HANDOUT V
MATH 273 CALCULUS III
DR. MCLOUGHLIN'S
HANDY DANDY GUIDE TO
INFINITE SERIES
PART II
INCLUDING: TESTS FOR CONVERGENCE OR DIVERGENCE

A few reminders:

A. If Γ is a series, say $\sum_{i=1}^{\infty} a_i$, and $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots$,

then $S_A = \{a_n\}_{n=1}^{\infty}$ is called the **sequence of terms** *from the series* Γ, and
$S_A = a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \quad \ldots$

whereas,

$P_A = a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4 \quad \ldots$ is called the **sequence of partial sums** of the series Γ and

Let us rewrite P_A as $P_A = p_1, \ p_2, \ p_3, \ \ldots$ where
$p_1 = a_1$,
$p_2 = a_1 + a_2$,
$p_3 = a_1 + a_2 + a_3$
.
.
.
$p_m = a_1 + a_2 + a_3 + \ldots + a_m$
.
.
.

Now, by definition Γ <u>converges</u> iff there exists a real number, A, such that $A = \sum_{i=1}^{\infty} a_i$

Γ does not converges (<u>diverges</u>) iff there does not exists a real number, B, such that $B = \sum_{i=1}^{\infty} a_i$

we say when Γ diverges it 'blows up,' 'goes to infinity,' goes to negative infinity,' 'it jumps back and forth and doesn't go to **<u>a</u>** number,' or the sum does not exist.

<u>Definition</u>: Let Γ be the series, say $\sum_{i=1}^{\infty} a_i$, and $\{p_k\}_{k=1}^{\infty}$ be the sequence of partial sums associated with Γ, then $\lim_{k\to\infty}(p_k) = \sum_{i=1}^{\infty} a_i$ when $\sum_{i=1}^{\infty} a_i$ converges (the limit of the sequence of partial sums is the same as sum of the series). When $\lim_{k\to\infty}(p_k)$ does not exist, we say $\sum_{i=1}^{\infty} a_i$ diverges.

B. One of our favourite series is $\sum_{j=1}^{\infty} \frac{1}{j}$ It is called the **harmonic series** and diverges (by the integral test, for one).

C. Also remember for the sequence of terms, $S_A = \{a_n\}_{n=1}^{\infty}$

Suppose $\lim_{n\to\infty}(a_n) = 0$.

This means ***nothing*** in terms of whether $\sum_{i=1}^{\infty} a_i$ converges or not!

Suppose $\sum_{i=1}^{\infty} a_i$ converges, then it must be the case that $\lim_{n\to\infty}(a_n) = 0$.

Suppose $\lim_{n\to\infty}(a_n) \neq 0$, then it must be the case that $\sum_{i=1}^{\infty} a_i$ diverges.

And suppose $\sum_{i=1}^{\infty} a_i$ diverges. This means ***nothing*** in terms of whether

$\lim_{n\to\infty}(a_n) \neq 0$ or $\lim_{n\to\infty}(a_n) = 0$

The previous notes about the limit of the sequence of terms is one piece of information we can sometimes use to determine whether a series converges or diverges. Let us call it <u>**method 1**</u>.

Method 7: The Ratio Test : [many students like this test - it is easy to execute]

If $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0 \quad \forall\ k \in \mathbb{N}$

then (1) $\sum_{k=1}^{\infty} a_k$ converges provided $\lim_{k\to\infty}\left(\frac{a_{(k+1)}}{a_k}\right) < 1.$

while (2) $\sum_{k=1}^{\infty} a_k$ diverges provided $\lim_{k\to\infty}\left(\frac{a_{(k+1)}}{a_k}\right) > 1.$

whereas (3) we ***do not know*** what $\sum_{k=1}^{\infty} a_k$ does when $\lim_{k\to\infty}\left(\frac{a_{(k+1)}}{a_k}\right) = 1.$

Note: if (3) happens we must use another method !!!!!!!

Method 8: The Root Test

If $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0 \quad \forall\ k \in \mathbb{N}$

then (1) $\sum_{k=1}^{\infty} a_k$ converges provided $\lim_{k\to\infty}\sqrt[k]{a_k} < 1.$

while (2) $\sum_{k=1}^{\infty} a_k$ diverges provided $\lim_{k\to\infty}\sqrt[k]{a_k} > 1.$

whereas (3) we *do not know* what $\sum_{k=1}^{\infty} a_k$ does when $\lim_{k\to\infty}\sqrt[k]{a_k} = 1.$

Note: if (3) happens we must use another method !!!!!!!

