# MATH 224 FOUNDATIONS
# DR. M. P. M. M. M[C]LOUGHLIN
# HAND-OUT XIV
# FUNCTIONS
# REVISED 2016

Recall:
Let $U$ be a well defined universe. Let $V$ be the well defined universe, $\mathcal{P}(U)$.

Let A be a set. We considered theorems on sets, power sets, $\mathcal{P}$(A), and sub-collections of power sets, $\Omega$.

Then we considered Let $U$ be a well defined universe. Let $V$ be the well defined universe..
Let A be a set within $U$. Let B be a set within $V$.
We defined the universe $W = U \times V$.

**<u>Definition 13.01</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that $A \subseteq U$ whilst $B \subseteq V$. The relation M from A to B is the set $M = \{ (a, b): a \in A, b \in B\}$.

Let R be any subset of $A \times B$, $R \subseteq A \times B$, R is called a relation. Big whoop.!
Please remember for relations
cor(R) = A,
cod(R) = B,
the range, ran(R) $\subseteq$ B such that ran(R) = $\{x \in B: \exists\, a \in A \ni (a, x) \in R\}$.
the domain, dom(R) $\subseteq$ A such that dom(R) = $\{x \in A: \exists\, b \in B \ni (x, b) \in R\}$.

**<u>Definition 14.01</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that $A \subseteq U$ whilst $B \subseteq V$. The relation $f$ from A to B is the set $f = \{ (a, b): a \in A, b \in B\}$. Let the relation $f$ have the following properties:

(1) $A \neq \varnothing$

(2) $\forall\, a \in A\ \exists\, b \in B \ni (a, b) \in f$; and,

(3) $(a, x) \in f \wedge (a, y) \in f \Rightarrow x = y$,

then it is the case that $f$ is called a **function** from A to B (or more precisely a **well-defined function** from A to B) and we symbolise it as $f\colon A \longrightarrow B$ or $A \xrightarrow{f} B$

**<u>Definition 14.01 (Alternate)</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that $A \subseteq U$ whilst $B \subseteq V$, $A \neq \varnothing$, and $B \neq \varnothing$. The relation $f$ from A to B is the set $f = \{ (a, b): a \in A, b \in B\}$. $f$ is a **function** from A to B if and only if:

(1) dom($f$) = cor($f$); and,

(2) $(a, x) \in f \wedge (a, y) \in f \Rightarrow x = y$.

Please note that symbols such as $f$: A $\longrightarrow$B, $f$(x), A $\xrightarrow{f}$ B, etc. do not create a function or guarantee that the referenced $f$ is a function – the only way that a function is a well-defined function is by proving such.

**<u>Definition 14.02</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. A function $f$ : A $\to$ B is **injective** (or one-to-one) iff for every x and for every y in A, $f$(x) = $f$(y) $\Rightarrow$ x = y.

Notation: Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. The function $f$ : A $\to$ B which is injective is denoted as $f$: A $\rightarrowtail$ B

**<u>Definition 14.03</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. A function $f$ : A $\to$ B is **surjective** (or onto B) *iff* for every y in B there exists an x in A such that $f$(x) = y

Notation: Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. The function $f$ : A $\to$ B which is surjective is denoted as $f$: A $\twoheadrightarrow$ B

**<u>Definition 14.04</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. A function $f$ : A $\to$ B is **bijective** *iff* $f$ is injective and surjective.

Notation: Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. The function $f$ : A $\to$ B which is bijective is denoted as $f$: A $\rightarrowtail\!\!\!\!\twoheadrightarrow$ B

**<u>Definition 14.05</u>** : Let the universe $W = U \times \mathbb{R}$ be defined such that A $\subseteq U$ whilst B $\subseteq \mathbb{R}$. The $f$ is called a function from A to B (or more precisely a well-defined function from A to B) is called a **real-valued function** (since the cod($f$) $\subseteq \mathbb{R}$ [one can just as easily let the codomain be the reals]).

**<u>Definition 14.06</u>** : Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined such that A $\subseteq \mathbb{N}$ whilst B $\subseteq \mathbb{R}$. The $f$ is called a **sequence** from A to B (or more precisely a well-defined function from A to B that has domain the naturals) since it is a real-valued function
(since the dom($f$) $\subseteq \mathbb{N}$ and the cod($f$) $\subseteq \mathbb{R}$).

