MATH 273 CALCULUS III
DR. MCLOUGHLIN
HANDOUT 2
CONCEPTS ABOUT SEQUENCES

Recall the definition of a sequence[1]: Let us consider the well defined sequence such that $f\colon \mathbb{N} \longrightarrow \mathbb{R}$ where f is defined by $\{f_n\}_{n=1}^{\infty}$ and it is a **sequence of real numbers** *from* $\mathbb{N}$ to $\mathbb{R}$ *or simply a sequence of real numbers* $\{f_n\}_{n=1}^{\infty} = f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, \ldots, f_n, \ldots$

Example 1: $\{f_n\}_{n=1}^{\infty} = \left\{\dfrac{1}{n^2+1}\right\}_{n=1}^{\infty}$

From Maple (see 273-2008-09-13.mws)

Sequences in Maple
Dr. McLoughlin
Fall 2008

```
> with(student):
```

You have to load a package, I will load the package student. I am going to produce the first five terms of the sequence in Example 1 of Handout 2.

```
> seq( 1/(i^2 + 1), i=1..5 );
>
```

$$\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}$$

Tah-dah! Big whoop. We can get even more terms . . . for example:

```
> seq( 1/(i^2 + 1), i=1..50 );
```

$$\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}, \frac{1}{37}, \frac{1}{50}, \frac{1}{65}, \frac{1}{82}, \frac{1}{101}, \frac{1}{122}, \frac{1}{145}, \frac{1}{170}, \frac{1}{197}, \frac{1}{226}, \frac{1}{257}, \frac{1}{290}, \frac{1}{325}, \frac{1}{362}, \frac{1}{401},$$
$$\frac{1}{442}, \frac{1}{485}, \frac{1}{530}, \frac{1}{577}, \frac{1}{626}, \frac{1}{677}, \frac{1}{730}, \frac{1}{785}, \frac{1}{842}, \frac{1}{901}, \frac{1}{962}, \frac{1}{1025}, \frac{1}{1090}, \frac{1}{1157}, \frac{1}{1226}, \frac{1}{1297},$$
$$\frac{1}{1370}, \frac{1}{1445}, \frac{1}{1522}, \frac{1}{1601}, \frac{1}{1682}, \frac{1}{1765}, \frac{1}{1850}, \frac{1}{1937}, \frac{1}{2026}, \frac{1}{2117}, \frac{1}{2210}, \frac{1}{2305}, \frac{1}{2402},$$
$$\frac{1}{2501}$$

We can opine (see) it is bounded and monotonic (it is decreasing) & the limit is zero. This Maple stuff PROVES NOTHING nor does it SHOW it is bounded and monotonic in the mathematical sense of showing something is just below proving it. Remember showing something is rough-sketch proving something or rough-sketch disproving something.

```
> limit(1/(i^2 + 1), i=infinity);
```

$$0$$

---

[1] An infinite sequence (finite sequences are uninteresting to a Calculus scholar. Without loss of generality, the domain of the sequence is the natural numbers and the codomain is the real numbers.

What we need to understand is

Theorem 1: Let f: $\mathbb{N} \longrightarrow \mathbb{R}$ (where f is defined by $\{f_n\}_{n=1}^{\infty}$) be a well defined sequence $\{f_n\}_{n=1}^{\infty}$ is **decreasing** *if and only if* $f_{(n)} > f_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Notice Example 1 show this nicely $(\frac{1}{2} > \frac{1}{5}) \wedge (\frac{1}{5} > \frac{1}{10}) \wedge (\frac{1}{10} > \frac{1}{26}) \wedge (\frac{1}{26} > \frac{1}{37}) \wedge \ldots$

But no matter how many exemplars from the sequence you produce to look at it does not prove or show[2] it.

Theorem 2: Let f: $\mathbb{N} \longrightarrow \mathbb{R}$ (where f is defined by $\{f_n\}_{n=1}^{\infty}$) be a well defined sequence $\{f_n\}_{n=1}^{\infty}$ is **non-increasing** *if and only if* $f_{(n)} \geq f_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Theorem 3: Let f: $\mathbb{N} \longrightarrow \mathbb{R}$ (where f is defined by $\{f_n\}_{n=1}^{\infty}$) be a well defined sequence $\{f_n\}_{n=1}^{\infty}$ is **constant** *if and only if* $f_{(n)} = f_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Theorem 4: Let f: $\mathbb{N} \longrightarrow \mathbb{R}$ (where f is defined by $\{f_n\}_{n=1}^{\infty}$) be a well defined sequence $\{f_n\}_{n=1}^{\infty}$ is **non-decreasing** *if and only if* $f_{(n)} \leq f_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Theorem 5: Let f: $\mathbb{N} \longrightarrow \mathbb{R}$ (where f is defined by $\{f_n\}_{n=1}^{\infty}$) be a well defined sequence $\{f_n\}_{n=1}^{\infty}$ is **increasing** *if and only if* $f_{(n)} < f_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Example 2: $\{f_n\}_{n=1}^{\infty} = \begin{cases} \frac{1}{n+1} & \exists j \in \mathbb{N} \ni n = 2j-1 \\ \frac{1}{n} & \exists j \in \mathbb{N} \ni n = 2j \end{cases}$

Notice $\{f_n\}_{n=1}^{\infty}$ is **non-increasing.**

[2] Remember showing something is rough-sketch proving something or rough-sketch disproving something.