MATH 255 SET THEORY - DR. MCLOUGHLIN'S CLASS
HANDOUT 15
AN INTRODUCTION TO ORDINALITY

Common custom characterises an ordinal number as an adjective which describes the numerical position of an object; e.g., first, second, third, etc. as opposed to the *common* custom that characterises a cardinal number as a number used in counting such as
0, 1, 2, 3, ...

Object list: a, b, c, d , ...
Ordinal: 1^{st}, 2^{nd}, 3^{rd}, 4^{th} , . . .

It is through custom that we understand what is meant; however, rigor is advised.

Recall a relation R on a non-empty set A is a partial order iff

(1) $(x, x) \in R \ \forall x \in A$
(2) $(x, y) \in R \Rightarrow (y, x) \notin R$ when $x \neq y$
(3) $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$

Def. 15.1: Let relation R on a non-empty set A be a partial order. We will write $x \underline{\ll} y$ to mean $(x, y) \in R$

Def. 15.2: The pair the set A and $\underline{\ll}$ is oft denoted as $(A, \underline{\ll})$ to represent a partially ordered system.

Recall the following: Let R be a relation on a non-empty set A be a partial order

(1) $x \in A$ is a minimal element if there does not exist a $y \in A$ such that $y \underline{\ll} x$
(2) $x \in A$ is the least element if there does not exist a $y \in A$ such that $y \underline{\ll} x$ and $x \underline{\ll} y \ \forall y \in A$
(3) $x \in A$ is a maximal element if there does not exist a $y \in A$ such that $x \underline{\ll} y$
(4) $x \in A$ is the greatest element if there does not exist a $y \in A$ such that $x \underline{\ll} y$ and $y \underline{\ll} x \ \forall y \in A$

Recall we defined $\mathbb{N}^* = \{0, 1, 2, . . . \} = \mathbb{N} \cup \{0\}$

Def. 15.3: Let relation R on a non-empty set A be a partial order. If x $\underline{\ll}$ y exor x $\underline{\ll}$ y $\forall$ x $\in$ A $\wedge$ y $\in$ A, then the partial order is called a total order. We say $\forall$ x $\in$ A $\wedge$ y $\in$ A, x and y are comparable.

Example 15.1: Let A = $\mathbb{R}$ and $\underline{\ll}$ be $\le$. ($\mathbb{R}$, $\le$) is a totally ordered system.

Exercise 15.1: Let A = $\mathbb{N}$ and $\underline{\ll}$ be $\le$. Determine if ($\mathbb{N}$, $\le$) is a totally ordered system.

Exercise 15.2: Let A = $\mathbb{Z}$ and $\underline{\ll}$ be $\le$. Determine if ($\mathbb{Z}$, $\le$) is a totally ordered system.

Def. 15.4: Let the relation $\underline{\ll}$ be on a non-empty set A. Let (A, $\underline{\ll}$) be a partially ordered system. $\underline{\ll}$ is called a well ordering iff every non-empty subset of A has a least element.

Def. 15.5: Let the relation $\underline{\ll}$ be on a non-empty set A. Let (A, $\underline{\ll}$) be a partially ordered system. (A, $\underline{\ll}$) is called a well ordered system iff every non-empty subset of A has a least element.

Example 15.2: Let A = $\mathbb{N}$ and $\underline{\ll}$ be $\le$. ($\mathbb{N}$, $\le$) is a well ordered system.

Exercise 15.3: Let A = $\mathbb{R}$ and $\underline{\ll}$ be $\le$. Determine if ($\mathbb{R}$, $\le$) is a well ordered system.

Exercise 15.4: Let A = $\mathbb{Z}$ and $\underline{\ll}$ be $\le$. Determine if ($\mathbb{Z}$, $\le$) is a well ordered system.

Exercise 15.5: Let A = $\mathbb{N}^*$ and $\underline{\ll}$ be $\le$. Determine if ($\mathbb{N}^*$, $\le$) is a well ordered system.