**Examples:**

**Example 14.01**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $f$ on A.

Consider A = {1, 2, 3, 4} Let $f$ = {(1, 1), (1, 2), (2, 2), (3, 2), (4, 3)}
$f$ **is not** a function from A to A since both $(1, 1) \in f \wedge (1, 2) \in f$

**Example 14.02**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $g \subseteq A \times A$.

Consider A = {1, 2, 3, 4} Let $f$ = {(1, 1), (2, 2), (3, 2), (4, 3)}
$g$ **is** a function from A to A since
dom($g$) $\neq \varnothing$;
1 $\in$dom($g$) and 1 $\in$cor($g$); 2 $\in$ dom($g$) and 2 $\in$cor($g$); 3 $\in$ dom($g$) and 3 $\in$cor($g$);
4 $\in$ dom($g$) and 4 $\in$ cor($g$)
$\Rightarrow$ cor(g) $\subseteq$ dom(g) $\Rightarrow$ dom(g) = cor(g) (we had dom(g) $\subseteq$ cor(g) since g was a relation)
whist for each x $\in$ dom($g$) there is a unique y $\in$ ran($g$).

**Example 14.03**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $h$ from A to B

where A = {1, 2, 3, 4} B = {1, 2, 3, 4, 5} Let $h$ = {(1, 1), (2, 4), (3, 2), (4, 3)}
$h$ **is** a function from A to B.
Moreover, $h$ **is** an **injective** function from A to B since (x, y) $\in h$ and (x, z) $\in h \Rightarrow$ y = z
But, $h$ **is not** a surjection from A to B since 5 $\in$ cod($h$) and $5 \notin$ ran ($h$)

**Example 14.04**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $j$ from B to A

where A = {1, 2, 3, 4} B = {1, 2, 3, 4, 5} Let $j$ = {(1, 1), (2, 4), (3, 2), (4, 3), (5, 3)}
$j$ **is** a function from B to A.
Moreover, $j$ **is not** an injective function from B to A since (4, 3) $\in j$ and (5, 3) $\in j$ *but* $4 \neq 5$
But, $j$ **is** a **surjective** function from B to A since cod($j$) = ran ($j$)

**Example 14.05**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $k$ from A to A

where A = {1, 2, 3, 4} Let $k$ = {(1, 1), (2, 4), (3, 2), (4, 3)}
$k$ **is** a function from A to A.
Moreover, $k$ is an **injective** (an **injection**) function from A to A and $k$ **is** a **surjective** (a **surjection**) function from A to A so $k$ is **bijective** (a **bijection).**

**Exercises:**

**Exercise 14.01**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $g$ on A.

Consider A = {1, 2, 3, 4} Let $g$ = {(1, 1), (2, 2), (3, 2), (4, 4)}
(1) Find cor($g$). (2) Find cod($g$). (1) Find ran($g$). (2) Find dom($g$).

**Exercise 14.02**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $g$ on A.

Consider A = {1, 2, 3, 4} Let g = {(1, 1), (2, 2), (3, 2), (4, 4)}
(1) Prove $g$ is a function from A to A.
(2) Prove or disprove $g$ is an injection from A to A.
(3) Prove $g$ is a surjection from A to A.

**Exercise 14.03**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation $h$ on A.

Consider A = $\mathbb{N}_4$ Let $h$ = {(1, 1), (2, 3), (3, 4), (4, 2)}

Prove $h$ is a function from A to A.

**Exercise 14.04**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation $f_1$ from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_4$ Let $f_1$ = {(1, 6), (2, 0), (3, -1), (4, π)}

Prove $f_1$ is a real-valued function.

**Exercise 14.05**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation $f_2$ from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_{25}$ Let $f_1 : A \longrightarrow \mathbb{R} \ni f_1(a) = \sqrt{a}$

Prove $f_2$ is a real-valued function.

**Exercise 14.06**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Consider the relation $f_3$ from $\mathbb{N}$ to $\mathbb{R}$.

Let $f_3 : \mathbb{N} \longrightarrow \mathbb{R} \ni f_3(a) = 3a + 1$

Prove $f_3$ is a sequence.

