HANDOUT IV
MATH 273 CALCULUS III
DR. MCLOUGHLIN'S
HANDY DANDY GUIDE TO
INFINITE SERIES
PART I
INCLUDING: TESTS FOR CONVERGENCE OR DIVERGENCE

A few reminders:

A. If Γ is a series, say $\sum_{i=1}^{\infty} a_i$, and $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \quad \ldots,$

then $S_A = \{a_n\}_{n=1}^{\infty}$ is called the **sequence of terms** *from the series* Γ , and
$S_A = a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \quad \ldots$

whereas,

$P_A = a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4 \quad \ldots$ is called the **sequence of partial sums** of the series Γ and

Let us rewrite P_A as $P_A = p_1, \; p_2, \; p_3, \; \ldots$ where

$p_1 = a_1,$
$p_2 = a_1 + a_2,$
$p_3 = a_1 + a_2 + a_3$
.
.
.
$p_m = a_1 + a_2 + a_3 + \ldots + a_m$
.
.
.

Now, by definition Γ <u>converges</u> iff there exists a real number, A, such that $A = \sum_{i=1}^{\infty} a_i$

Γ does not converges (<u>diverges</u>) iff there does not exists a real number, B, such that $B = \sum_{i=1}^{\infty} a_i$

we say when Γ diverges it 'blows up,' 'goes to infinity,' goes to negative infinity,' 'it jumps back and forth and doesn't go to **<u>a</u>** number,' or the sum does not exist.

<u>Definition</u>: Let Γ be the series, say $\sum_{i=1}^{\infty} a_i$, and $\{p_k\}_{k=1}^{\infty}$ be the sequence of partial sums associated with Γ, then $\lim_{k\to\infty}(p_k) = \sum_{i=1}^{\infty} a_i$ when $\sum_{i=1}^{\infty} a_i$ converges (the limit of the sequence of partial sums is the same as sum of the series). When $\lim_{k\to\infty}(p_k)$ does not exist, we say $\sum_{i=1}^{\infty} a_i$ diverges.

B. One of our favourite series is $\sum_{j=1}^{\infty} \frac{1}{j}$ It is called the **harmonic series** and diverges (by the integral test, for one).

C. Also remember for the sequence of terms, $S_A = \{a_n\}_{n=1}^{\infty}$

Suppose $\lim_{n\to\infty}(a_n) = 0$.

This means ***nothing*** in terms of whether $\sum_{i=1}^{\infty} a_i$ converges or not!

Suppose $\sum_{i=1}^{\infty} a_i$ converges, then it must be the case that $\lim_{n\to\infty}(a_n) = 0$.

Suppose $\lim_{n\to\infty}(a_n) \neq 0$, then it must be the case that $\sum_{i=1}^{\infty} a_i$ diverges.

And suppose $\sum_{i=1}^{\infty} a_i$ diverges. This means ***nothing*** in terms of whether

$\lim_{n\to\infty}(a_n) \neq 0$ or $\lim_{n\to\infty}(a_n) = 0$

The previous notes about the limit of the sequence of terms is one piece of information we can sometimes use to determine whether a series converges or diverges. Let us call it <u>**method 1**</u>.

Some of the others are:

Method 2: The Geometric Series Test

if $\sum_{k=1}^{\infty} a_k$ is a series such that $a_k = c(r^{k-1})$, $k \in \mathbb{N}$, $c \in \mathbb{R}$

and $|r| < 1 \Rightarrow$ the series converges. Further, it converges to $\frac{c}{1-r}$.

$|r| \geq 1 \Rightarrow$ the series diverges

Method 3: The Telescoping Series Test[1]

If $\sum_{k=1}^{\infty} a_k$ is a series such that a_k can be decomposed using partial fractions into so "nice" sum of fractions which telescope, then the series converges if the infinite sum reduces to a form that is a finite sum. This method is not a method of high choice simply because it so nebulous in its application / definition. Most of the time, I would say, it is to be used if you are looking to compute the actual value of the series.

