Set Theory Math 255
Handout 4
Standard Sets to Memorise
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$\mathbb{N}$ is the set of natural numbers, it is $\{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{N}^*$ is the set of natural numbers and the special number zero, it is $\{0, 1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ is the set of integers, it is $\{ 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \}$

$\mathbb{Q}$ is the set of rationals, it is $\{ \frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{Z}, \wedge n \neq 0 \}$

$\mathbb{R}$ is the set of reals it is $(-\infty, \infty)$

$\mathbb{I}$ is the set of irrationals, it contains all the elements of the reals that are not rational.

$\mathbb{C}$ is the set of complex numbers, it is $\{(m + ni) \mid m \in \mathbb{R},\ n \in \mathbb{R} \wedge i \text{ is defined as } \sqrt{-1}\}$

<u>Definition 1.1</u>: Let A be a well defined subset of the reals, then $A^{(\neg(-))}$ is the set of elements in A such that they are non-negative.

<u>Definition 1.2</u>: Let A be a well defined subset of the reals, then $A^{(+)}$ is the set of elements in A such that they are positive.

<u>Definition 1.3</u>: Let A be a well defined subset of the reals, then $A^{(\neg(+))}$ is the set of elements in A such that they are non-positive.

<u>Definition 1.4</u>: Let A be a well defined subset of the reals, then $A^{(-)}$ is the set of elements in A such that they are negative.

$\mathbb{N}^* = \mathbb{Z}^{(\neg(-))}$ is the set of non-negative integers, it is { 0, 1, 2, 3, 4, 5 . . . }

$\mathbb{N} = \mathbb{Z}^{(+)}$ is the set of positive integers, it is {1, 2, 3, 4, 5 . . . }

Notice $\mathbb{Z}^{(-)}$ is the set of negative integers, it is {-1, -2, -3, -4, -5 . . . }

Notice $\mathbb{Z}^{(\neg(+))}$ is the set of non-positive integers, it is {0, -1, -2, -3, -4, -5 . . . }

Some of the relationships:

$$\mathbb{N} \subseteq \mathbb{N}^* \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$$\mathbb{I} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

If you need more remediation on these sets, go to a reference. I note that the following comes from my e-book, The Principles of Mathematics, that you are free to read on the web or download and use at your convenience.

From page 44:

There are some special standard sets and symbols for them to denote sets that we use often. The sets under discussion are formulated from the real line (are either points of the line or are generalisations of the line. You knowledge of high school geometry will, no doubt, be of use in making concrete these abstract ideas that follow. As previously stated in the text, one of the most basic of sets is called the **natural numbers**. It has been with us since antiquity, and we will denote it as $\mathbb{N}$ or $\mathrm{I\!N} = \{1, 2, 3, 4, \ldots\}$ where the set never ends and includes all the whole or counting numbers that the student learnt in kindergarten or before. We shall denote the set of natural numbers along with zero as the set $\mathbb{N}^* = \mathbb{N}_{\aleph} = \{0, 1, 2, 3, 4, \ldots\}$.[1] We shall denote the set $\{1\}$ as $\mathbb{N}_1$, the set $\{1, 2\}$ as $\mathbb{N}_2$, the set $\{1, 2, 3\}$ as $\mathbb{N}_3$, and so forth so that $\mathbb{N}_k = \{1, 2, 3, 4, \ldots, (k - 1), k\}$. This definition is known as a **recursive definition** since we are inductively defining a myriad of sets at once; the three dots signify that the enumeration of the elements continues.[2] Likewise, we shall denote the set $\{0, 1\}$ as $\mathbb{N}^*_1$, the set $\{0, 1, 2\}$ as $\mathbb{N}^*_2$, the set $\{0, 1, 2, 3\}$ as $\mathbb{N}^*_3$, and so forth so that $\mathbb{N}^*_p = \{0, 1, 2, 3, \ldots, (p - 1), p\}$.

Another of the most basic of sets is called the **integers**. It has been with us quite a long time (the people of India invented the symbol of zero and were really the first to use it and negative numbers (in fact the number system that we use is of course the Hindu-Arabic number system since the Hindus created it, the Arabs adopted it and brought it west); it should also be noted that the Mayans also invented a zero independent of the Hindus). We will denote the set of integers as $\mathbb{Z}$, $\mathbf{Z}$, or $\mathcal{Z}$ such that $\mathbf{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$.

Generalising from the integers, we have the **rational numbers**. The rationals are denoted as $\mathbb{Q}$ or $\mathcal{Q}$ such that $\mathbb{Q} = \{x \mid x = \frac{a}{b} \text{ where } a \in \mathbf{Z},\ b \in \mathbf{Z},\ \wedge\ b \neq 0\}$. This statement is read as the rationals are the set consisting of elements x such that x is equal to a divided by b where a is an integer, b is an integer, and b is not zero. Thus, the symbol "|" in this context means **such that**. Indeed, as I type this opus I am getting very tired of writing the words, "such that," (I suppose it is an occupational hazard, I think we mathematicians are a lazy lot so we have invented many symbols to create a short hand to ease the amount of words necessary to communicate). A more general symbol for **such that** is, "∍," and will be used liberally from this point onward. Typically it is not used in the notation inside the braces for a set only as a free-

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[1] The symbol ℵ is not a stylised "x" (though seemingly written as such) it is aleph the first letter of the Hebrew alphabet.