<u>Method 9: The Alternating Series Test</u>

If $\sum_{k=1}^{\infty} a_k$ is a series such that if $a_k = (-1)^k b_k$ and we know something about $S_B = \{b_n\}_{n=1}^{\infty}$,

then we can possibly tell if $\sum_{k=1}^{\infty} a_k$ converges or diverges.

$\sum_{k=1}^{\infty} a_k$ converges if

1) $\lim_{k\to\infty}(b_k) = 0$ ***and***

2) $b_{k+1} \le b_k \quad \forall k$

$\sum_{k=1}^{\infty} a_k$ diverges if condition 1 **or** 2 is not satisfied.

Finally, the key to understanding all of these methods is to do multiple drill exercises and to consider drill exercises such that the method is not specified so you can determine if you have understood these methods or not. Further, you must be careful when doing proofs not to:

1) assume the conclusion and
2) write out a careful organised argument.

Also, what we do in freshmen level calculus courses (Calculus I, II, and III) are more computational than a course such as Real Analysis (called Advanced Calculus here at Kutztown (Math 351)) or other higher courses. However, it is important to get a firm understanding of the concepts, too. So, consider the following:

Food for thought (a taste of what is discussed in Math 351):

<u>Claim</u>: If $\sum_{i=1}^{\infty} a_i$ converges, then $\forall\ M \in \mathbb{N} \quad \sum_{i=M}^{\infty} a_i$ converges

If $\exists\ M \in \mathbb{N} \quad \sum_{i=M}^{\infty} a_i$ diverges, then $\sum_{i=1}^{\infty} a_i$ diverges.

Produce an argument as to why you think the claim is true or produce an argument as to why you think the claim is false.

MATH 273 CALCULUS III FALL 2008 WORKSHEET III NAME:_______________________
(please print legibly)

1. Consider the following series. Show each either converges or diverges. Cite the test used.

A. $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$

B. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

C. $\sum_{n=2}^{\infty} \frac{n-1}{n^6-1}$

D. $\sum_{n=1}^{\infty} \frac{6^n}{7^n+1}$

E. $\sum_{n=0}^{\infty} \frac{(n+1)!}{n!}$

F. $\sum_{n=0}^{\infty} \frac{n!}{(n+1)!}$

G. $\sum_{n=1}^{\infty} \frac{2^n}{3^n n^2}$

G. $\sum_{n=1}^{\infty} (-1)^{(2n-1)} \frac{(\frac{\pi}{4})^n}{(2n-1)!}$

H. $\sum_{n=1}^{\infty} \frac{2^{(n-1)}}{(n-1)!}$

I. $\sum_{n=1}^{\infty} \sin(\frac{(2n-1)\pi}{2})(\frac{n+1}{n^2+1})$

J. $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$

K. $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+13n^2+26n+15}$

L. $\sum_{n=7}^{\infty} \frac{n!}{(n+1)!}$

M. $\sum_{n=8,073}^{\infty} \frac{n-1}{n^6-1}$

2. Consider $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ Show it converges and find bounds for $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$.

3. Consider $\sum_{n=1}^{\infty} \frac{5}{n^2+7n+6}$ Show it converges and find the sum.

4. Consider $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ Prove by a *Direct Comparison Test* that $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges.

5. Consider $\{g_n\}_{n=1}^{\infty} \ni \quad g_n = \dfrac{3}{n(n+1)}$

A). Find the first four terms of $\{g_n\}_{n=1}^{\infty}$ in simplified form.

B). Now consider $\sum_{n=1}^{\infty} g_n \ni \quad g_n = \dfrac{3}{n(n+1)}$. Does $\sum_{n=1}^{\infty} g_n$ converge or diverge? If it converges, find the sum.

C) Let P be the sequence of partial sums from $\sum_{n=1}^{\infty} g_n$ Find the first four terms of the sequence of partial sums (reduce numerical results)

6. Consider the series $\sum_{k=1}^{\infty} \dfrac{5(-1)^k}{(k+1)^2}$. A. Show it converges.

B Assume $\sum_{k=1}^{\infty} \dfrac{5(-1)^k}{(k+1)^2}$ converges to A where A is a real number. Since it converges, find the *least* natural number n such that the partial sum P_n of $\sum_{k=1}^{\infty} \dfrac{5(-1)^k}{(k+1)^2}$ approximates A within $.001$

(e.g.: $|A - P_n| \le .001$)

4. Consider $\sum_{k=1}^{\infty} 4(-1)^k (k+1)^{-1}$ Test for divergence, conditional convergence, or absolute convergence of the series and cite the test(s) used.

End, Handout 5