**<u>Definition 14.07</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. Let $f$ : A $\longrightarrow$ B be a function from A to B. Let C $\subseteq$ A. Define $f|_C$ as the function $f|_C$ : C $\longrightarrow$ B $\ni$ $f|_C$ (c) = $f$(c) defined by

$f$ : A $\longrightarrow$ B $f|_C$ is called the restriction function of $f$ on C.

**<u>Definition 14.08</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. Let $f$ : A $\longrightarrow$ B be a function from A to B. Let A $\subseteq$ M $\subseteq$ $U$. Define $f|^M$ as a function $f|^M$ : M $\longrightarrow$ B $\ni$ $f|^M$ (m) = $f$(m) defined by

$f$ : A $\longrightarrow$ B and such that $\forall$ x $\in$ (M – A) $f|^M$ is defined properly (who cares how just properly) so that $f|^M$ is a function from M to B. $f|^M$ is called an extension function of $f$ on M.

**Examples:**

**<u>Example 14.06</u>**. Let the universe $W = \mathbb{Z} \times \mathbb{R}$. Let A $\subseteq \mathbb{N}$. Consider the relation $f_5$ from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_{25}$ Let $f_5$ : A $\longrightarrow \mathbb{R} \ni f_5$ (a) = $\sqrt{a}$

Note $f_5$ : A $\longrightarrow \mathbb{R} \ni$ f5 (a) = $\sqrt{a}$ is a well defined function from A to $\mathbb{R}$.

**Exercises:**

**Problems 7 – 10** Let C = $\mathbb{N}_{10}$ ; M = $\mathbb{N}_{28}$.; L = {1, 4, 7, 10};. and, J = {-4, -3, -2, -1, 0, 1, 2, 3, 4}

**<u>Exercise 14.07</u>**. Find $f_5|_C$ (note that it is unique).

**<u>Exercise 14.08</u>**. Find a $f_5|^M$ (note that it is not unique).

**<u>Exercise 14.09</u>**. Find $f_5|_L$ (note that it is unique).

**<u>Exercise 14.10</u>**. Try to create $f_5|^J$ - - note it cannot be defined.

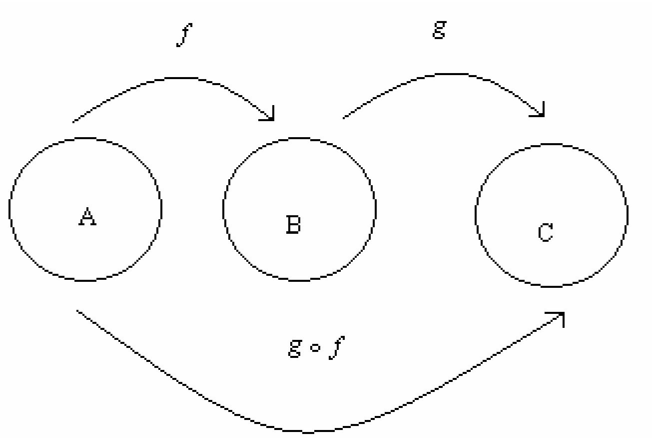
**<u>Definition 14.09</u>** : Let the universe $W = U \times V \times X$ be defined from the well defined universes $U$ , $V$, $X$ such that A $\subseteq U$ , B $\subseteq V$, whilst C $\subseteq X$. Let $f$ be a function from A to B and $g$ be a function from B to C. Define the composition function[1] $g \circ f$: A $\longrightarrow$ C such that

$g \circ f$ (x) =$g$($f$(x)) .

[1] Same definition as composition relation but just subsumed to functions.

So, since $f$: A $\longrightarrow$ B is a well defined function $\exists$ y $\in$ B $\ni$ $f$(x) = y. Since $g$: B $\longrightarrow$ C is a well defined function, dom($g$) = B. Hence, $\exists$ z $\in$ C $\ni$ $g$(y) = z.

