $J = \{x | x \in S \wedge \exists m \in \mathbb{N} \ni x = 2 \cdot m\}$.

A. Find $Pr(E)$. B. Let $E = [1, 3]$ C. Find $Pr(E^c)$. D. Find $Pr(F)$.
E. Let $G = E - F$ Find $Pr(G)$. F. Find $Pr(H)$. G. Find $Pr(J)$.

Exercise 4.1.7. Suppose $U = \mathbb{R}$ and $S = [1, \pi)$. Let $E = [2, 3]$. Let $F = (2, 3)$. Let $G = (3, 3)$. Let $H = (e, \pi)$. Let $J = (1, \frac{\pi}{2})$. Let $K = (1, \frac{\pi - 1}{2}]$. Let $M = [3, 3]$.

A. Find $Pr(E)$. B. Find $Pr(E^c)$. C. Find $Pr(F)$. D. Find $Pr(G)$.
E. Find $Pr(H)$. F. Find $Pr(J)$. G. Find $Pr(M)$. H. Find $Pr(K)$.

Exercise 4.1.8. Suppose $U = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and let $S = \{(x,y)| \ \ (x-1)^2 + (y+2)^2 = 9\}$.
A. Let k be the point $(2,0)$. Find $Pr(k \in S)$. B. Let p be the point $(0,2)$. Find $Pr(p \notin S)$.
C. Let C be the set of all points in S such that $x > 0$. Find $Pr(C)$.
D. Let D be the set of all points in S such that $x \geqslant 0 \wedge y \geqslant 0$. Find $Pr(D)$.
E. Let E be the set of all points in S such that $(x-1)^2 + (y+2)^2 = 9$. Find $Pr(E)$.
F. Let $F = \{(x,y)| \ \ (x-1)^2 + (y-4)^2 < \frac{1}{2}\} \bigcap S$. Find $Pr(F)$.
G. Let $G = \{(x,y)| \ \ y = -2x\} \bigcap S$. Find $Pr(G)$.
H. Let $J = \{(x,y)| \ \ y = -2x\} \bigcap S$. Find $Pr(H)$.
J. Let $J = \{(x,y)| \ \ y \leq -2x\} \bigcap S$. Find $Pr(J)$.

Exercise 4.1.9. Suppose $U = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and let $S = \{(x,y)| \ \ (x-1)^2 + (y+2)^2 \leqslant 9\}$.
With this problem realise that we are referencing the set-theoretic universe for definitions in the exercises and the probabilities computed are those sets intersect the sample space. This is not a 'good' way to state a problem; but, as we shall see it is a way we will sometimes encounter problems (e.g.: see any book).
A. Let k be the point $(2,0)$. Find $Pr(k \in S)$. B. Let p be the point $(0,2)$. Find $Pr(p \notin S)$.
C. Let C be the set of all points such that $x > 0$. Find $Pr(C)$.
D. Let D be the set of all points such that $x \geqslant 0 \wedge y \geqslant 0$. Find $Pr(D)$.
E. Let E be the set of all points such that $(x-1)^2 + (y+2)^2 = 9$. Find $Pr(E)$.

Chapter 5

Introduction to Probability Theory

5.1 Introduction

Mathematics is predicated on logic (for the rules of inference) and on Set Theory. From those foundations we build geometry, real analysis, abstract algebra, the theory of differential equations and difference equations, probability theory, point-set topology, general topology, numerical analysis, number theory, combinatorics, complex variables, dynamical systems, graph theory, game theory, statistics, cryptology, etc.

Recall from Set Theory that every set, A, is a subset of some well defined universe, U. So, for probability theory we rename the universe as a sample space [the space from whence a sample may be chosen]; an arbitrary set is called an event; and an element of that set an outcome.

Logic	Set Theory	Probability Theory
Domain of definition	Universe, U example $U = \{1, 2, 3, 4, 5\}$	Sample space, S example $S = \{1, 2, 3, 4, 5\}$
Statements p, q, and r	Sets A, B, and C. example $A = \{1, 2\}$ $B = \{2, 3, 4\}$ $C = \{1, 4\}$	Events E, F, and G example $E = \{1, 2\}$ $F = \{2, 3, 4\}$ $G = \{1, 4\}$
Simple statements **true** statements like 1 is in A **false** statements like 3 is in A	Elements example $1 \in A$ and $1 \notin B$ $1 \in B$, $\{2\} \subseteq C$	Outcomes example $\{2, 4\} \subseteq F$ $\{2, 4\} \subseteq E$, $4 \in E$

etcetera

What is 'tricky' in doing basic probability proofs is you are working with sets; functions; and real numbers. So, before tackling these claims think about what is the claim. Is it a numeric, set-theoretic, or both? How so? And do you opine it is true or not?