[2] The three dot symbol (the ellipsis) is most problematic. Many students think that the ellipsis establishes a pattern. It does not. Consider π. π is not 3.14. It is not 3.141592. It is 3.14159265359 . . . (the decimal does not repeat or have a pattern); thus, the three dot symbol means on and on but not *necessarily* in a pattern.

standing symbol. However, it is not incorrect to use it. Therefore, it is technically correct to write

$$\mathbb{Q} = \{x \ni x = \frac{a}{b} \text{ where } a \in \mathbb{Z}, b \in \mathbb{Z}, \wedge\ b \neq 0\}.$$

We can simplify the definition of the rationals so that it does not have to depend solely on the integers. Note the definition $\mathbb{Q} = \{x : x = \frac{a}{b} \text{ where } a \in \mathbb{Z}, b \in \mathbb{N}\}$ is logically equivalent to the previous definition of the rationals and in this case the colon means **such that** (which is the third of the standard notations for such that).

Generalising from the rational numbers, we have the **real numbers**. However, this is axiomatically executable, but practically most difficult to do in a basic introduction to sets. Therefore, we shall consider the set of reals from a geometric standpoint. The set of real numbers are denoted as $\mathbb{R}$, $\mathbf{IR}$, or $\mathcal{R}$ such that $\mathbb{R} = \{x \mid x \text{ is a point on the line}\}$. One could also define the reals from a sequential (or decimal) perspective by defining the reals to be $\mathbb{R} = \{x \mid$ x is a number where x is an integer followed by a decimal and then a sequence of digits where each digit belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$. Yes, this is a rather cumbersome definition, but one can prove that the sequential definition and the geometric definitions are equivalent. One other way to write $\mathbb{R}$ is by writing $(-\infty, \infty)$. The symbols ∞ and $-\infty$ are **not numbers**, they simply represent that the line goes on ad infinitum to the left in the case of $-\infty$ and to the right in the case of ∞.

Note, gentle reader that I skipped another standard set; that is because in a basic introduction to sets it is oft easier to 'jump' up to the reals, then return back to describe another set. That set is, of course, the irrationals. By the very nature of its name one can understand that it is composed of elements that are not rational. But, recall our discussion of the need for defining a domain of discourse or a universe. To say what something is not presupposes that everything has been specified! Putting it another way saying that the irrationals are numbers that are not rational is wrong since $\sqrt{-1}$ is a number that is not rational; but, it is also not irrational. Thus, the set of **irrational numbers** will be denoted as $\mathbf{II}$, $\mathbb{I}$, or $\mathcal{I}$ such that $\mathbf{II} = \{x \mid x \in \mathbb{R}, x \notin \mathbb{Q}\}$. So, the irrationals are the set of all real numbers that are not rational.

Note that this is a definition of something such that it is defined by what it is not. This definition by negation is oft quite useful; but one must understand what the first thing is (a real number) and the second thing is (a rational number) in order to understand what the third thing is (an irrational number) by way of what it isn't.

Constructing sets from this perspective leaves us with the feeling that all is known and specified previously, but consider people before they thought of these sets. Consider the man or woman who first thought of these sets. Is it not rather astonishing to think that such was not known nor conceived, but someone thought of these ideas first? A facile way of considering the wonderful experience it must have been is by specify a set that is not contained within the set of reals. Laying aside the important principle of consideration of the specification of a universe for the moment, let us look at the idea of set from a construction standpoint. Note that we did this by specifying $\mathbf{IN}$ then $\mathbb{Z}$ then $\mathbb{Q}$ then $\mathbf{IR}$. We deviated from it when we specified $\mathbf{II}$.

Let us do it again. Define $\mathbb{C}$ or $\mathcal{C}$ to be the **complex numbers** such that

$\mathbb{C} = \{ x \mid x = a + bi \text{ where } a \in \mathbb{R}, b \in \mathbb{R}, \wedge\ i = \sqrt{-1} \}$. Note that the complex numbers are really the plane (the horizontal axis consists of all points corresponding to the real part of the complex number and the vertical axis consists of all points corresponding to the i ('imaginary') part). So, why do so many people call complex number imaginary numbers when they correspond to things not so imagined but real? Indeed, if one argues that the reals correspond to real things and the complex numbers are 'not real' then why are both simply concepts corresponding to geometric forms (which recall are axiomatically given [point, line, and plane]). So, how real are the reals and imaginary are the complex numbers? But, I digress.[3]

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[3] Hopefully this rudimentary waxing upon the nature of mathematics and ideas is not something which causes you, dear reader, to be distressed. If it does cause you discomfort, I am sorry. It might indicate (dare I say?) a lack of enthusiasm for studying mathematics. If this is the case, then perhaps one should spend time asking oneself why exactly he is majoring in mathematics.