MATH 224 FOUNDATIONS OF MATHEMATICS
SET THEORY
DR. MCLOUGHLIN'S CLASS
HANDOUT 10
GENERALISED POINT METHOD OF PROOF

Suppose we have a well defined universe U and sets X and Y that are subsets of the universe and we wish to prove that $X \subseteq Y$

We have two ways to prove this under the generalised point method (direct and indirect).

Direct: This usually involves two cases: the null case and non-null case. We must consider the null case if there is no justification (in the hypothesis or premises) for X not being null.

Null case: This is easy. Note that null is a subset of any set (which was proven in class indirectly).

Non-null case: Since the set is not null, it must contain at least one point. Call the generalised point whatever you so desire (x will do). Assume $x \in X$. Analytically (step - by - step) argue until you have shown that $x \in Y$.

Therefore, $X \subseteq Y$.

Indirect: This does ***not*** involve two cases. We suppose that X is *not* a subset of Y. We simply must consider the case that there is an element in X that is not in Y. Call the generalised point whatever you so desire (w will do). Assume $w \in X \wedge w \notin Y$. Analytically (step - by - step) argue until you have a contradiction. Thus, the supposition must be false; therefore, $z \in X \Rightarrow z \in Y$. Therefore, $X \subseteq Y$.

A Well Thought Out and Done Set Theoretic Proof (I hope)

Consider the following claim:

<u>Claim 1</u>: Let U be a well defined universe such that A, B, and C are non-empty subsets of U. It is the case that $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

You must first READ the claim and decide whether or not you think it is true (you may be wrong, but you have to practice this step; it is based on your prior experience and knowledge). It is an inductive step; hence, there is no guarantee that you are right.

Next, after considering claim 1, suppose we think it true. Thinking it is true is not proving it is true. Hence, we need to construct a proof. We must announce it is a proof and frame it at the beginning (Proof:) and at the end (Q. E. D. [Quod Erat Demonstratum]).

<u>Proof</u>: *{comment: justify each line yourself at home!}*

Assume the premises.
Let U be a well defined universe such that A, B, and C are non-empty subsets of U.

{comment: $A \neq \varnothing$, $B \neq \varnothing$, $\wedge\ C \neq \varnothing$ does not imply that $(A \cup B) - C \neq \varnothing$!}

Case 1: Suppose $(A \cup B) - C = \varnothing$.
$\varnothing$ is a subset of any set nested in U.
Thus, $\varnothing \subseteq (A - C) \cup (B - C)$.
Hence, $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

Case 2: Suppose $(A \cup B) - C \neq \varnothing$.
Thus, $\exists\ x \in (A \cup B) - C$.
So, $x \in (A \cup B) \cap C^C$.
Therefore, $x \in (A \cup B) \wedge x \in C^C$.
So, $(x \in A \vee x \in B) \wedge x \in C^C$.
Consequently, $(x \in A \wedge x \in C^C) \vee (x \in B \wedge x \in C^C)$.
As a result, $(x \in (A - C)) \vee (x \in (B - C))$.
Ergo, $x \in (A - C) \cup (B - C))$.
Hence, $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

Q. E. D.

Consider the following claim:

<u>Claim 2</u>: Let *U* be a well defined universe such that A, B, and C are non-empty subsets of U. It is the case that $(A - C) \cup (B - C) \subseteq (A \cup B) - C$

<u>Proof</u>: *{comment: justify each line yourself at home!}*

Assume the premises.
Let U be a well defined universe such that A, B, and C are non-empty subsets of U.

Suppose $(A - C) \cup (B - C) \not\subseteq (A \cup B) - C$

Thus, $\exists\ y \in (A - C) \cup (B - C) \ni y \notin (A \cup B) - C$.
But, $y \in (A \cap C^C) \cup (B \cap C^C)$.
So, $(y \in A \wedge y \in C^C) \vee (y \in B \wedge y \in C^C)$.
Consequently, $(y \in A \vee y \in B) \wedge y \in C^C$.
Therefore, $y \in (A \cup B) \wedge y \in C^C$.
As a result, $y \in (A \cup B) - C$.
However, $y \notin (A \cup B) - C \wedge y \in (A \cup B) - C$ which is a contradiction.
Hence, $\exists\ y \in (A - C) \cup (B - C) \ni y \notin (A \cup B) - C$ is false
which implies $z \in (A \cup B) - C \Rightarrow z \in (A - C) \cup (B - C)$.
Ergo, that $(A - C) \cup (B - C) \subseteq (A \cup B) - C$

Q. E. D.

Note therefore:

<u>Claim 3</u>: Let *U* be a well defined universe such that A, B, and C are non-empty subsets of U. It is the case that $(A - C) \cup (B - C) = (A \cup B) - C$.

<u>Proof</u>: ATP (Assume the premises). Apply claim 1 and claim 2. **Q. E. D.**

Finally, as with all the discussions, examples, proofs, counterexamples, claims, etc. that we encounter; it is my opinion that few can do well in this class through just attending and watching others do the work. I opine that only through doing can we understand and KNOW. Hence, my advise is: "practice, practice, practice."