5.2 The Kolmogorov Axioms of Probability

[1]

Let S denote the sample space, E, E_i, F, etc. events and the notation $Pr(\cdot)$ the probability of whatever.

Axiom 5.2.1. *(The Space Axiom) S is the space* $\Rightarrow Pr(S) = 1$.

Axiom 5.2.2. *(The Event Axiom) E is an event* $\Rightarrow 0 \leq Pr(E) \leq 1$.

Axiom 5.2.3. *(The Collection of M. E. E.[2] Axiom) Let I be an index set. The collection* $\{E_i\}_{i \in I}$ *being mutually exclusive* $\Rightarrow \Pr(\bigcup\limits_{i \in I} E_i) = \sum\limits_{i \in I} \Pr(E_i)$.

5.2.1 Some Theorems, Lemmas, or Corollaries of Probability

Lemma 5.2.1. *Let S be a well defined sample space; whilst E and F are events.*

1. *E is an event* $\Rightarrow E^c$ *is an event.*
2. *E and F are events* $\Rightarrow E \cap F$ *is an event.*
3. *E and F are events* $\Rightarrow E \cup F$ *is an event.*
4. *E and F are events* $\Rightarrow E - F$ *is an event.*

Corollary 5.2.1. $\mathbb{B}$ $\mathbb{D}1$ *(The Null Corollary) Let S be a well defined sample space whilst* $E = \emptyset$. *It is the case that* $Pr(E) = 0$.

Corollary 5.2.2. $\mathbb{B}$ $\mathbb{D}1$ *(The Complement Corollary) E is an event* $\Rightarrow Pr(E^c) = 1 - Pr(E)$.

Corollary 5.2.3. $\mathbb{B}$ $\mathbb{D}1$ *(The Subset Corollary) E and F are events* $\ni E \subseteq F \Rightarrow Pr(E) \leq Pr(F)$.

Corollary 5.2.4. $\mathbb{B}$ $\mathbb{D}5$ *(The Spanning Corollary) Let I be an index set. The collection* $\{E_i\}_{i \in I}$ *being mutually exclusive and exhaustive* $\Rightarrow \Pr(\bigcup\limits_{i \in I} E_i) = \sum_{i \in I} \Pr(E_i) = 1$.

Corollary 5.2.5. $\mathbb{B}$ $\mathbb{D}1$ *(The Same Event Corollary) E and F are events* $\ni E = F \Rightarrow Pr(E) = Pr(F)$.

Theorem 5.2.1. $\mathbb{B}$ $\mathbb{D}2$ *(The Union Theorem) E and F are events.* $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$.

Note some of the other axioms lists from other classes that are of great use to us in this class.

[1] The Kolmogorov Axioms of Probability are named after the creator of the axioms: the Russian mathematician Kolmogorov. These are so named since he created them as an answer to one of the famous Hilbert 20^{th} century 10 questions. The axioms of probability are one of the shortest lists of axioms I can recall for an area of mathematics.

[2] M. E. E. : Mutually Exclusive Events

5.2.2 The Axioms of Set Theory

Axiom 5.2.4. *(The Axiom of Extension) Two sets are equal iff they have the same elements.*

Axiom 5.2.5. *(The Axiom of Null) There exists a set with no elements, call it* $\emptyset$

Axiom 5.2.6. *(The Axiom of Pairing) Given any sets* A *and* B*, there exists a set* C *whose elements are* A *and* B*.*

Axiom 5.2.7. *(The Axiom of Union) Given any set* A*, the union of all elements in* A *is a set.*

Axiom 5.2.8. *(The Axiom of Power Set) Given any set* A*, there exists a set* B *consisting of all the subsets of* A*.*

Axiom 5.2.9. *(The Axiom of Separation) Given any set* A *and a sentence* $p(a)$ *that is a statement for all* $a \in A$*, then there exists a set* $B = \{a \in A : p(a) \text{ is true}\}$*.*

Axiom 5.2.10. *(The Axiom of Replacement) Given any set* A *and a function* f *defined on* A*, the image* $f[A]$ *is a set.*

Axiom 5.2.11. *(The Axiom of Infinity) There exists a set* A *such that* $\emptyset \in A$*, and whenever* $a \in A$*, it follows that* $a \cup \{a\} \in A$

Axiom 5.2.12. *(The Axiom of Regularity) Given any non-empty set* A*, there exists an* $ainA$ *such that* $a \cap A = \emptyset$*.*

Axiom 5.2.13. *(The Axiom of Choice) Given any non-empty set* A *whose members are pair-wise disjoint non-empty sets, there exists a set* B *consisting of exactly one element taken from each set belonging to* A*.*

5.2.3 The Peano Axioms for $\mathbb{N}$

The properties of addition of natural numbers are be derived from a short set of axioms which also form the basis for mathematical induction. The axioms are called **the Peano Axioms**: There exists a set, P, which is defined by the following four axioms.

