MATH 273
HANDOUT III: TELESCOPING SERIES
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A. If Γ is a series, say $\sum_{i=1}^{\infty} a_i$, and $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots$,

then $S_A = \{a_n\}_{n=1}^{\infty}$ is called the **sequence of terms** *from the series* Γ , and
$S_A = a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots$

whereas,

$P_A = a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4 \ldots$ is called the **sequence of partial sums** of the series Γ and

Let us rewrite P_A as $P_A = p_1, p_2, p_3, \ldots$ where

$p_1 = a_1,$
$p_2 = a_1 + a_2,$
$p_3 = a_1 + a_2 + a_3$
.
.
.
$p_m = a_1 + a_2 + a_3 + \ldots + a_m$
.
.
.

Now, by definition Γ <u>converges</u> iff there exists a real number, A, such that $A = \sum_{i=1}^{\infty} a_i$

Γ does not converges (<u>diverges</u>) iff there does not exists a real number, B, such that $B = \sum_{i=1}^{\infty} a_i$

we say when Γ diverges it 'blows up,' 'goes to infinity,' goes to negative infinity,' 'it jumps back and forth and doesn't go to **<u>a</u>** number,' or the sum does not exist.

Definition: Let Γ be the series, say $\sum_{i=1}^{\infty} a_i$, and $\{p_k\}_{k=1}^{\infty}$ be the sequence of partial sums associated with Γ, then $\lim_{k\to\infty}(p_k) = \sum_{i=1}^{\infty} a_i$ when $\sum_{i=1}^{\infty} a_i$ converges (the limit of the sequence of partial sums is the same as sum of the series). When $\lim_{k\to\infty}(p_k)$ does not exist, we say $\sum_{i=1}^{\infty} a_i$ diverges.

Technique: Let $\sum_{i=1}^{\infty} b_i$ be a series and suppose it is the case that we can use partial fraction decomposition to say that $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} c_i + \sum_{i=1}^{\infty} d_i$ (in the 2 fraction case [it generalizes so that you can have finitely many sums]) $\sum_{i=1}^{\infty} b_i$ converges if $\sum_{i=1}^{\infty} c_i + \sum_{i=1}^{\infty} d_i$ have the property that only finitely many terms remain when the sums are rewritten and infinitely many of the terms cancel due to the axiom of the additive inverse.

Example (Mr. Frey): Let us consider $\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)} = \sum_{k=1}^{\infty}\frac{1}{(k+2)} + \sum_{k=1}^{\infty}\frac{-1}{(k+3)}$

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)} = \quad & \sum_{k=1}^{\infty}\frac{1}{(k+2)} \\
& + \sum_{k=1}^{\infty}\frac{-1}{(k+3)} \\
& = \left(\frac{1}{3}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{5}\right)+\left(\frac{1}{6}\right)+\left(\frac{1}{7}\right)+\left(\frac{1}{8}\right)+\left(\frac{1}{9}\right)+\left(\frac{1}{10}\right)+\dots \\
& + (0)+\left(\frac{-1}{4}\right)+\left(\frac{-1}{5}\right)+\left(\frac{-1}{6}\right)+\left(\frac{-1}{7}\right)+\left(\frac{-1}{8}\right)+\left(\frac{-1}{9}\right)+\left(\frac{-1}{10}\right)+\dots \\
& = {}^{1}/_{3}
\end{aligned}$$

Things 'line up' well so we were able to determine the sum of the series.

<u>An ‘Unfortunate’ Example:</u> Let us consider $\sum_{k=1}^{\infty}\frac{-6}{(k+1)(2k+5)}$ verify that through partial fraction decomposition we get $\sum_{k=1}^{\infty}\frac{-6}{(k+1)(2k+5)} = \sum_{k=1}^{\infty}\frac{-2}{(k+1)} + \sum_{k=1}^{\infty}\frac{4}{(2k+5)}$

So,
$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{-6}{(k+1)(2k+5)} &= \sum_{k=1}^{\infty}\frac{-2}{(k+1)} \\
&\quad + \sum_{k=1}^{\infty}\frac{4}{(2k+5)} \\
&= \left(\frac{-2}{2}\right)+\left(\frac{-2}{3}\right)+\left(\frac{-2}{4}\right)+\left(\frac{-2}{5}\right)+\left(\frac{-2}{6}\right)+\left(\frac{-2}{7}\right)+\left(\frac{-2}{8}\right)+\left(\frac{-2}{9}\right)+\ldots \\
&\quad +\left(\frac{4}{7}\right)+\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)+\left(\frac{4}{11}\right)+\left(\frac{4}{13}\right)+\left(\frac{4}{15}\right)+\left(\frac{4}{17}\right)+\left(\frac{4}{19}\right)+\ldots
\end{aligned}$$

Things **DO NOT** ‘line up’ well so we are unable to determine the sum of the series (indeed we cannot determine if it converges or diverges from this work); hence, the reason we must study this topic of series in more detail as the semester progresses.