Recall form the axioms of set theory:

Axiom 2) The Axiom of Null There exists a set with no elements, call it $\varnothing$.

So, let us begin construction of the ordinals.

Define 0 as the ordinal associated with $\varnothing$.

Def. 15.6: Let A be a set. The successor of A, denoted S(A), is the set $A \cup \{A\}$.

Example 15.3: $S(\varnothing) = \varnothing \cup \{\varnothing\}$.

Define 1 as the ordinal associated with $S(\varnothing) = \varnothing \cup \{\varnothing\} = \{\varnothing\}$.

Note $\varnothing \subseteq \{\varnothing\} = S(\varnothing)$.

Example 15.4: $S(S(\varnothing)) = S(\varnothing \cup \{\varnothing\}) = S(\{\varnothing\}) = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$

Define 2 as the ordinal associated with $S(S(\varnothing)) = \{\varnothing, \{\varnothing\}\}$

Note $\varnothing \subseteq \{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$, so $\varnothing \subseteq S(\varnothing) \subseteq S(S(\varnothing))$.

Notice the successor system is similar, but not the *same* as the power set system because:
$\varnothing \subseteq S(\varnothing) \subseteq S(S(\varnothing))$ and $\varnothing \subseteq \wp(\varnothing) \subseteq \wp(\wp S(\varnothing))$ which is
$\varnothing \subseteq \{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$ and $\varnothing \subseteq \{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$
but,

Example 15.5: $S(S(S(\varnothing))) = S(S(\{\varnothing\})) = S(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} =$
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

Define 3 as the ordinal associated with $S(S(S(\varnothing))) = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

So, the successor system is not the *same* as the power set system because:
$\varnothing \subseteq S(\varnothing) \subseteq S(S(\varnothing)) \subseteq S(S(S(\varnothing)))$ and $\varnothing \subseteq \wp(\varnothing) \subseteq \wp(\wp S(\varnothing)) \subseteq \wp(\wp(\wp S(\varnothing)))$ which is
$\varnothing \subseteq \{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\} \subseteq \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ & $\varnothing \subseteq \{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\} \subseteq \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}$

The successor system gives us, 0, 1, 2, 3, etc. whilst the power system gives us 0, 1, 2, 4, etc.

Exercise 15.6: Find $S(S(S(S(\varnothing))))$

Exercise 15.7: Find $S(S(S(S(S(\varnothing)))))$

Exercise 15.8: Find $S(S(S(S(S(\varnothing))))))$

Def. 15.6: Let $\mathfrak{I} = \{ \emptyset,\ S(\emptyset),\ S(S(\emptyset)),\ S(S(S(\emptyset))),\ S(S(S(S(\emptyset)))),\ S(S(S(S(S(\emptyset))))),\ S(S(S(S(S(S(\emptyset)))))), \ldots \}$

Exercise 15.9: $(\mathfrak{I}, \subseteq)$ is a well ordered system.

Def. 15.7: Let A be a set that is partially ordered by $\ll_A$ and B be a set that is partially ordered by $\ll_B$. An order-isomorphism is a function, *f*, which is bijective from A to B {e, g. : f: A >——>> B } such that $f(a) \ll_B f(b)$ iff $a \ll_A b$.

Exercise 15.10: $(\mathfrak{I}, \subseteq)$ is order isomorphic to $(\mathbb{N}^*, \leq)$.

Exercise 15.11: $(\mathfrak{I}, \subseteq)$ is order isomorphic to $(\mathbb{N}, \leq)$.

Since they are order isomorphic we interchange at this stage $\mathfrak{I}$ and $\mathbb{N}^*$ (sloppy, but bear with me).

Def. 15.8: Let ω_0 be the ordinal of $\mathbb{N}$.

.

Note the similarity to how we defined $\aleph_0$.

However, we have a 'strange' difference.

Consider $S(\mathbb{N}) = \mathbb{N} \cup \{\mathbb{N}\}$

this is the ordinal of $S(\mathbb{N})$,

and it is $\omega_0 + 1 \neq \omega_0$!

