

Handout 2 $\frac{1}{2}$
PROOFS, COUNTEREXAMPLES, AND SOME BASIC STANDARD SETS
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Definition 2.5.1: $\mathbb{R} = \{x \mid \text{where } x \text{ is a point on the line}\}$.

Let $U = \mathbb{R}$ for the line and let $U = \mathbb{R} \times \mathbb{R}$ for the plane.

Definition 2.5.2: $\mathbb{N}^* = \{0, 1, 2, 3, \dots, (k-1), k, \dots\}$

Definition 2.5.3: $\mathbb{N} = \{1, 2, 3, \dots, (k-1), k, \dots\}$

Definition 2.5.4: $\mathbb{N}_k^* = \{0, 1, 2, 3, \dots, (k-1), k\}$

Definition 2.5.5: $\mathbb{N}_k = \{1, 2, 3, \dots, (k-1), k\}$

Definition 2.5.6: Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that $a < b$.

Then a **segment** is (a, b) and an **interval** is $[a, b]$.

A **half-segment or half-interval** is $(a, b]$ or $[a, b)$.

Definition 2.5.7: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

Definition 2.5.8: $\mathbb{Q} = \{x \mid x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\}$

Definition 2.5.9: $\mathbb{I} = \{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$

To prove or not to prove, that is the Question.

Definition 2.5.10: Certain principles, ideas, etc. that form the foundation of an area of mathematics that we call fundamental primitive true¹ statements in a system are called axioms. Such statements are agreed upon to be assumed true.

Definition 2.5.11: Statements derived from the axioms are called lemmas, theorems, or corollaries. A theorem is a further statement proven from the axioms. If the theorem is of a "sufficiently" small scale and is used to prove a larger claim, then it is called a lemma (consider it a 'helper' theorem). If a theorem follows so clearly and obviously from another theorem or axiom, then it is called a corollary.

Definition 2.5.12: A proof or mathematical argument is an argument such that it consists of a finite sequence of statements, each of which is either a premise, an axiom, or a previously proven lemmas, theorems, or corollaries, or follows from the premises, axioms, or previously proven theorems *by application of correct modes of inference*(logic). The last statement is the conclusion that follows from the given set of premises.

Note: It shall be the case that a proof is announced by writing 'Proof' before the argument and is closed by writing 'QED' (which means quod erat demonstratum² loosely translated means, 'so it has been demonstrated') at the end. The application of the correct modes of inference is the "map" of the proof and the word Proof and the letters QED are the frame to announce to a reader where the proof begins and where it ends. Furthermore, the claim being proven should be succinctly stated (otherwise oft one will be left with a very confused audience).

¹Technically assumed to be true or agreed to be true since they cannot be *proven* true.

²"Thus it has been proved."

A proof is to be clear, hopefully concise, and correct. A proof is not to be some sort of magic trick where slight of hand, misdirection, etc. are employed. A proof should be understandable (assuming the reader has the requisite background). A magician pulls a rabbit out of a hat because of a concealed compartment, a gambler wins a hand of poker because of an ace up his sleeve, a businessman gets a contract because of "connections." None of these are reasonable concepts for the mathematician. The mathematician objectively seeks the truth. The mathematician justifies his inferences. The mathematician explains his work. And the way the mathematician does these things is by constructing sound proofs or counter-arguments.

Definition 2.5.13: A counterexample is an argument such that it consists of a finite sequence of statements which constructs a demonstration that a claim is false such that the premises all hold true and the consequence is denied.

Note: It shall be the case that a counterexample is announced by writing "Counter-example" before the argument and is closed by writing 'EEF' (which means *exemplum est factum*³)

A proof (to prove a claim true) or 'dis-proof' (to prove a claim is not true) are the highest forms of reasoning known to the human race. We are indeed doing something marvelous when we prove or disprove something. There are weaker forms of argument that we humans put forward; for example:

To show: Lower in 'logical strength' than a proof or mathematical argument is to show something, we use diagrammes, pictures, graphs, sketches or outlines of a proof or some other non-rigorous demonstration of the *seeming* veracity or lack thereof of a claim.

To opine: Lower in 'logical strength' than to prove or to show is when one opines. One opines that a person offers some justification for the opinion that something is true or not. We use diagrammes, pictures, graphs, etc. also but not as thoroughly as when we show something.

To believe: Lower in 'logical strength' than to prove, to show, or to opine is when one believes. One believes something based on being told something or seeing something, perhaps, but in mathematics such is CRAP-ola. One might as well just give up and do something other than math because much that is believed is found to be false when we delve deeply into mathematics.

To feel: Lowest of the low: it isn't proof, showing, opining, or believing. To feel something is WORTHLESS and if one wishes to feel something in math, well, let us just say it won't lead them toward truth or success.

³The example is fact. The counter-example is an example but should be accompanied by a demonstration that it indeed shows the premises fulfilled and the consequent denied.