Axiom 5.2.14. *There exists a natural number, call it 1, that is not the successor of any other natural number.*

Axiom 5.2.15. *Every natural number has a unique successor. If* $k \in P$ *, then let* $k\prime$ *denote the successor of* k*.*

Axiom 5.2.16. *Every natural number except one is the successor of exactly one natural number.*

Axiom 5.2.17. *If* M *is* ***a*** *set of natural numbers such that*

1. $1 \in M$ *and*

2. *for each* $k \in P$*, if* $k \in M$*, then* $k' \in P$*,*

then $P = M$*.*

P, of course is $\mathbb{N}$.

So, the Peano axioms assert the uniqueness of the naturals that this successor property along with the element 1 creates the entirety of the natural numbers. No matter how you name the set (you can call it Ray, or you can call it Jay, . . .) if it has these properties then it really is the naturals. From these axioms arise the natural numbers by defining what addition by one means.

Definition 5.2.1. For every $k \in \mathbb{N}$, define $k + 1 = k'$.

Then, note inductively, the entire understanding of addition flows from this definition (likewise multiplication, etc.). Also we note the Archimedean Principle of $\mathbb{N}$ states that $\mathbb{N}$ is unbounded above; in other words, there does not exist a greatest natural number. It is worth repeating that the Peano axioms and the Archimedean principle provide a justification for the concept of mathematical induction.

5.2.4 The Axioms of the Reals: Field, Order, and Completeness

Axiom 5.2.18. *(Closure of Addition)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x + y) \in \mathbb{R}$ *and* $(x = w \wedge y = v) \Rightarrow (x + y = w + v)$

Axiom 5.2.19. *(Commutativity of Addition)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x + y) = (y + x)$

Axiom 5.2.20. *(Associativity of Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x + y) + z = x + (y + z)$

Axiom 5.2.21. *(Existence of Identity of Addition)* $\exists$ *a unique number* $0 \ni x + 0 = 0 + x \forall\ x \in \mathbb{R}$.[3]

Axiom 5.2.22. *(Existence of Additive Inverse)* $\forall\ x \in \mathbb{R}$ *it is the case that there exists a unique number* $-x$ *such that* $x + (-x) = (-x) + x = 0$.

Axiom 5.2.23. *(Closure of Multiplication)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x \cdot y) \in \mathbb{R}$ *and* $(x = w \wedge y = v) \Rightarrow (x \cdot y = w \cdot v)$

Axiom 5.2.24. *(Commutativity of Multiplication)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x \cdot y) = (y \cdot x)$

Axiom 5.2.25. *(Associativity of Multiplication)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Axiom 5.2.26. *(Existence of Identity of Multiplication)* $\exists$ *a unique number* $1 \ni x \cdot 1 = 1 \cdot x \forall\ x \in \mathbb{R}$. *Moreover,* $0 \neq 1$.

Axiom 5.2.27. *(Existence of Multiplicative Inverse)* $\forall\ x \in \mathbb{R} \ni x \neq 0$ *it is the case that there exists a unique number* x^{-1} *such that* $x \cdot (x^{-1}) = (x^{-1}) \cdot x = 1$.

Axiom 5.2.28. *(Axiom of the Distribution of Multiplication over Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Axiom 5.2.29. *(Trichotomy)* $\forall\ x, y \in \mathbb{R}$ *it is the case that* $(x < y) \vee (x = y) \vee (x > y)$ *and moreover it is not the case that any two of these conditions can simultaneously hold.*

[3] The existence of a unique element is also denoted as $\exists!$.

Axiom 5.2.30. *(Transitivity)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x < y) \wedge (y < z) \implies (x < z)$.

Axiom 5.2.31. *(Preservation of Order under Addition)* $\forall\ x, y, z \in \mathbb{R}$ *it is the case that* $(x < y) \implies (x + z) < (y + z)$.