But, $\aleph_0 = \aleph_0 + 1$!

So, what is ∞? $\aleph_0$? $\aleph_0 + 1$? $\aleph_1$? ω_0? $\omega_0 + 1$? Or is it just a concept?

You can easily see why the authors of the text denote {0, 1, 2, . . . } as $\mathbb{N}$, and {1, 2, 3, . . . } as $\mathbb{Z}^+$ Whereas, other authors text denote {1, 2, 3, . . . } as $\mathbb{N}$, and {0, 1, 2, . . . } as $\mathbb{N}^*$

Whether cardinality or ordinality the similarities are striking.

Oy, the confusion. Oy, the pain! Oy, isn't Math interesting, fun, and as we can see Set Theory is indeed a deep, rich, and subtle subject.

Indeed, what I hope for you to gain from this handout is that, though you might think (and I would agree) Set Theory is a challenging and difficult course. There is much, much more to Math that what we thought before we entered this course and it is much more intriguing and rich than we have even discussed in this course.

END OF REVISIONS TO 29 APRIL 2003 \\Warning: the rest of this is gibberish - - it has not been completed or revised.

In set theory, an ordinal number (an "ordinal" also the "ordinality") is one of the numbers in Georg Cantor's extension of the natural numbers. An ordinal number is defined as the order type of a well ordered set. Finite ordinal numbers are commonly denoted using arabic numerals, whilst transfinite ordinals as denoted using lower case Greek letters.

It is facile to see that every finite totally ordered set is well ordered. Any two totally ordered sets with elements (for a nonnegative integer) are order isomorphic, and therefore have the same order type (which is also an ordinal number). The ordinals for finite sets are denoted 0, 1, 2, 3, ..., i.e., the integers one less than the corresponding nonnegative integers.

The first transfinite ordinal, denoted ω_0 (or simply ω), is the order type of the set of nonnegative integers. This is the 'smallest' of Cantor's transfinite numbers, defined to be the smallest ordinal number greater than the ordinal number of the whole numbers.

From the Def. 15.of ordinal comparison, is follows that the ordinal numbers are a well ordered set. In order of increasing size, the ordinal numbers are 0, 1, 2, . . . , ω_0, $\omega_0 + 1$, $\omega_0 + 2$, . . . , ω_1 , $\omega_1 + 1$, . . ., ω_2 , . . . , ω_ω , . . .

The notation of ordinal numbers can be a bit counterintuitive as was the notation of the cardinal numbers.

John von Neumann defined a set to be an ordinal number iff

1. If is a member of , then is a proper subset of
2. If and are members of then one of the following is true: , is a member of , or

is a member of .
3. If is a nonempty proper subset of , then
there exists a member of such that the
intersection is empty.

Since for any ordinal , the union is a bigger ordinal , there is no largest ordinal, and the class of all ordinals is therefore a proper class (as shown by the Burali-Forti paradox).

Ordinal numbers have some other rather peculiar properties. The sum of two ordinal numbers can take on two different values, the sum of three can take on five values. The first few terms of this sequence are 2, 5, 13, 33, 81, 193, 449, , , , , , ... The sum of ordinals has either or possible answers for .
is the same as , but is equal to . is larger than any number of the form , is larger than , and so on.

There exist ordinal numbers which cannot be constructed from smaller ones by finite addition, multiplication, or exponentiation. These ordinals obey Cantor's equation.
The first such ordinal is

The next is

then follow , , ..., , , ..., ,
..., , , ..., , , ..., ,
..., , ..., , ..., , ..., , ...,
, ..., , ..., , ..., , ...,
, ...

Ordinal addition, ordinal multiplication, and ordinal multiplication can be defined. Although these Def. 15.s also work perfectly well for order types, this does not seem to be commonly done. There are two methods common used to define operations on the ordinals: one is using sets, and the other is inductively.

Last revised 29 April 2003