MATH 182
HANDOUT III: SEQUENCES
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Basic definitions we will use on $\mathbb{R}$ and $\mathbb{R}^2$ (and the rest of the semester).

Let $U = \mathbb{R}$ for all definitions and $A \subseteq \mathbb{R}$ (of course).

Definition 1: Let A = [a, b] such that $a \le b$. then A is called an **interval**.
Definition 2: Let A = (a, b) such that $a < b$. then A is called a **segment**.
Definition 3: Let A = [a, b) such that $a < b$. then A is called **a half segment (or a half interval).**
Definition 4: Let A = (a, b] such that $a < b$. then A is called **a half segment (or a half interval).**
Definition 5: $\mathbb{R}$ can be denoted as $(-\infty, \infty)$ where $-\infty$ and ∞ are just symbols to mean on to the left ad infinitum and on to the right ad infinitum (meaning $-\infty$ and ∞ are *not* numbers).

Definition 6: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined such that $A = \mathbb{N}$ whilst $B \subseteq \mathbb{R}$. The f is called a sequence from $\mathbb{N}$ to B (or more precisely a well-defined function that is called a sequence from $\mathbb{N}$ to B but we will allow for less precise language and call it a sequence) since it is a real-valued function (since the dom(f) = cor(f) = $\mathbb{N}$ and the cod(f) $\subseteq \mathbb{R}$).

Notation: $f\colon \mathbb{N} \longrightarrow B$ or f_n

Definition 7: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called an *increasing* sequence *iff* $f_{(n)} < f_{(n+1)}\ \forall\ n \in \mathbb{N}$

Definition 8: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *decreasing* sequence *iff* $f_{(n)} > f_{(n+1)}\ \forall\ n \in \mathbb{N}$

Definition 9: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *non-increasing* sequence *iff* $f_{(n)} \ge f_{(n+1)}\ \forall\ n \in \mathbb{N}$

Definition 10: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *non-decreasing* sequence *iff* $f_{(n)} \le f_{(n+1)}\ \forall\ n \in \mathbb{N}$

Definition 11: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *constant sequence iff* $f_{(n)} = f_{(n+1)}\ \forall\ n \in \mathbb{N}$

Definition 12: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).
f_n is called a *monotone sequence iff* it is increasing, decreasing, non-increasing, or non-decreasing.

Definition 13: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *bounded above sequence iff* $\exists\ p \in \mathbb{R} \ni p \geq f_{(n)}\ \forall\ n \in \mathbb{N}$

Definition 14: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *bounded below* sequence *iff* $\exists\ q \in \mathbb{R} \ni q \leq f_{(n)}\ \forall\ n \in \mathbb{N}$

Definition 15: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *bounded* sequence *iff* $\exists\ p \in \mathbb{R} \wedge \exists\ q \in \mathbb{R} \ni q \leq f_{(n)} \leq p\ \forall\ n \in \mathbb{N}$

Definition 15 (alternate): Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

f_n is called a *bounded* sequence *iff* $\exists\ k \in \mathbb{R} \ni \left|f_{(n)}\right| \leq k\ \forall\ n \in \mathbb{N}$

Theorem 1: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

Every monotonic and bounded sequence converges (the $\lim_{n \longrightarrow \infty} f_n = L$ ($L \in \mathbb{R}$) exists).

Definition 16: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

We say $\lim_{n \longrightarrow \infty} f_n = L$ ($L \in \mathbb{R}$) if $\forall\ \varepsilon > 0\ \exists\ m \in \mathbb{R} \ni \left|f_{(n)} - L\right| < \varepsilon\ \forall\ n \in \mathbb{N}$ where $n > m$ (the sequence f converges to L).

Definition 17: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n).

We say $\lim_{n \longrightarrow \infty} f_n$ does not exist (DNE) if $\nexists$ L ($L \in \mathbb{R}$) to which f converges.

Definition 18: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n). We say

$\lim_{n \longrightarrow \infty} f_n$ DNE in a 'blow up' sense if $\forall\ y \in \mathbf{R}\ \exists\ m \in \mathbb{R} \ni f_{(n)} > y\ \forall\ n \in \mathbb{N}$ where $n > m$.
(the sequence f diverges).

Definition 19: Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined and let f be a sequence (f_n). We say

$\lim_{n \longrightarrow \infty} f_n$ DNE in a 'blow down' sense if $\forall\ y \in \mathbf{R}\ \exists\ m \in \mathbb{R} \ni f_{(n)} < y\ \forall\ n \in \mathbb{N} \ni n > m$.
(the sequence f diverges).