Axiom 5.2.32. *(Preservation of Order for a Positive Multiplier)* $\forall\ x, y \in \mathbb{R}\ \wedge z \in \mathbb{R}\ \ni 0 < z$ *it is the case that* $(x < y) \implies (x \cdot z) < (y \cdot z)$.

Axiom 5.2.33. *(Completeness)* $\forall A \subseteq \mathbb{R} \ni A$ *is bounded above*[4] $\exists\ m \in \mathbb{R}$ *such that it is the supremum*[5] *of the set* A.

5.2.5 Important Lemmas of Use

We need certain basic properties of sets and real numbers so we may assume:
Let our universe be $\mathbb{R}$.

Lemma 1: $0 < 1$.

Lemma 2: Let $x \in \mathbb{R}$ It is the case that $x \cdot 0 = 0$.

Lemma 3: $(-1) \cdot (-1) = 1$.

Lemma 4: Let $x \in \mathbb{R}$. It is the case that $(-1) \cdot x = -x$

Lemma 5: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x - y = x + (-y) = x + -y$.

Definition 5.2.2. Let $x \in \mathbb{R}$. It is the case that $x \cdot x = x^2$.

Definition 5.2.3. Let $x \in \mathbb{R} \wedge n \in \mathbb{N}$. It is the case that $x \cdot x = x^2$ whilst $\underbrace{x \cdot \ldots \cdot x}_{n}$ is x^n

Definition 5.2.4. (Law of Exponents) Let $x \in \mathbb{R}$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$.

1. $x^a \cdot x^b = x^{(a+b)}$
2. $x^a \cdot b^a = (x \cdot b)^a$
3. $x^a \div x^b = x^{(a-b)}$ when $x^b \neq 0$
4. $(x^a)^b = x^{a \cdot b}$

Definition 5.2.5. Let $x \in \mathbb{R}$. It is the case that x is **positive** if and only if $x > 0$.

Definition 5.2.6. Let $x \in \mathbb{R}$. It is the case that x is **non-negative** if and only if $x \geq 0$.

Definition 5.2.7. Let $x \in \mathbb{R}$. It is the case that x is **non-positive** if and only if $x \leq 0$.

Definition 5.2.8. Let $x \in \mathbb{R}$. It is the case that x is **negative** if and only if $x < 0$.

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[4] A set being bounded above is to be defined later. The inclusion of this axiom is due to my desire to put the axioms together at the beginning of the tome.

[5] A set having a supremum is to be defined later.

5.2.6 For Number Theoretic Claims

Properties of Natural, Integers, or Rational Numbers that you may assume:

Definition 5.2.9. (Closure of addition in $\mathbb{N}$) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m + n) \in \mathbb{N}$.

Definition 5.2.10. (Closure of multiplication in $\mathbb{N}$) Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m \cdot n) \in \mathbb{N}$.

Definition 5.2.11. (Closure of addition in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a + b) \in \mathbb{Z}$.

Definition 5.2.12. (Closure of subtraction in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a - b) \in \mathbb{Z}$.

Definition 5.2.13. (Closure of multiplication in $\mathbb{Z}$) Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(m \cdot n) \in \mathbb{Z}$.

Definition 5.2.14. (Closure of addition in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p + q) \in \mathbb{Q}$.

Definition 5.2.15. (Closure of subtraction in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p - q) \in \mathbb{Q}$.

Definition 5.2.16. (Closure of multiplication in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p \cdot q) \in \mathbb{Q}$.

Definition 5.2.17. (Closure of non-zero division in $\mathbb{Q}$) Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ where $q \neq 0$, then $\frac{p}{q} \in \mathbb{Q}$.

5.2.7 For Claims About Odd or Even Natural Numbers

Definition 5.2.18. Let $m \in \mathbb{N}$. m is even if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j$.

Definition 5.2.19. Let $m \in \mathbb{N}$. m is odd if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j - 1$.

5.2.8 For Claims About Odd or Even Integers

Definition 5.2.20. Let $w \in \mathbb{Z}$. w is even if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p$.

Definition 5.2.21. Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p + 1$.
Version 2 of 5.2.21: Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer q (meaning $q \in \mathbb{Z}$) such that $w = 2 \cdot q - 1$.

If there are **any other** seemingly ‘obvious’ definitions, lemmas, theorems, corollaries, laws, etc. *you wish to cite for a proof for class, please ask about it as soon as possible.*

5.3 Basic Claims To Prove or Disprove

Note: These claims are to be proven or disproven **individually** - there is to be no conferring on a claim until **someone presents a particular claim**.[6] Therefore there is to be no discussing problems on this with other people, a professor other than me, using a tutor (good luck with that), or working with another person.

