HANDOUT 6 ½
THE PEANO AXIOMS
MATH 224 FOUNDATIONS OF MATHEMATICS

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The properties of addition of natural numbers can be derived from a short set of axioms. The axioms are called **the Peano Axioms**:

There exists a set, P, which is defined by the following four axioms.

Axiom 1: There exists a natural number, call it 1, that is not the successor of any other natural number.

Axiom 2: Every natural number has a unique successor. If $k \in P$, then let k' denote the successor of k.

Axiom 3: Every natural number except one is the successor of exactly one natural number.

Axiom 4: If M is **a** set of natural numbers such that
(i) $1 \in M$ and
(ii) for each $k \in P$, if $k \in M$, then $k' \in P$,
then $P = M$.

P, of course is $\mathbb{N}$.

So, the Peano axioms assert the uniqueness of the naturals that this successor property along with the element 1 creates the entirety of the natural numbers. No matter how you name the set (you can call it Ray, or you can call it Jay, . . .) if it has these properties then it really is the naturals.

From these axioms arise the natural numbers by defining what addition by one means.

Definition 3.6.1: For every $k \in \mathbb{N}$, define $k + 1 = k'$.

Then, note inductively, the entire understanding of addition flows from this definition (likewise multiplication, etc.).

THE ARCHIMEDEAN PRINCIPLE OF $\mathbb{N}$

$\mathbb{N}$ is unbounded above; in other words, there does not exist a greatest natural number.

It is worth noting that the Peano axioms and the Archimedean principle provide a justification for the concept of mathematical induction.