So, we have $A \xrightarrow{f} B \wedge B \xrightarrow{g} C \Rightarrow A \xrightarrow{g \circ f} C$




Diagramme[2] depicting $A \xrightarrow{f} B \wedge B \xrightarrow{g} C \Rightarrow A \xrightarrow{g \circ f} C$

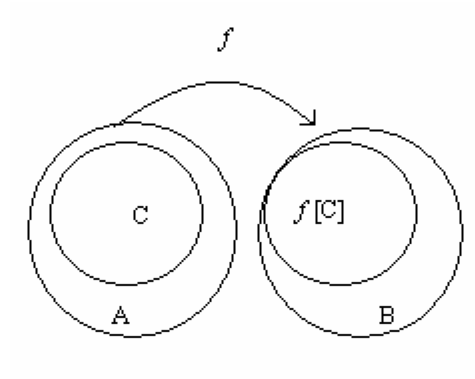
**<u>Theorem 14.01</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. Let the function $f$ : A $\to$ B be bijective. It is the case that the relation $f^{-1} \subseteq$ B $\times$ A is also a bijective function and note that $f^{-1}$ : B $\to$ A.
So, as $f$: A $\rightarrowtail\!\!\!\twoheadrightarrow$ B $\Rightarrow f^{-1}$ : B $\rightarrowtail\!\!\!\twoheadrightarrow$ A.

**<u>Definition 14.10</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. Let C $\subseteq$ A. Let $f$ be a function from A to B.
The set $f$[C] is a subset of B and is called the image set of C under $f$
$f$[C] = {d $\in$ B: $f$(c) = d where c $\in$ C}.

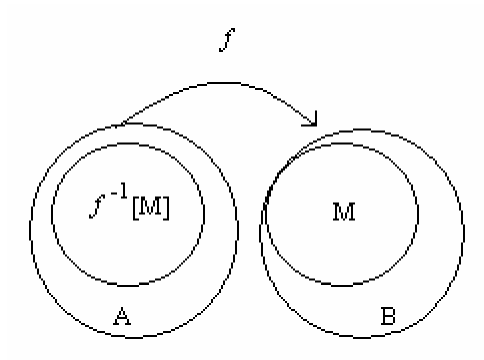
**<u>Definition 14.11</u>** : Let the universe $W = U \times V$ be defined from the well defined universes $U$ and $V$ such that A $\subseteq U$ whilst B $\subseteq V$. Let M $\subseteq$ B. Let $f$ be a function from A to B.
The set $f^{-1}$[M] is a subset of A and is called the inverse image set of M under $f$
$f^{-1}$[M] = {a $\in$ A: $f$(a) = m where m $\in$ M}.

Note: The existence of the set $f^{-1}$[M] does not imply that $f^{-1}$ is a function from B to A. We know that since $f$ is a function from A to B it was a relation from A to B; hence, $f^{-1}$ is a relation from B to A but is not necessarily a function from B to A.

[2] Notice this is not a Venn Diagramme since there is not a drawing of the universe.




Diagramme[3] depicting image set $f[C]$




Diagramme[4] depicting inverse image set $f^{-1}[M]$

**Examples:**

**Example 14.07**.Let the universe $W = \mathbb{Z} \times \mathbb{R}$. Let $A \subseteq \mathbb{Z}$. Consider the function $f_6$ from A to $\mathbb{R}$.

Consider A = { -2, -1, 0, 1, 2}   Let $f_6 : A \longrightarrow \mathbb{R} \ni f_6(a) = a^2$

Let C = {-1, 0, 1, 2} Notice that $f[C] = \{d \in B: f(c) = d \text{ where } c \in C\}$ so
$f[C] = \{0, 1, 4\}$.

Let $M = [\frac{1}{\sqrt{2}}, 9)$. Notice that $f^{-1}[M] = \{a \in A: f(a) = m \text{ where } m \in M\}$ so
$f^{-1}[M] = \{-1, 1, 2\}$.
Let K = [ ¼ , ¾). Notice that $f^{-1}[K] = \varnothing$.
Let C = {-1, 0, 1, 2} Notice that $f^{-1}[C] = \{-1, 0, 1\}$.
Let $L = (-\infty, 0)$ Notice that $f^{-1}[L] = \varnothing$.

3 Notice this is not a Venn Diagramme since there is not a drawing of the universe.
4 Notice this is not a Venn Diagramme since there is not a drawing of the universe.