Method 4: The Integral Test

If $\sum_{k=1}^{\infty} a_k$ is a series such that $\exists\, M \in \mathbb{N} \ni \forall\, k \geq M \quad a_k = f(k)$ and $f\colon D \longrightarrow \mathbb{R}$ where $[M,\infty) \subseteq D$ where $f(x)$ is a continuous, non-increasing function $\forall \quad x \in [M,\infty)$

and if $\int_M^{\infty} f(x)dx$ exists (i.e. is L (a real number)), then $\sum_{k=1}^{\infty} a_k$ converges

whilst if $\int_M^{\infty} f(x)dx$ 'does not exist,' or 'goes to ∞,' then $\sum_{k=1}^{\infty} a_k$ diverges.

[1] Note: There is a more detailed discussion in the telescoping handout.

Method 4A (really a corollary of the integral test): bound on a convergent series via the integral test.

If $\sum_{k=1}^{\infty} a_k$ is a series such that $\forall \quad k \geq 1 \quad a_k = f(k)$ and f: $D \longrightarrow \mathbb{R}$ where $[M, \infty) \subseteq D$ where $f(x)$ is a continuous, non-increasing function $\forall \quad x \in [1, \infty)$

and if $\int_1^{\infty} f(x)dx$ exists (i.e. = L (a real number)), then $\sum_{k=1}^{\infty} a_k$ converges

and further $0 \leq \left(\sum_{k=1}^{\infty} a_k\right) \leq \left(a_1 + \int_1^{\infty} f(x)dx\right)$ (a very nice result since we have a bound).

Corollary 4A: the p - series trick

If $\sum_{k=1}^{\infty} a_k$ is a series such that $a_k = \frac{c}{k^p} \ni c \in \mathbb{R}, \ p \in \mathbb{R} \wedge p > 0$

then when $|p| > 1 \quad \Rightarrow$ the series converges.

$|p| \leq 1 \quad \Rightarrow$ the series diverges

Important note: **p-series citation is something one uses within the context of a comparison or limit comparison test to show that he knows that the series he picks converges or diverges (as he claims). Free-standing it is not a test for convergence or divergence and should not be used as a free-standing test but only contextually as noted above[2].**

[2] Even though the authors of many texts claim it is a test - - it is not *unless* one proves it, then uses it. If one desires to take the time on a test and prove it (correctly, of course), fine then it can be used. I do not expect a student in Calculus III to prove such therefore I suggest most adamantly to heed the warning.

<u>Method 5: The Direct Comparison Test</u>

If $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0$ $\forall$ $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ is a series that *you choose* which is convergent (you must justify the assertion it converges[3]) such that you show

$\exists$ $M \in \mathbb{N} \ni \forall$ $k \geq M$ $b_k \geq a_k$ then since $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=1}^{\infty} a_k$ converges.

Whereas, if $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0$ $\forall$ $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ is a series that you choose which is divergent (you must justify the assertion it converges[4]) such that you show

$\exists$ $M \in \mathbb{N} \ni \forall$ $k \geq M$ $b_k \leq a_k$ then since $\sum_{k=1}^{\infty} b_k$ diverges, $\sum_{k=1}^{\infty} a_k$ diverges.

<u>Method 6: The Limit Comparison Test</u> [many students like this test - it is easy to execute]

If $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0$ $\forall$ $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ is a series that you choose which is convergent (you must justify the assertion it converges[5])
such that $\lim_{k\to\infty}\left(\frac{a_k}{b_k}\right)$ exists

(i.e. = L (a real number)) then since $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=1}^{\infty} a_k$ converges.

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[3] Here justification for the assertion is by any other test or citation of the p-value for a p-series. The use of the p-series assertion is justified by the fact that you are not proving $\sum_{k=1}^{\infty} b_k$ converges or diverges , but $\sum_{k=1}^{\infty} a_k$ converges or diverges. Please note the subtlety of this point.

[4] Ibid.

[5] Ibid.

Whereas, if $\sum_{k=1}^{\infty} a_k$ is a series $\ni$ $a_k > 0$ $\forall$ $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ is a series that you choose which is divergent (you can prove the assertion it converges) such that $\lim_{k \to \infty} (\frac{a_k}{b_k})$ exists ***and is non-zero*** (i.e. = L > 0 (a real number)) **OR** does not exit : 'blows up,' or 'goes to ∞' then, since $\sum_{k=1}^{\infty} b_k$ diverges, $\sum_{k=1}^{\infty} a_k$ diverges.

End, Handout 4