All claims in this section are board-worthy. They are of a varying degree of difficulty that shan't be defined (yet).

Claim 5.3.1. Let S be a well defined sample space (w.d.s.s.) whilst $E = \emptyset$. So, $Pr(E) = 0$.

Claim 5.3.2. Let S be a well defined sample space (w.d.s.s.) whilst $Pr(E) = 0$.
It is the case that $E = \emptyset$.

Claim 5.3.3. Let S be a w.d.s.s. whilst E is an event. It is the case that $Pr(E^c) = 1 - Pr(E)$.

Claim 5.3.4. Let S be a w.d.s.s. whilst E is an event. $Pr(E) > \frac{1}{2} \Longrightarrow Pr(E^c) < \frac{1}{2}$

Claim 5.3.5. Let S be a w.d.s.s. whilst E is an event. $Pr(E) > \frac{1}{2} \Longrightarrow Pr(E^c) \leqslant \frac{1}{2}$

Claim 5.3.6. Let S be a w.d.s.s. whilst E is an event. $Pr(E) \geqslant \frac{1}{2} \Longrightarrow Pr(E^c) < \frac{1}{2}$

Claim 5.3.7. Let S be a w.d.s.s. whilst E, F are events such that $E \subseteq F$.
It is the case that $Pr(E) \leqslant Pr(F)$

Claim 5.3.8. Let S be a w.d.s.s. whilst E, F are events $\ni E \subset F$.
It is the case that $Pr(E) < Pr(F)$.

Claim 5.3.9. Let S be a w.d.s.s. whilst E, F are events $\ni E = F$.
It is the case that $Pr(E) = Pr(F)$.

Claim 5.3.10. Let S be a w.d.s.s. whilst E, F are events $\ni Pr(E) = Pr(F)$.
It is the case that $E = F$.

Claim 5.3.11. Let S be a w.d.s.s. whilst E, F are events. $Pr(E) > \frac{1}{2} \Longrightarrow Pr(F) < \frac{1}{2}$

Claim 5.3.12. Let S be a w.d.s.s. whilst E, F are mutually exclusive events.
$Pr(E) > \frac{1}{2} \Longrightarrow Pr(F) < \frac{1}{2}$

Claim 5.3.13. Let S be a w.d.s.s. whilst E, F are events. Let $G = E - F$.
It is the case that $Pr(G) = Pr(E) - Pr(F)$.

Claim 5.3.14. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cap F) = Pr(E) + Pr(F)$

Claim 5.3.15. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Claim 5.3.16. Let S be a w.d.s.s. whilst E, F are events.
It is the case that $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

Claim 5.3.17. Let S be a w.d.s.s. whilst $E, F, \wedge G$ are events.
It is the case that $Pr(E \cup F \cup G) = Pr(E) + Pr(F) + Pr(G) - Pr(E \cap F \cap G)$

Claim 5.3.18. Let S be a w.d.s.s. whilst $E, F, \wedge G$ are events where E and F are mutually exclusive and E and G are mutually exclusive.
It is the case that $Pr(E \cup F \cup G) = Pr(E) + Pr(F) + Pr(G) - Pr(F \cap G)$

[6] Pretend it is day 5 and someone presents claim 3.12 on the board and 3.6 has not been presented on this day or any day previous. So, you may discuss claim 3.12 **since it is after** it has been presented. But you can't discuss, let us say, 3.6 since it has not been presented yet.

5.4 More Claims To Prove or Disprove

Claim 5.4.1. There exists a w.d.s.s. S and events E, F such that $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Claim 5.4.2. There exists a w.d.s.s. S and events E, F such that $Pr(E \cup F) \geqslant 1$

Claim 5.4.3. There exists a w.d.s.s. S and events E, F such that $Pr(E \cup F) > 1$

Claim 5.4.4. $\exists$ a w.d.s.s. S and event E such that $Pr(E) \in \mathbb{I}$

Claim 5.4.5. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E) + Pr(F) > 1$

Claim 5.4.6. $\exists$ a w.d.s.s. S and event E such that $E \neq S$ but $Pr(E) = Pr(S)$

Claim 5.4.7. $\exists$ a w.d.s.s. S and event E such that $Pr(E) > Pr(S)$

Claim 5.4.8. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E \cup F) > Pr(E) + Pr(F)$

Claim 5.4.9. $\exists$ a w.d.s.s. S and events E, F such that $Pr(E \cup F) < Pr(E) + Pr(F)$

References:
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Advanced Calculus, 1956, Buck, R. C.