**Exercises:**

**11 – 14**. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation $g$ on A.

Consider A = {1, 2, 3, 4} Let $f$ = {(1, 1), (2, 2), (3, 2), (4, 4)}

**Exercise 14.11**. Let C = {2, 3}. Find the set $f$[C].
**Exercise 14.12**. Let M = {3, 4}. Find the set $f^{-1}$[M].
**Exercise 14.13**. Let K = {3, 4}. Find the set $f$[K].
**Exercise 14.14**. Let L = {2, 3}. Find the set $f^{-1}$[L].

**15 – 18**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation $h$ on A.

Consider A = $\mathbb{N}_4$ Let $h$ = {(1, 1), (2, 3), (3, 4), (4, 2)}

**Exercise 14.15**. Let C = {2, 3}. Find the set $h$[C].
**Exercise 14.16**. Let M = {3, 4}. Find the set $h^{-1}$[M].
**Exercise 14.17**. Let K = {3, 4}. Find the set $h$[K].
**Exercise 14.18**. Let L = {2, 3}. Find the set $h^{-1}$[L].

**19 – 22**. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation $f_2$ from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_{25}$ Let $f_1 : A \longrightarrow \mathbb{R} \ni f_1(a) = \sqrt{a}$

**Exercise 14.19** Let C = {2, 3}. Find the set $f_1$[C].
**Exercise 14.20**. Let M = {3, 4}. Find the set $f_1^{-1}$[M].
**Exercise 14.21**. Let K = {3, 4}. Find the set $f_1$[K].
**Exercise 14.22** Let L = {2, 3}. Find the set $f_1^{-1}$[L].

**23 – 26** Let the universe $W = \mathbb{N} \times \mathbb{R}$. Consider the relation $f_3$ from $\mathbb{N}$ to $\mathbb{R}$.

Let $f_3 : \mathbb{N} \longrightarrow \mathbb{R} \ni f_3(a) = 3a + 1$.

23. Let J = $\mathbb{N}_4$. Find the set $f_3$[J].

24. Let M = $\mathbb{N}_4$. Find the set $f_3^{-1}$[M].

25. Let K = {3, 8}. Find the set $f_3$[K].
26. Let L = [3, 8]. Find the set $f_3^{-1}$[L].

**From this point on the discussion is purely one of enrichment; it is NOT part of the course but is 'future excitement.'**

We could study functions in even more detail. If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

<u>**Definition Set**</u>: Assume the universe $W = \mathbb{R} \times \mathbb{R}$ be defined.

Let that $A \subseteq \mathbb{R}$ whilst $B \subseteq \mathbb{R}$.

<u>Definition F.1</u>: A function $f : A \to B$ is **even** iff for every $x \in A$, $f(-x) = f(x)$

<u>Definition F. 2</u>: A function $f$ is **periodic** iff there exists a $k > 0$ such that for every $x \in A$ x, $f(x + k) = f(x)$

<u>Definition F. 3</u>: A function $f$ is **non-decreasing** iff for every $x \in A$ and for every $y \in A$, $x \leq y \Rightarrow f(x) \leq f(y)$

<u>Definition F. 4</u>: A function $f$ is **increasing** iff for every $x \in A$ and for every $y \in A$, $x < y \Rightarrow f(x) < f(y)$

<u>Definition F. 5</u>: A function $f$ is **non-decreasing** iff for every x and for every y, $x \leq y \Rightarrow f(x) \geq f(y)$

<u>Definition F. 6</u>: A function $f$ is **decreasing** iff for every x and for every y, $x < y \Rightarrow f(x) > f(y)$

<u>Definition F. 7</u>: A function $f$ is **constant** iff for every x and for every y, $x < y \Rightarrow f(x) = f(y)$

<u>Definition F. 8</u>: The number L is the **limit** of the function $f : A \to B$ **at a ∈ A** iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $| f(x) - L | < \varepsilon$ whenever $x \in A$ and $0 < | x - a | < \delta$.

<u>Definition F. 9</u>: A function $f : A \to B$ is **continuous** at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $| f(x) - f(a) | < \varepsilon$ whenever $| x - a | < \delta$.

<u>Definition F. 10</u>: A function $f : A \to B$ is **uniformly continuous** on the set $C \subseteq A$ at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $| f(x) - f(y) | < \varepsilon$ whenever $x \in C$, $y \in C$, and $| x - y| < \delta$.

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